How interpreting Whittaker functions as lattice models led to an unexpected duality

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Workshop on the representation theory of p-adic groups
and connections to quantum groups, geometry and combinatorics

University of Amsterdam

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Slides available at https://hgustafsson.se

Papers

Joint work with Ben Brubaker, Valentin Buciumas and Daniel Bump

(0. Vertex operators, solvable lattice models and metaplectic Whittaker functions
Communications in Mathematical Physics 380 (Dec, 2020), 535–579)

1. Colored five-vertex models and Demazure atoms
Journal of Combinatorial Theory, Series A 178 (Feb, 2021)

2. Colored vertex models and Iwahori Whittaker functions
arXiv:1906.04140

3. Metaplectic Iwahori Whittaker functions and supersymmetric lattice models
arXiv:2012.15778

4. Iwahori-metaplectic duality (recently updated)
arXiv:2112.14670

Include both pure representation theoretical and lattice model results

focus today

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Outline

Origin: study p-adic Whittaker functions using lattice models.

- Construct first toy lattice model describing Schur polynomials.
- Define the spherical Whittaker functions we study.
- Refine to Iwahori Whittaker functions by adding colors to lattice model.
- Metaplectic covers and Whittaker functions.
- Iwahori–metaplectic duality.

Why lattice models?

- Powerful toolbox from statistical mechanics to manipulate models and prove identities.
- Building new bridges between widely different mathematical objects. (See also Paper 0).
- Surprisingly effective at describing these representation theoretical objects: bijection of data, highly constrained by solvability conditions.
- Generator of ideas and conjectures.

First toy lattice model

Schur polynomials

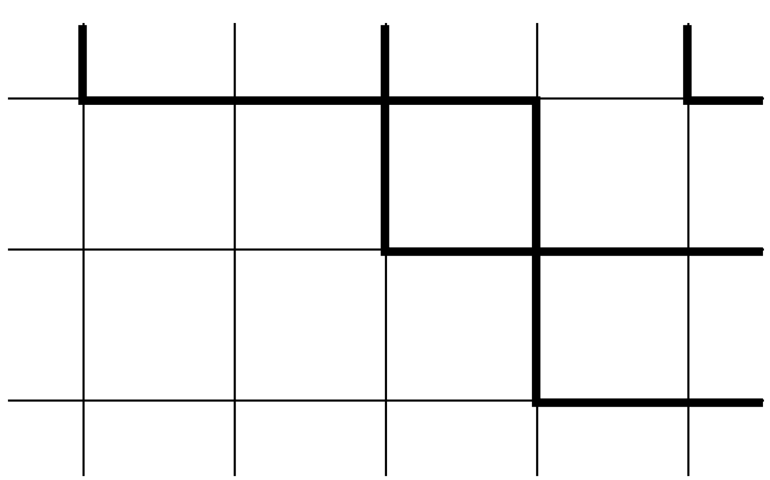
First toy lattice model

Construct lattice model describing Schur polynomials
"Half-way" to Whittaker functions.

Achieved by using an already known combinatorial description.
(This is not the case in our papers – we use solvability of the lattice model)

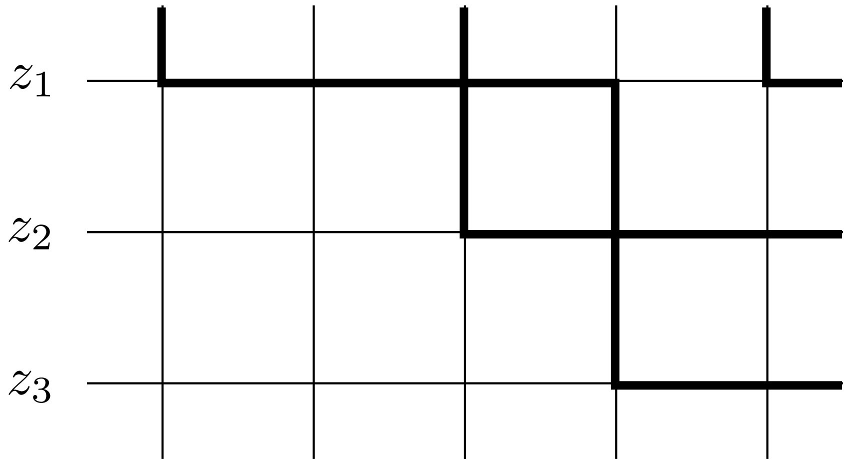
The lattice model consists of a two-dimensional grid with r rows, sufficiently many columns, and each vertex has four adjacent edges.

We will assign data to these edges according to certain rules, and in this first example the data is binary: the edge is **filled in**, or not.



These edges will form paths on the grid, and for given boundary conditions there is a finite number of configurations called states.

First toy lattice model



These edges will form paths on the grid, and for given boundary conditions there is a finite number of configurations called states.

A state $\mathfrak{s}$ is assigned a Boltzmann weight $\beta(\mathfrak{s}) \in \mathbb{C}[\mathbf{z}]$ depending on parameters $\mathbf{z} = (z_1, z_2, \ldots, z_r) \in \mathbb{C}^r$ (one for each row).

The partition function, given som fixed boundary conditions:

$$Z := \sum_{\substack{\text{state } \mathfrak{s} \\ \text{with given b.c.}}} \beta(\mathfrak{s})$$

Goal: any Schur polynomial in $\mathbf{z}$ = such a partition function.

Schur polynomials

Let $\lambda = (\lambda_1, \ldots \lambda_r)$ be a partition ~~of~~ $\cancel{r}$ padded with zeroes to length r. We define the Schur polynomial $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by

$$s_\lambda(\mathbf{z}) = \frac{\det(z_i^{(\lambda+\rho)_j})_{ij}}{\det(z_i^{\rho_j})_{ij}}$$

where $\mathbf{z} = (z_1, \ldots, z_r)$ and $\rho = (r-1, r-2, \ldots, 1, 0)$.

Combinatorial description using Semi-Standard Young Tableaux of shape λ

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

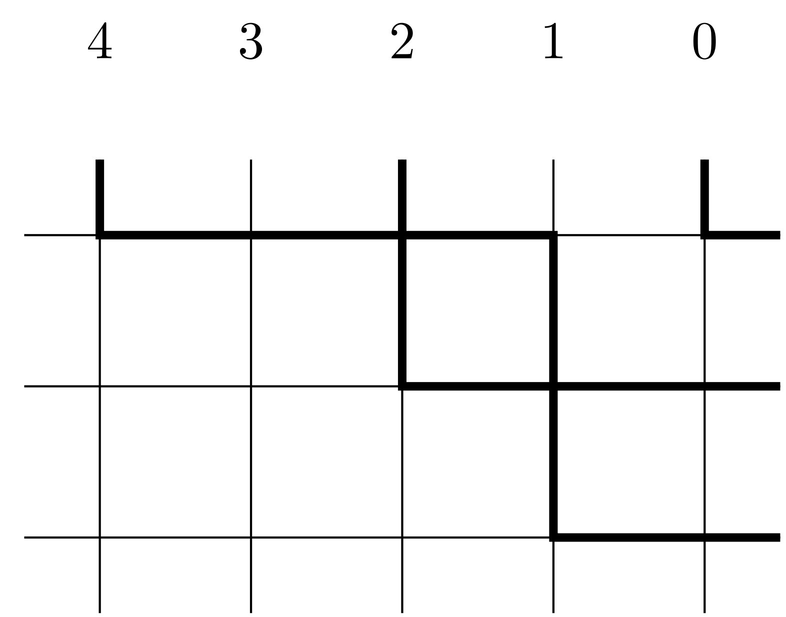
(#ones, #twos, #threes, ...)

$\lambda = (3, 1, 1)$ $\quad$ $\mathrm{SSYT}(\lambda) \ni T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$ $\quad$ $\mathrm{wt}(T) = (2, 2, 0, 0, 1)$

Lattice paths

SSYT $\overset{\sim}{\longleftrightarrow}$ south-east moving lattice paths
(certain)

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$$



$\lambda \vdash r$

r rows

Let $\lambda^{(i)}(T) \in \mathbb{Z}^i$ be the shape of T after removing labels larger than i

$$\lambda^{(3)}(T) = \mathrm{shape}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}\right) = (2,1,0)$$

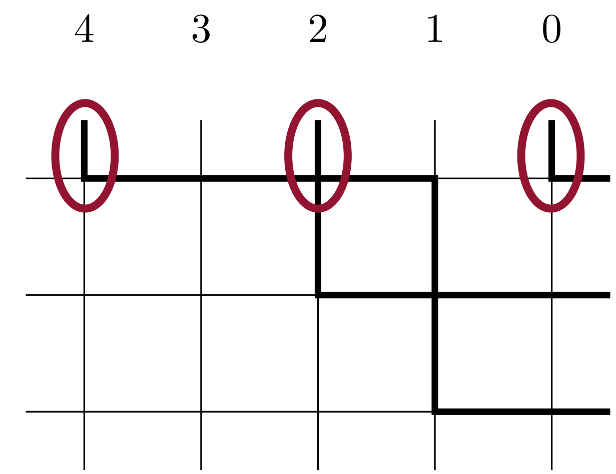
$$\lambda^{(2)}(T) = \mathrm{shape}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = (1,1)$$

$$\lambda^{(1)}(T) = \mathrm{shape}\left(\begin{array}{|c|}\hline 1 \\ \hline \end{array}\right) = (1)$$

These will label which columns are filled in for each row.

Lattice paths

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$



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These will label which columns are filled in for each row.

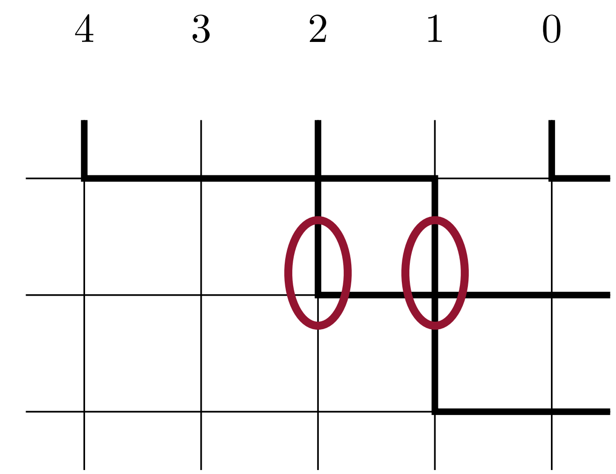
To avoid overlapping edges we add $\rho^{(r)} = (r-1, r-2, \ldots, 1, 0)$ to each shape:

$$\left\{\begin{array}{c} \lambda^{(3)}(T) + \rho^{(3)} \\ \lambda^{(2)}(T) + \rho^{(2)} \\ \lambda^{(1)}(T) + \rho^{(1)} \end{array}\right\} = \left\{\begin{array}{ccccc} 4 & & 2 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array}\right\}$$

Gelfand-Tsetlin pattern

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

Lattice paths



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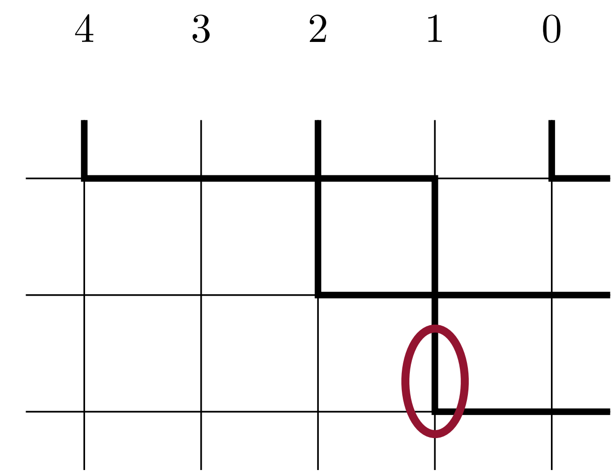

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Lattice paths

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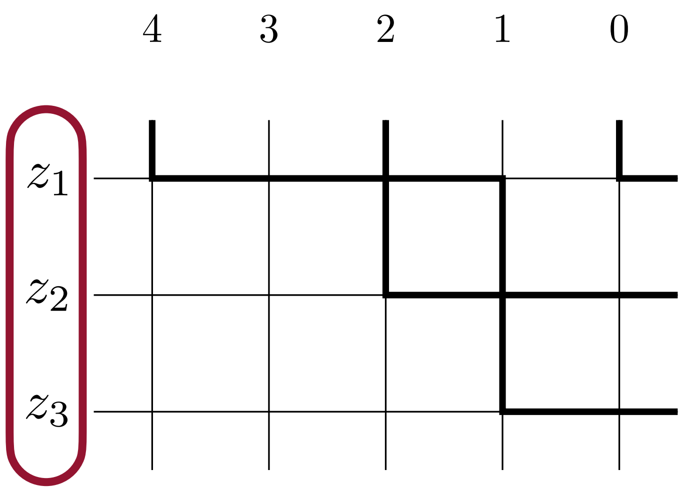
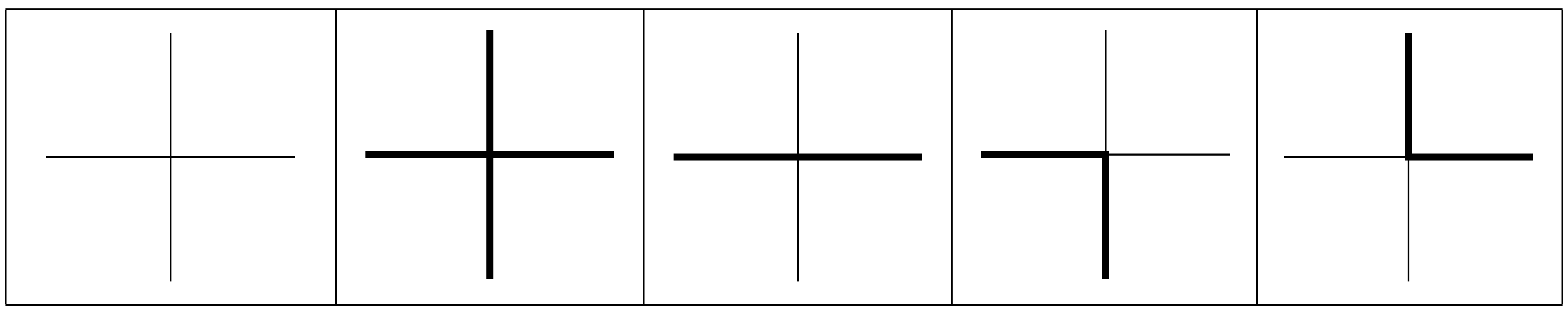


Five different vertex configurations:



$\mathrm{SSYT} \overset{\sim}{\longleftrightarrow}$ lattice paths using these vertex configurations

shape λ filled in top boundary edges at columns $\lambda + \rho$

Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

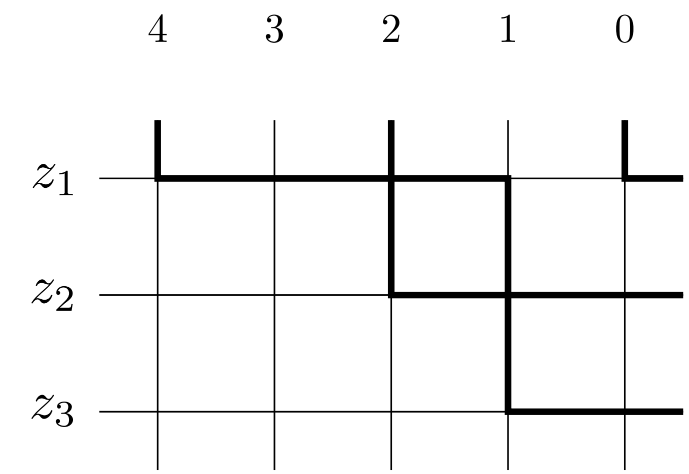
$\mathrm{wt}(T)$ counts the number of filled in left-edges in each row

Introduce row parameters $z_1, \ldots, z_r \in \mathbb{C}$ and vertex weights at row i

Lattice paths

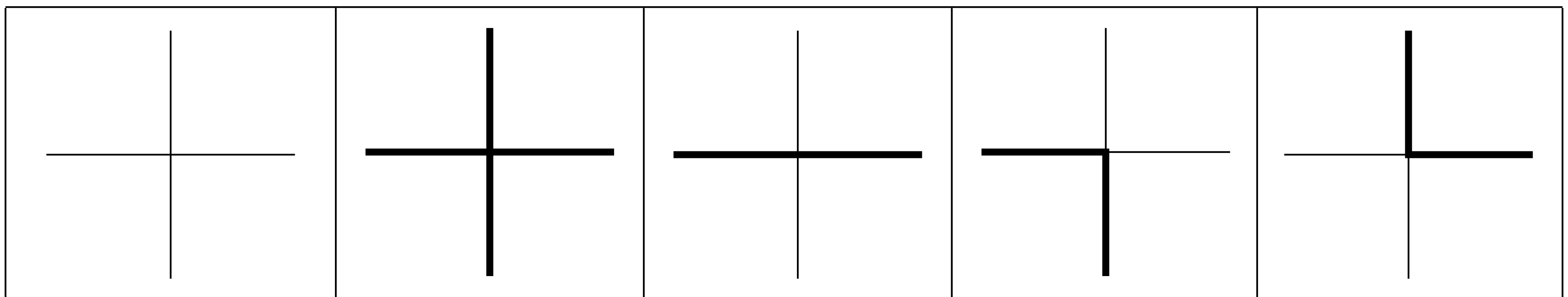
$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

state $\mathfrak{s}$ $\longrightarrow$



$\beta(\mathfrak{s}) = z_1^3 z_2^2 z_3$

Five different vertex configurations:



1	z_i	z_i	z_i	1

Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

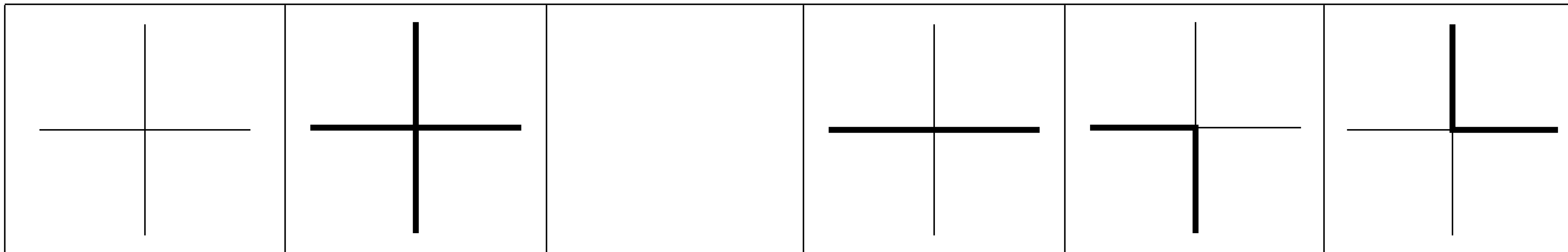
Boltzmann weight $\beta(\mathfrak{s}) := \prod_{\text{vertex}}$ vertex weights $= \mathbf{z}^\rho \cdot (w_0 \mathbf{z})^{\mathrm{wt}(T)}$

$w_0(z_1, z_2, \ldots, z_r) = (z_r, \ldots, z_2, z_1)$

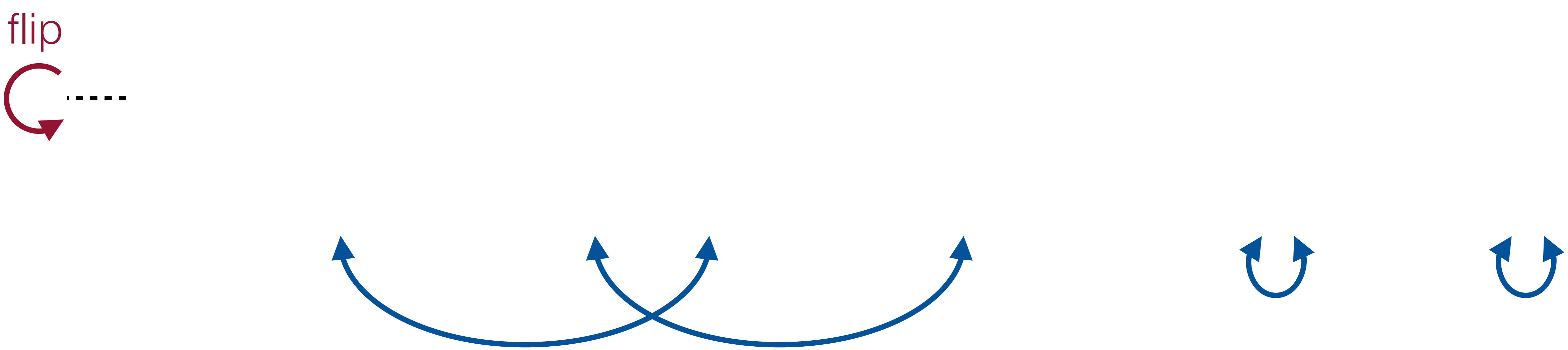
Partition function $Z(\lambda, \mathbf{z}) := \sum_{\mathfrak{s} \text{ with top } \lambda+\rho} \beta(\mathfrak{s}) = \mathbf{z}^\rho s_\lambda(w_0 \mathbf{z}) = \mathbf{z}^\rho s_\lambda(\mathbf{z})$

From 5 to 6 vertex configurations

Symmetry of vertex configurations using arrow description

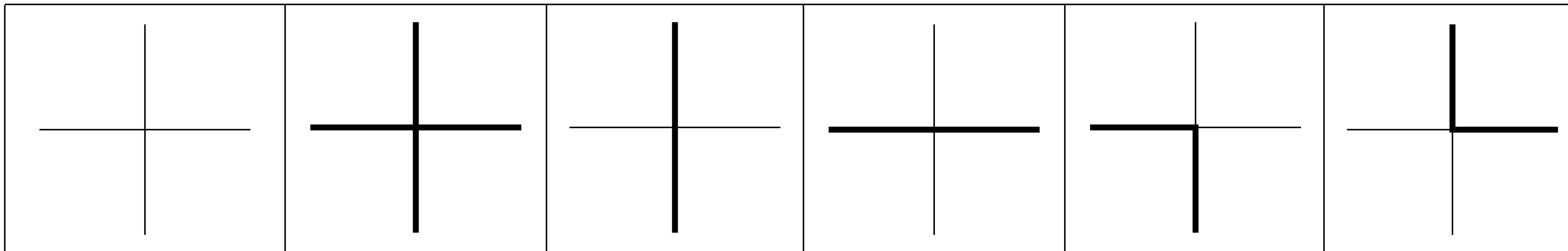


1	z		z	z	1



Weights are adjusted for solvability (to satisfy Yang–Baxter equation).

From 5 to 6 vertex configurations



1	z	$-v$	z	$(1-v)z$	1

These new weights introduce a slight deformation of the partition function

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^{\rho}\prod_{i<j}(1 - v\tfrac{z_j}{z_i})s_{\lambda}(\mathbf{z})$$

[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

If $v=-1$ then a flip preserves the Boltzmann weight of the state.
The flip can be used to prove Cauchy identity for Schur polynomials.

Whittaker functions

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^\rho \prod_{i<j}(1 - v\frac{z_j}{z_i})s_\lambda(\mathbf{z})$$

is a Whittaker function

Whittaker functions

$G = \mathrm{GL}_r(F)$ $\quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$ $\quad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

Non-archimedean field.
Here $F = \mathbb{Q}_p$ for simplicity.

Character
$\psi : N \to \mathbb{C}^\times$
(principal; standard)

Whittaker model $\quad \pi \xhookrightarrow{\sim} \mathcal{W}_\psi(\pi) \subset \mathrm{Ind}_N^G(\psi)$

Image of G-equivariant embedding in $\{f : G \to \mathbb{C} \mid f(ng) = \psi(n) f(g)\}$

Whittaker function $\in \mathcal{W}_\psi(\pi)$

We will consider:

Unramified principal series representation $\pi_{\mathbf{z}}$ given by $\mathbf{z} \in (\mathbb{C}^\times)^r$

$f : G \to \mathbb{C}$ induced from B using an unramified character determined by $\mathbf{z}$

Whittaker functions

$$G = \mathrm{GL}_r(F) \qquad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \qquad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

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Whittaker function $\in \mathcal{W}_\psi(\pi)$

Unramified principal series representation $\pi_{\mathbf{z}}$ given by $\mathbf{z} \in (\mathbb{C}^\times)^r$

Embedding given by

$$\pi_{\mathbf{z}} \ni f : G \to \mathbb{C} \qquad \mathcal{W}_\psi(f) \; : \; g \longmapsto \int_N f(w_0 n g)\psi(n)^{-1} dn$$

long Weyl group element

The Whittaker model is unique if it exists [Gelfand–Kazhdan 1972, Rodier 1973].

Whittaker functions

$G = \mathrm{GL}_r(F)$ $\quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$ $\quad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

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Whittaker model $\quad \pi \xhookrightarrow{\sim} \mathcal{W}_\psi(\pi) \subset \mathrm{Ind}_N^G(\psi)$

Whittaker function $\in \mathcal{W}_\psi(\pi)$ $\quad \mathcal{W}_\psi(f) \; : \; g \longmapsto \int_N f(w_0 n g)\psi(n)^{-1} dn$

Unramified principal series representation $\pi_{\mathbf{z}}$ given by $\mathbf{z} \in (\mathbb{C}^\times)^r$

Right-invariant under $K := \mathrm{GL}_r(\mathbb{Z}_p)$

There is a unique spherical vector $f_{\mathbf{z}}^\circ$ in $\pi_{\mathbf{z}}$ up to normalization.

The corresponding spherical Whittaker function $\mathcal{W}_\psi(f_{\mathbf{z}}^\circ)$ is determined by its values on $g = p^\lambda := \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_r})$ with $\lambda \in \mathbb{Z}^r$ as

$$\mathcal{W}_\psi(f_{\mathbf{z}}^\circ)(p^\lambda) = \prod_{i<j}\left(1 - p^{-1}\frac{z_j}{z_i}\right)s_\lambda(\mathbf{z}) = \mathbf{z}^{-\rho}\, Z(\lambda; \mathbf{z}) \text{ with } v = p^{-1}$$

lattice model partition function

[Casselman 1980, Casselman–Shalika 1980]

Generalizations

Lattice models for other Whittaker functions

Generalizations

Lattice models for other Whittaker functions

spherical vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Number theory generalization
[Brubaker–Bump–Chinta–Friedberg–Gunnells 2012]

Representation theory refinement
Papers 1 & 2

spherical vector
for metaplectic
n-cover $\tilde{G}$ of G

Iwahori fixed vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

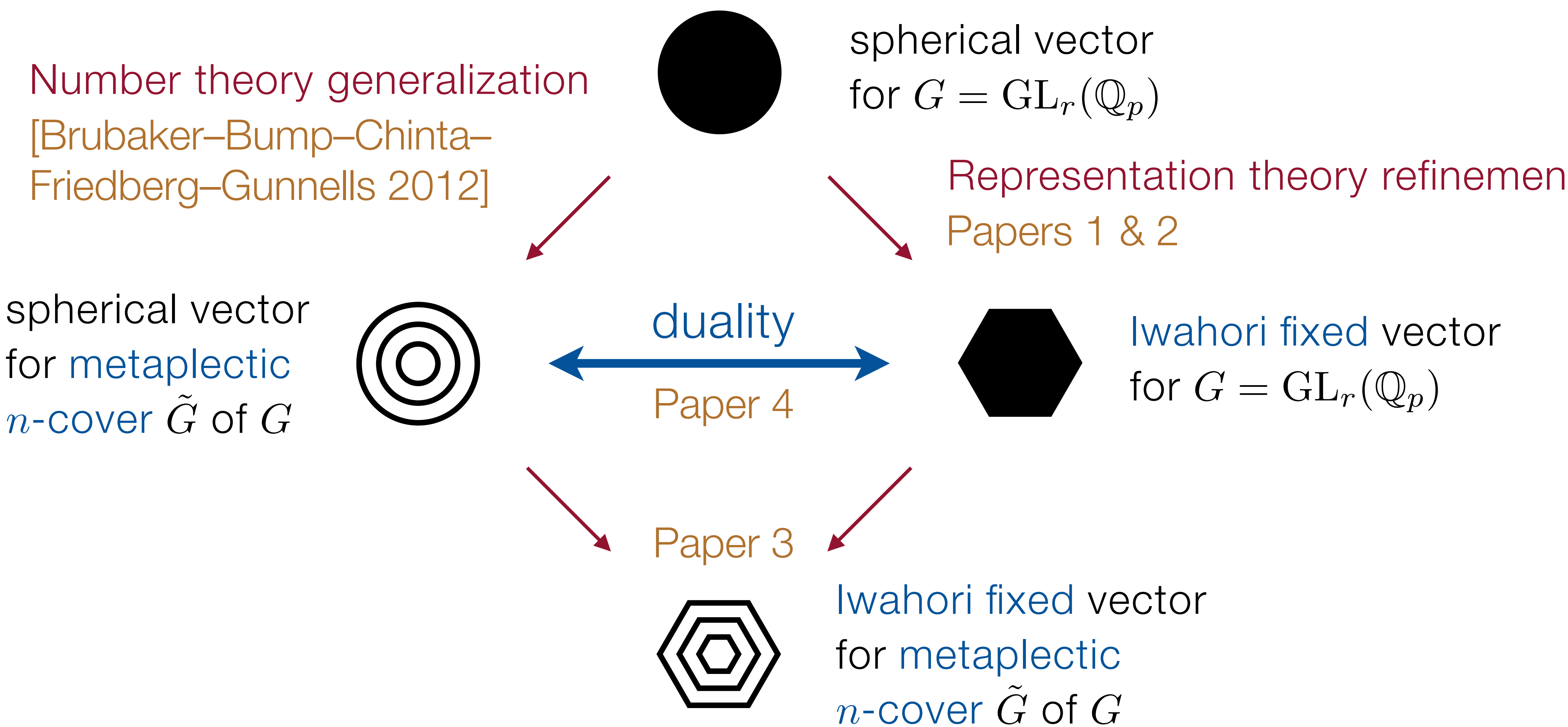
Paper 3

Iwahori fixed vector
for metaplectic
n-cover $\tilde{G}$ of G

Blue terms will be defined in the next slides

Generalizations

Lattice models for other Whittaker functions



Blue terms will be defined in the next slides

Generalizations

spherical vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

unique up to normalization

$f^\circ(gk) = f^\circ(g)$ for
$k \in K := \mathrm{GL}_r(\mathbb{Z}_p)$
maximal compact

$$G = BK$$

$$B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$$

$$B^- = \begin{pmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix}$$

Iwahori fixed vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

basis enumerated by $W = S_r$

$f(gk) = f(g)$ for
$k \in J \equiv B^- \pmod{p} \subset K$
Iwahori subgroup

$$G = \bigsqcup_{w \in W = S_r} BwJ$$

Refinement: $f^\circ_{\mathbf{z}} = \sum_{w \in W} f^{(w)}_{\mathbf{z}}$ each supported only on BwJ

Generalizations

On the lattice model side this refinement corresponds to assigning a different color to each path, making them distinct.

Schematically (with details to follow):

Paper 1 (5-vertex; $v = 0$)

Schur polynomial

colorize

Duplicate colors:
Demazure characters

Demazure atoms

Paper 2 (6-vertex; $v \neq 0$)

spherical Whittaker function

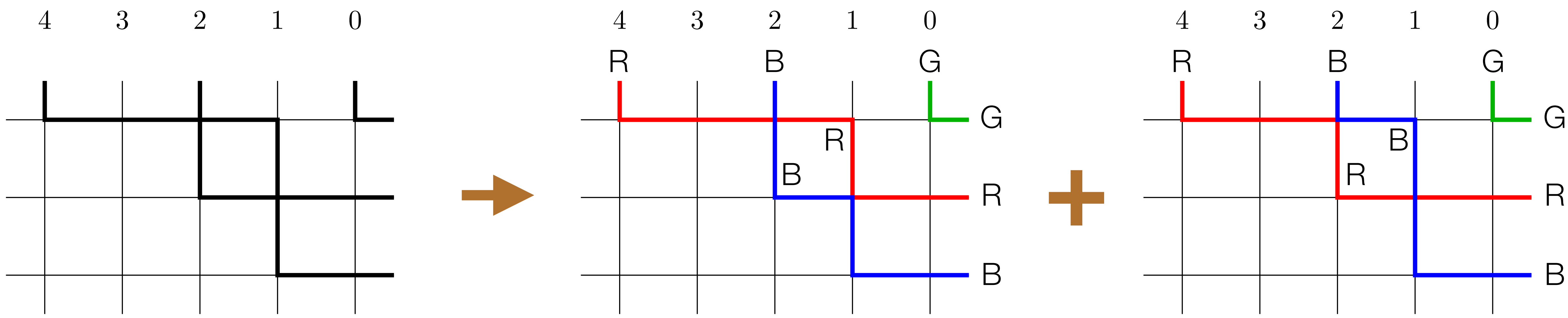
colorize

Duplicate colors:
parahoric Whittaker functions

Iwahori Whittaker functions

Color refinement

Ordered palette of r colors: R $>$ B $>$ G



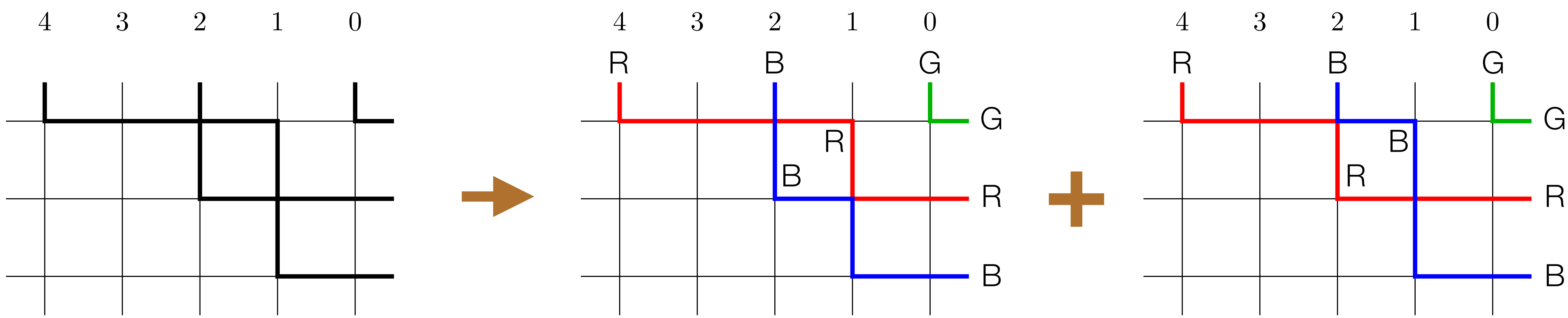
New right boundary data: permutation $w \in S_r$ of (R, B, G)

Have constructed vertex configuration weights such that the partition function is refined to:

uncolored $$Z(\lambda; \mathbf{z}) = \sum_{w \in S_r} Z(\lambda, w; \mathbf{z})$$ colored

[Papers 1 & 2] Concept based on [Borodin–Wheeler 2018]

Color refinement



uncolored $\quad Z(\lambda;\mathbf{z}) = \sum_{w\in S_r} Z(\lambda, w; \mathbf{z}) \quad$ colored

In more detail:

Bijection of data

Paper 1 (5-vertex; $v=0$)

states $\longleftrightarrow$ crystal Demazure atoms

Theorem:

$Z(\lambda, w;\mathbf{z})_{v=0} =$ Demazure atom

$\sum_{w\in S_r} \longrightarrow$ Schur polynomial

Paper 2 (6-vertex; $v\neq 0$)

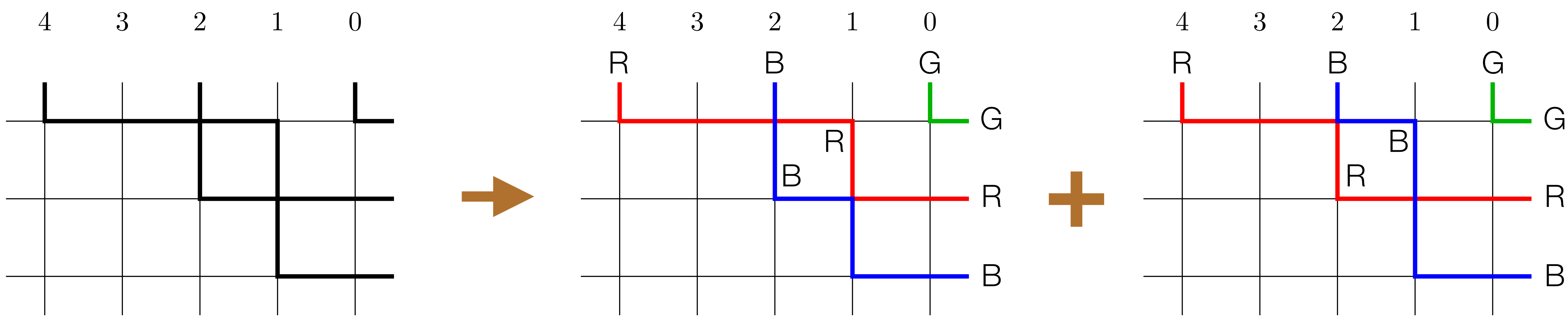
boundary data $\longleftrightarrow$ Whittaker data

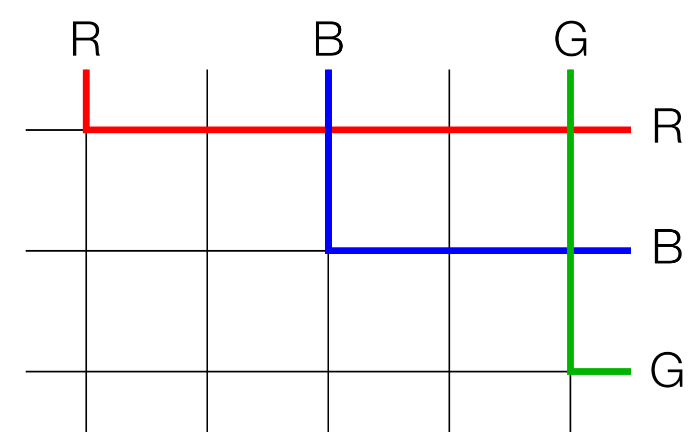
Theorem:

$Z(\lambda, w;\mathbf{z})_{v=p^{-1}} =$ Iwahori Whittaker function $\mathcal{W}_\psi(f_{\mathbf{z}}^{(w)})(p^\lambda)$

$\sum_{w\in S_r} \longrightarrow$ Spherical Whittaker function

Color refinement





When $w = 1$ there is only one allowed state, and the partition function can easily be computed to be $\mathbf{z}^{\lambda+\rho}$.

Lattice model is solvable, i.e. satisfies Yang–Baxter equations from underlying quantum group, which gives:

Theorem: [Papers 1, 2] $$Z(\lambda, s_{i_1} \cdots s_{i_r}; \mathbf{z}) = T_{i_1} \cdots T_{i_r} \mathbf{z}^{\lambda+\rho}$$

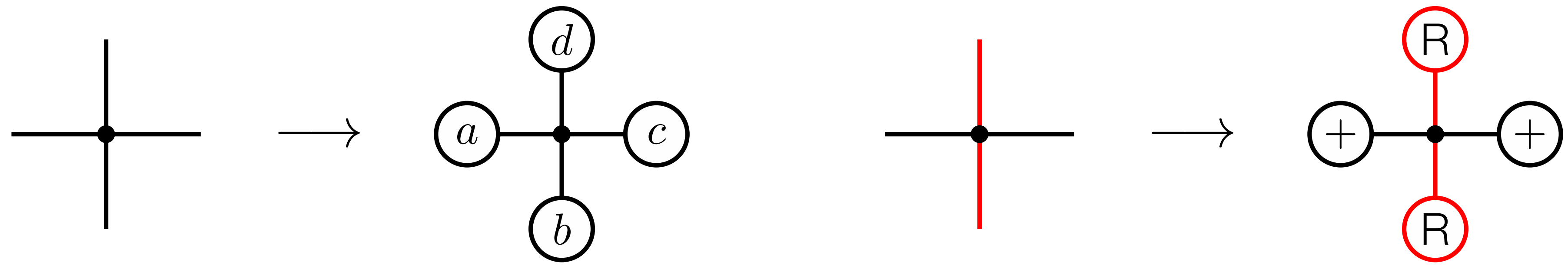
Divided difference Demazure operators

$$T_i f(\mathbf{z}) = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} f(s_i \mathbf{z}) + \frac{v-1}{1-\mathbf{z}^{\alpha_i}} f(\mathbf{z})$$

Same relations as for Whittaker functions in [Brubaker–Bump–Licata 2015] (non-metaplectic)

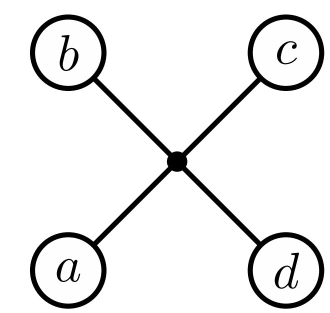
Yang-Baxter equations

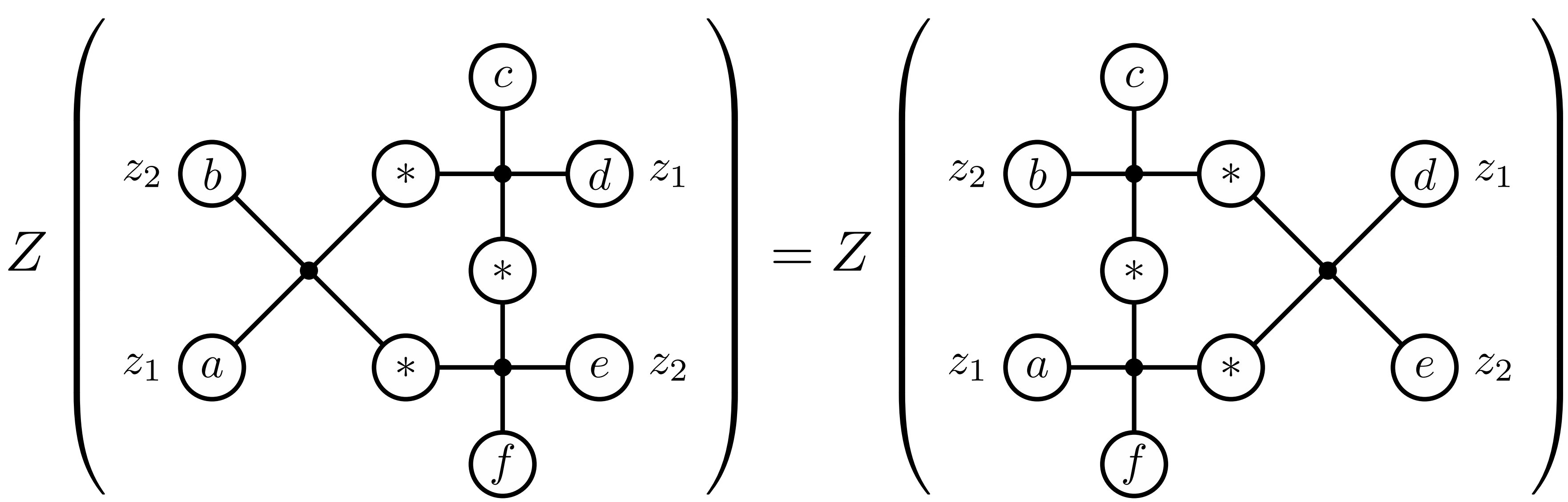
Label edges:



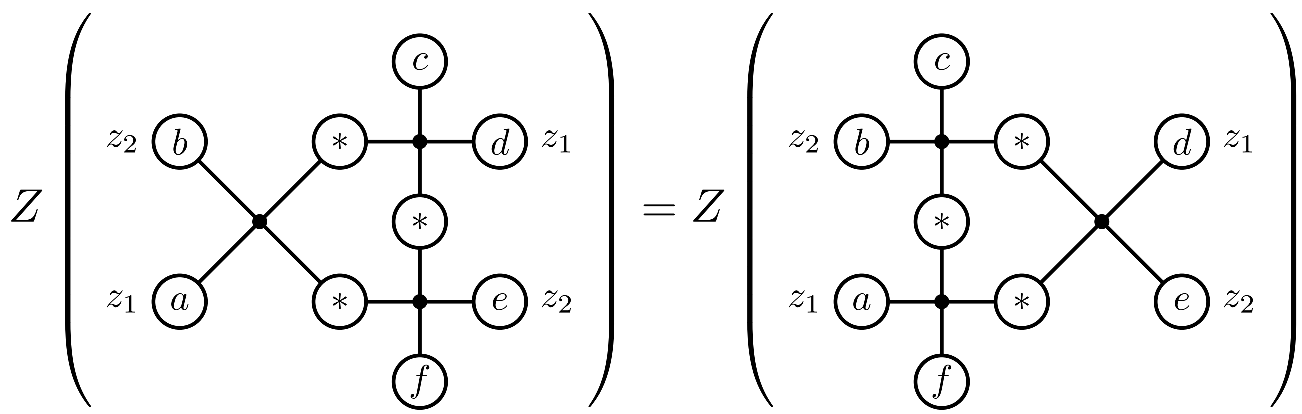
What happens when two rows of the lattice are switched?

The Yang–Baxter equation gives the answer for one column and includes a new type of vertex between rows:

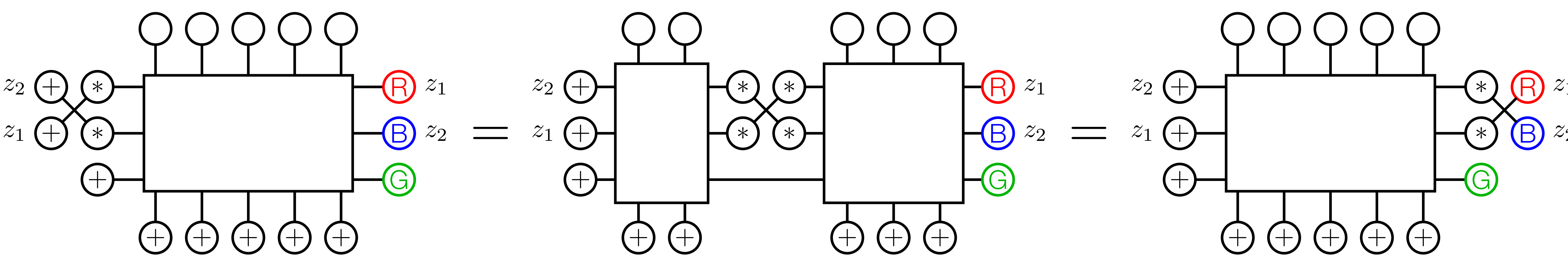




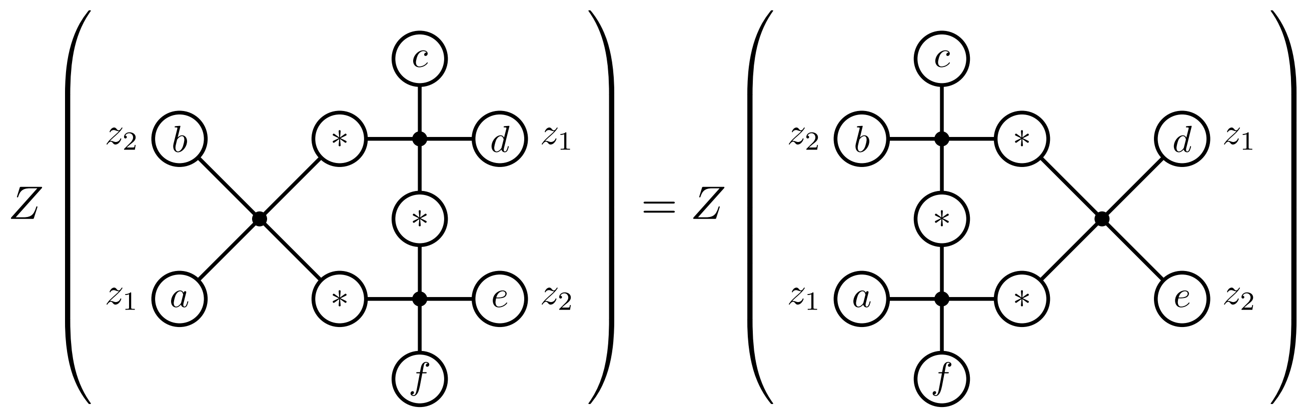
Yang-Baxter equations

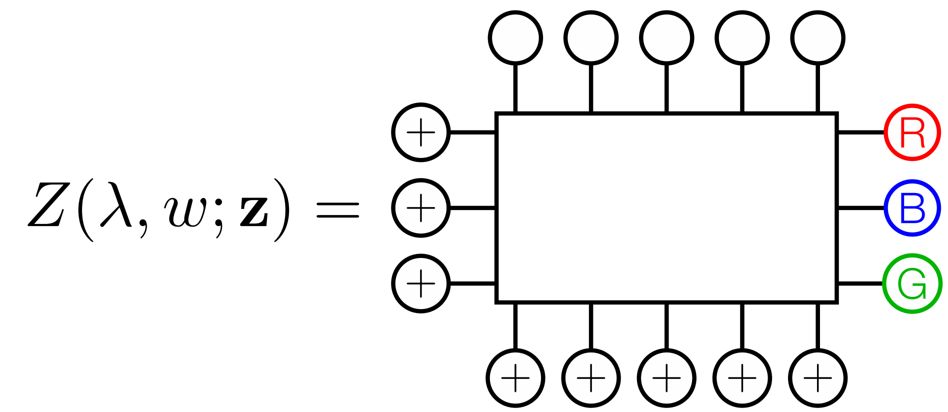


Train argument

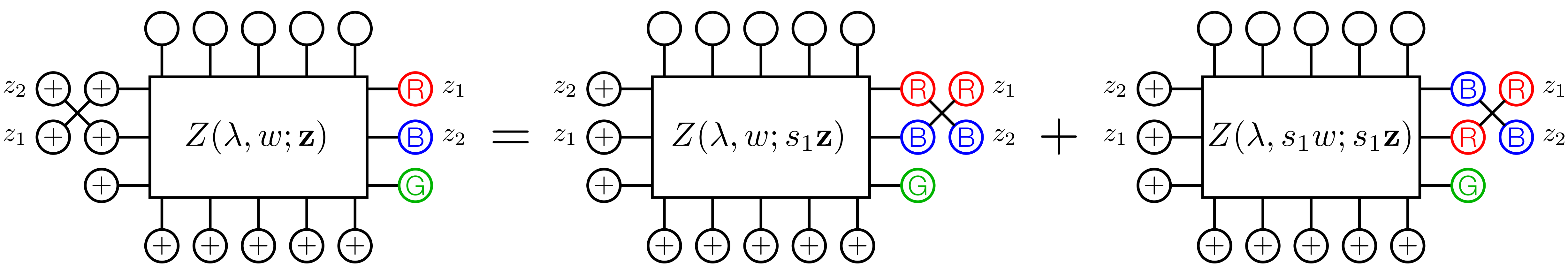


Yang-Baxter equations





Train argument



We can solve for $Z(\lambda, s_1 w; s_1\mathbf{z})$ in terms of $Z(\lambda, w; s_1\mathbf{z})$ and $Z(\lambda, w; \mathbf{z})$

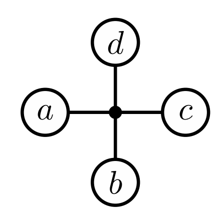
Recursion relation in terms of Demazure (divided difference) operators

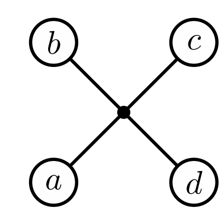
Quantum groups

Solutions to the Yang-Baxter equations arise from quantum groups. For the lattice models these are the following q-deformations of universal enveloping algebras:

	$\mathrm{GL}_r(F)$	metaplectic n-cover of $\mathrm{GL}_r(F)$
spherical	$U_q\big(\widehat{\mathfrak{gl}}(1\|1)\big)$	$U_q\big(\widehat{\mathfrak{gl}}(1\|n)\big)$
Iwahori (colored)	$U_q\big(\widehat{\mathfrak{gl}}(r\|1)\big)$	$U_q\big(\widehat{\mathfrak{gl}}(r\|n)\big)$

If the quantum group modules are known for the horizontal and vertical edge configurations then one can compute the Boltzmann weights and the R-matrix directly from the quantum group.

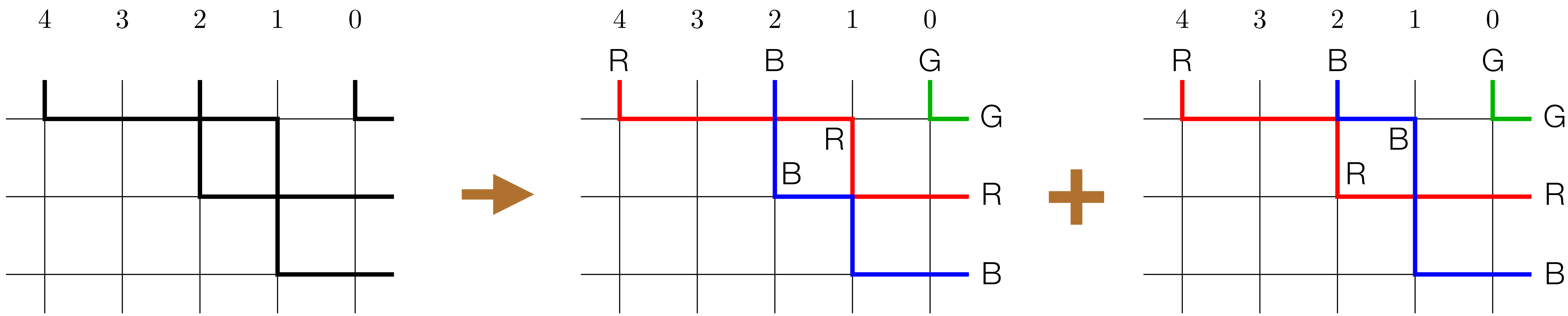




Automatically satisfy Yang–Baxter equations.

The module for the vertical edges is not known for any of the lattice models in this talk. The weights had to be constructed, and the Yang-Baxter equations had to be checked by hand.

Color refinement



Paper 3 (metaplectic)

Theorem: [Papers 1, 2] $Z(\lambda, s_{i_1} \cdots s_{i_r}; \mathbf{z}) = T_{i_1} \cdots T_{i_r} \mathbf{z}^{\lambda+\rho}$

Divided difference Demazure operators

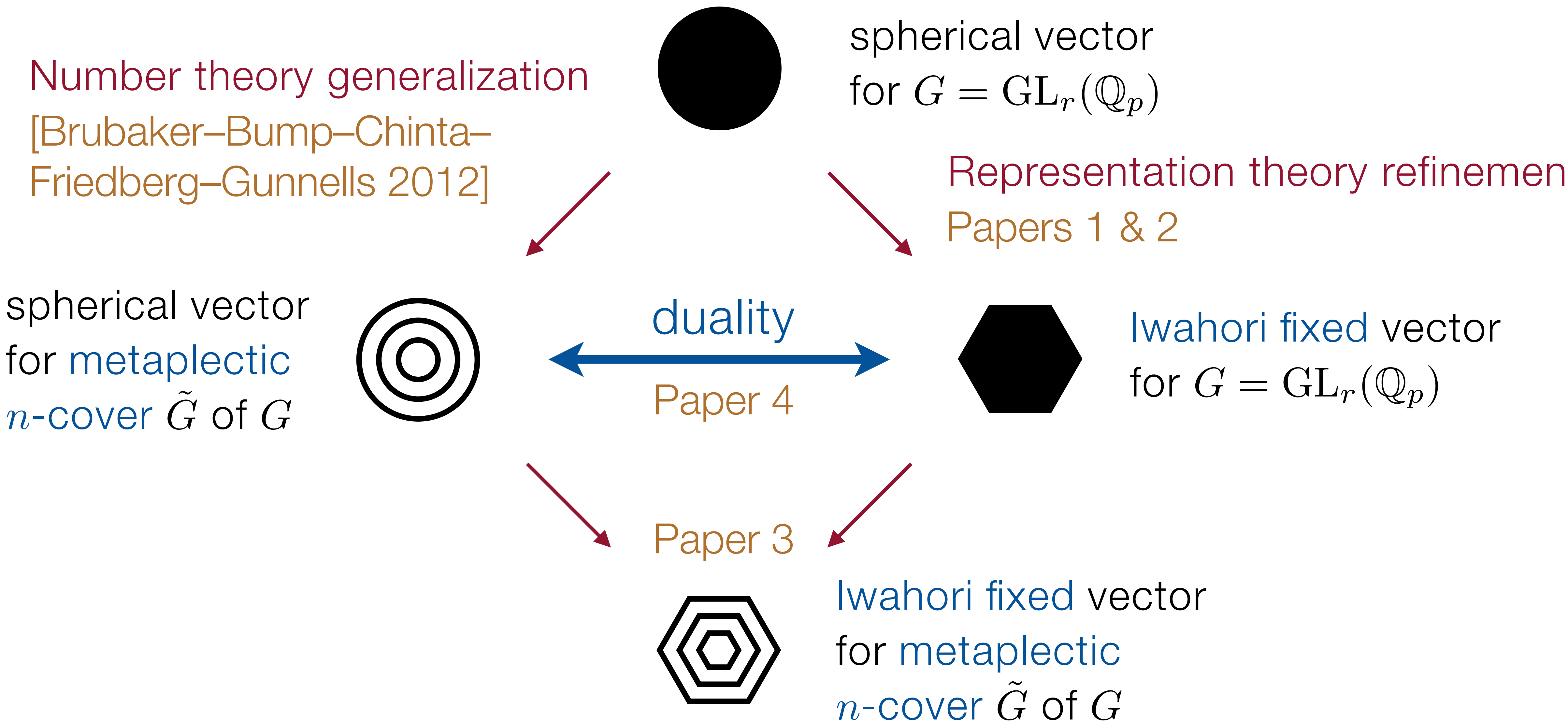
$$T_i f(\mathbf{z}) = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} f(s_i \mathbf{z}) + \frac{v-1}{1-\mathbf{z}^{\alpha_i}} f(\mathbf{z})$$

Same relations as for Whittaker functions in [Brubaker–Bump–Licata 2015] (non-metaplectic)

[Chinta–Gunnells–Puskás 2017, Patnaik–Puskás 2017] (metaplectic)

Metaplectic groups

Metaplectic groups



Blue terms will be defined in the next slides

Metaplectic Whittaker functions

The metaplectic n-cover $\tilde{G}$ of G is a central extension:

$$1 \longrightarrow \langle e^{2\pi i/n} \rangle \longrightarrow \tilde{G} \xrightarrow{\text{proj}} G \longrightarrow 1$$

$\tilde{T} := \text{proj}^{-1}(T)$ not abelian

group of n-th roots of unity

The particular cover of $\mathrm{GL}_r(F)$ we consider for lattice model interpretations is such that the group multiplication on $\tilde{T}$ is given by $[\widetilde{\lambda(x)}, \widetilde{\mu(y)}] = (x, y)^{\lambda \cdot \mu}$ where $x, y \in F^\times$, $\lambda, \mu \in X_*(T) \cong \mathbb{Z}^r$ and $(\ ,\ )$ is the n-th Hilbert symbol.

The principal series representation $\pi_{\mathbf{z}}$ with $\mathbf{z} \in (\mathbb{C}^\times)^r$ is constructed similarly, but is now vector-valued of dimension n^r.

See for example [Savin 04, McNamara 16]

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \subset G$$

abelian $\longrightarrow$ its irreps are 1-dimensional

$\tilde{T}/\text{max abelian} \cong \Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^r$

weight lattice

[Matsumoto 1969, Kazhdan–Patterson 1984, Brylinski–Deligne 2001, McNamara 2012]

Metaplectic Whittaker functions

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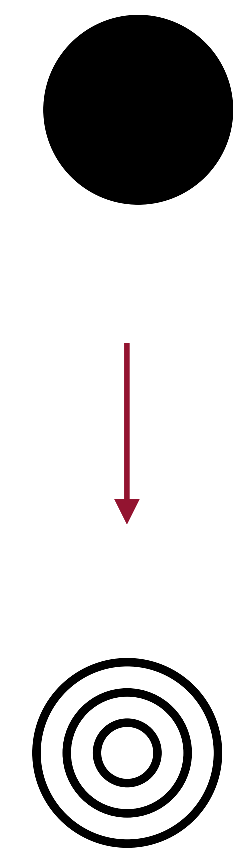
$$1 \longrightarrow \langle e^{2\pi i/n} \rangle \longrightarrow \tilde{G} \xrightarrow{\mathrm{proj}} G \longrightarrow 1 \qquad \tilde{T} := \mathrm{proj}^{-1}(T) \text{ not abelian}$$



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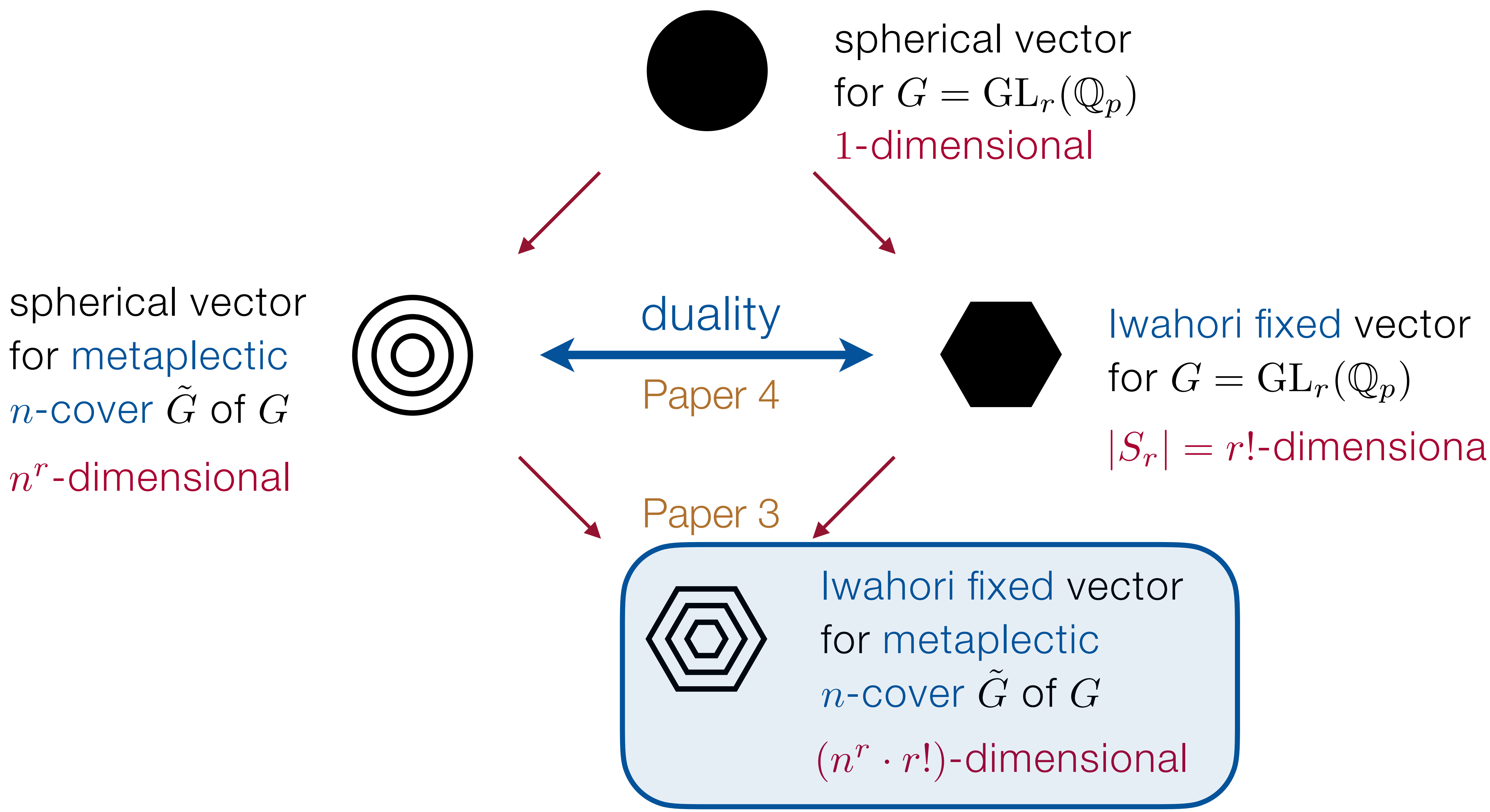
Whittaker module no longer unique; project to component $\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$.

Thus, we get a basis of n^r metaplectic spherical Whittaker functions.

Often, (e.g. [Chinta–Gunnells–Puskás 2017, Patnaik–Puskás 2017, McNamara 2016, Sahi–Stokman–Venkateswaran 2022]) the σ-average is considered.

Metaplectic groups



Includes the others as subcases

Metaplectic Iwahori lattice model

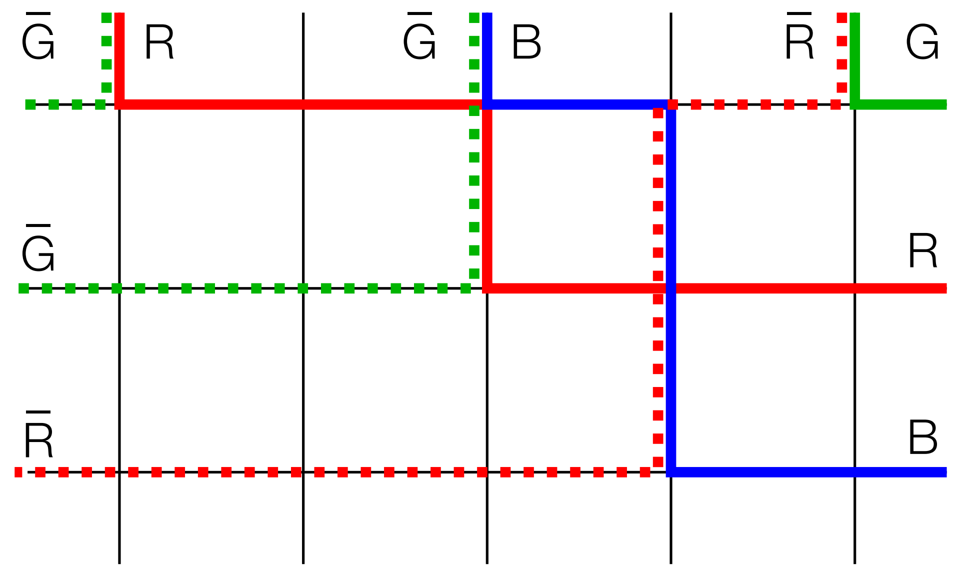
Two sets of r paths assigned colors from two different palettes:

- r colors — distinct south-east moving paths
- n supercolors — south-west moving paths (dotted)

Boltzmann weights with Gauss sums

Boundary data: group argument $g = p^\lambda$

Whittaker model
$\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$



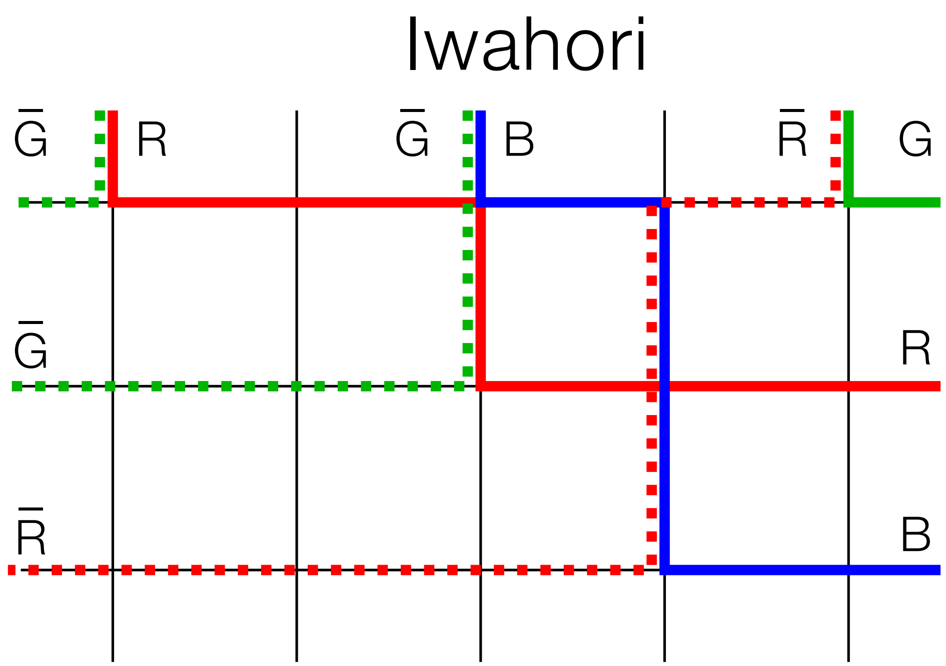
Iwahori basis
$w \in S_r$

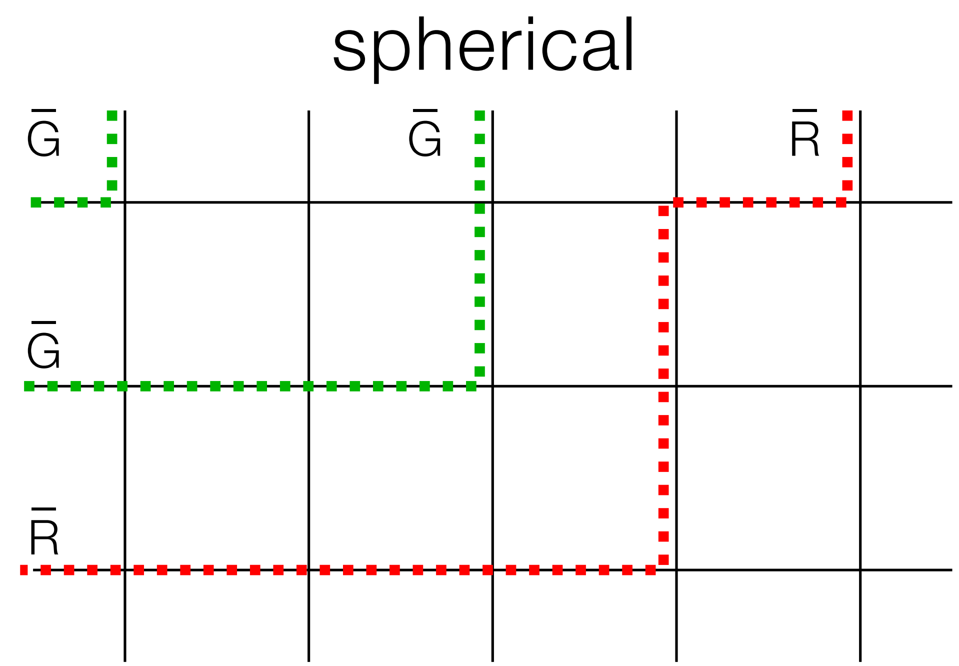
Theorem: [Paper 3]

$Z(\lambda, w, \sigma; \mathbf{z})$ = metaplectic Iwahori Whittaker function $\mathcal{W}_\psi^\sigma(f_{\mathbf{z}}^{(w)})(p^\lambda)$

Bijection: boundary data $\longleftrightarrow$ Whittaker function data

Metaplectic Iwahori lattice model





Theorem: [Paper 3]

$Z(\lambda, w, \sigma; \mathbf{z})$ = metaplectic Iwahori Whittaker function $\mathcal{W}^{\sigma}_{\psi}(f^{(w)}_{\mathbf{z}})(p^{\lambda})$

Bijection: boundary data $\longleftrightarrow$ Whittaker function data

Theorem: [Papers 2 & 3]

Summing over color permutations $w \longleftrightarrow$ equating colors

Metaplectic spherical Whittaker function obtained by equating all colors after which south-east moving paths become superfluous. Gives a reinterpretation of the metaplectic spherical lattice models by [Brubaker–Bump–Chinta–Friedberg–Gunnels 2012]

Iwahori–metaplectic duality

[Paper 4]

Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G

duality

Paper 4

Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

$\tilde{\phi}^{\circ}_{\sigma}(\mathbf{z}; g) \qquad \sigma \in (\mathbb{Z}/n\mathbb{Z})^r$

$\phi_w(\mathbf{z}; g) \qquad w \in S_r$

Theorem: Their associated lattice models are part of a parametric family of lattice models related by so-called Drinfeld twists of the underlying quantum group.

Theorem:
If the parts of σ are distinct then $\tilde{\phi}^{\circ}_{\sigma}(\mathbf{z}; g) = (\text{Gauss sums}) \cdot \phi_w(\mathbf{z}^n; g')$.

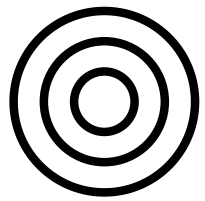
If the parts of σ are identical then $\tilde{\phi}^{\circ}_{\sigma}(\mathbf{z}; g) = \phi^{\circ}(\mathbf{z}^n; g') := \sum_{w \in S_r} \phi_w(\mathbf{z}^n; g')$

Conjecture: in general $\tilde{\phi}^{\circ}_{\sigma}(\mathbf{z}; g) \approx$ (non-metaplectic parahoric Whittaker function) with more complicated insertions of Gauss sums.

[Paper 4]

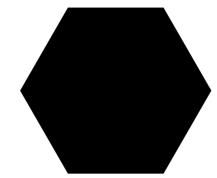
Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G



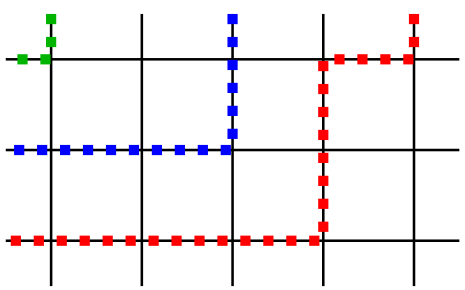
duality

Paper 4



Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Idea:

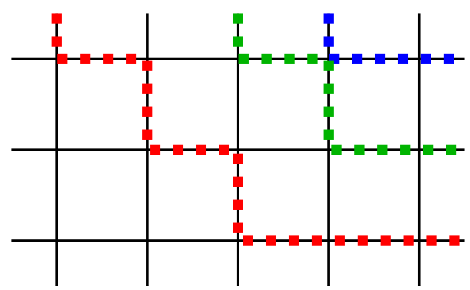


metaplectic spherical

Γ-Δ correspondence

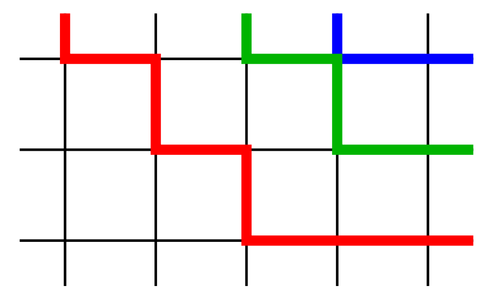
equality for partition functions (not for individual states)

[Brubaker–Bump–Friedberg 2011, Brubaker–Buciumas–Bump 2019]



Drinfeld twist

equality for states (changes Boltzmann weights and partition function)



non-metaplectic Iwahori

There is also a representation theoretical version using Demazure operators

[Paper 4]

Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G

duality

Paper 4

Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Metaplectic Iwahori lattice model and R-matrix:

dual

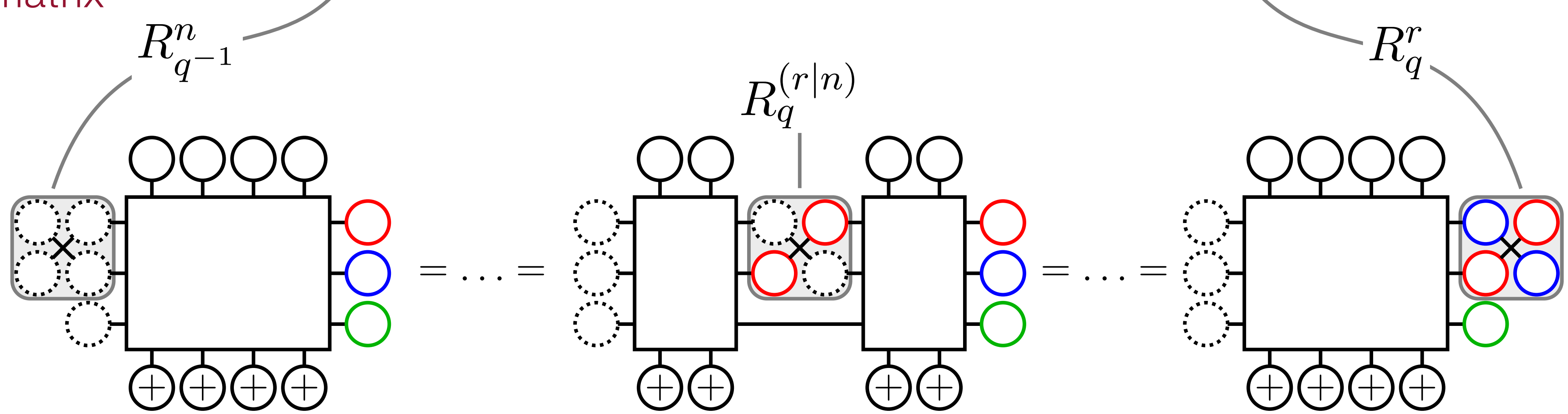
Kazhdan–Patterson scattering matrix

Whittaker functional $\Omega(\pi(g)f) = \mathcal{W}(f)(g)$

intertwiner

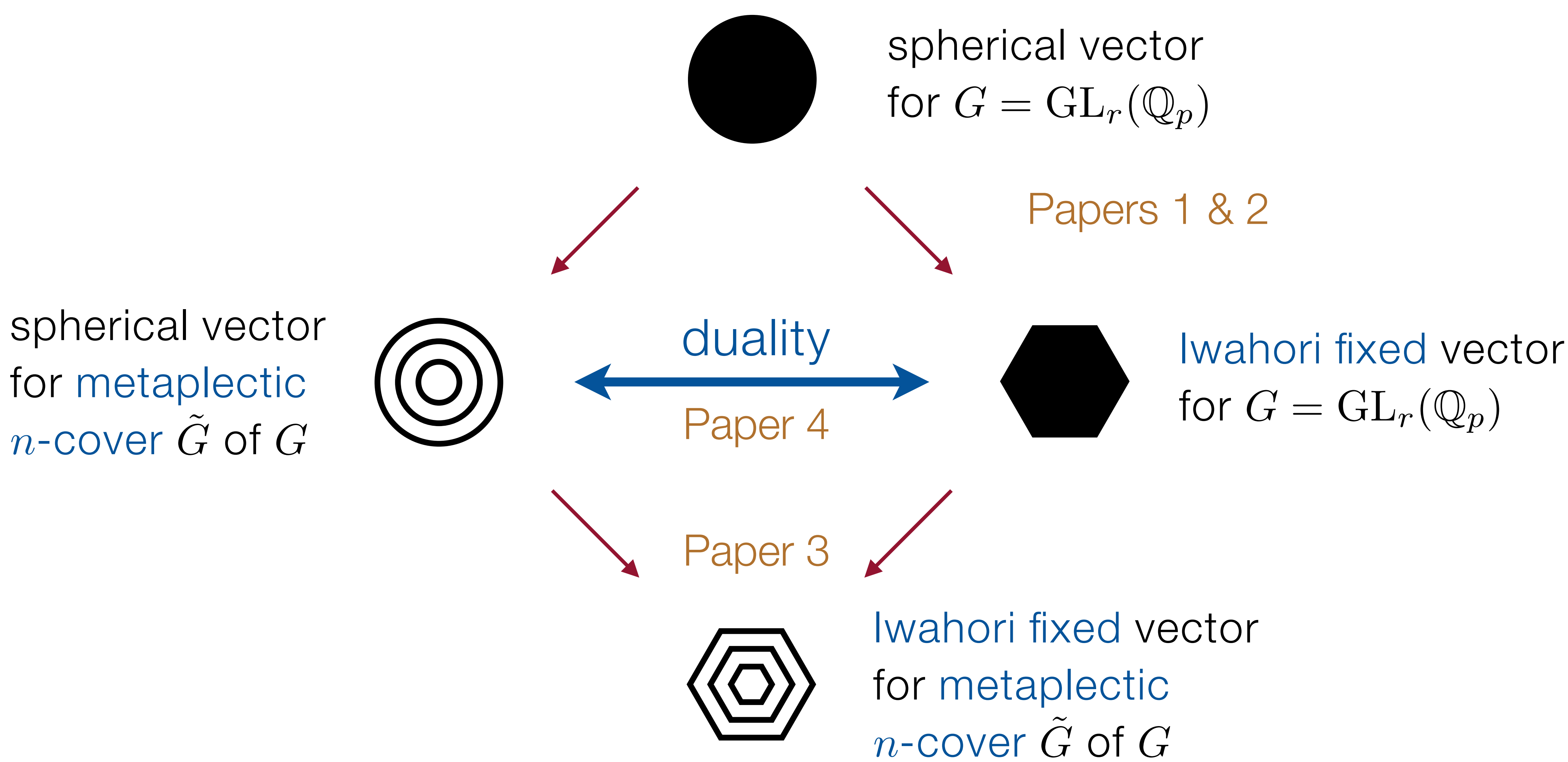
intertwining matrix

$$\left(\boxed{\Omega_\sigma^{s_i \mathbf{z}^{-1}} \mathcal{A}_{s_i}^{\mathbf{z}^{-1}}}\right)\left(\pi(g) f_w^{\mathbf{z}^{-1}}\right) = \Omega_\sigma^{s_i \mathbf{z}^{-1}}\left(\pi(g) \boxed{\mathcal{A}_{s_i}^{\mathbf{z}^{-1}} f_w^{\mathbf{z}^{-1}}}\right)$$



Summary

partition functions $\overset{\sim}{\longleftrightarrow}$ Whittaker functions



PhD position in representation theory and number theory
https://umu.varbi.com/en/what:job/jobID:616296/
Deadline May 14

Thank you!

Slides are available at

https://hgustafsson.se