Colored lattice models and p-adic representation theory

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Slides available at https://hgustafsson.se

Outline

Aim: give an overview of lattice models and *p*-adic representation theory and point to as many of the other talks at this conference as possible.

- First look at 5-vertex and 6-vertex lattice models
- Refine by adding colors
- Lattice models for various Whittaker functions
- Dualities
- Work in progress

Lattice model	Associated object in representation theory
5-vertex model (uncolored)	character for $\mathrm{GL}_r(\mathbb{C})$, Schur polynomial
6-vertex model (uncolored)	spherical Whittaker function on $\mathrm{GL}_r(F)$
5-vertex model (colored)	Demazure atoms and characters for $\mathrm{GL}_r(\mathbb{C})$
6-vertex model (colored)	Iwahori Whittaker function on $\mathrm{GL}_r(F)$
6-vertex model (supercolored)	Metaplectic spherical Whittaker function on $\widetilde{\mathrm{GL}}_r(F)$

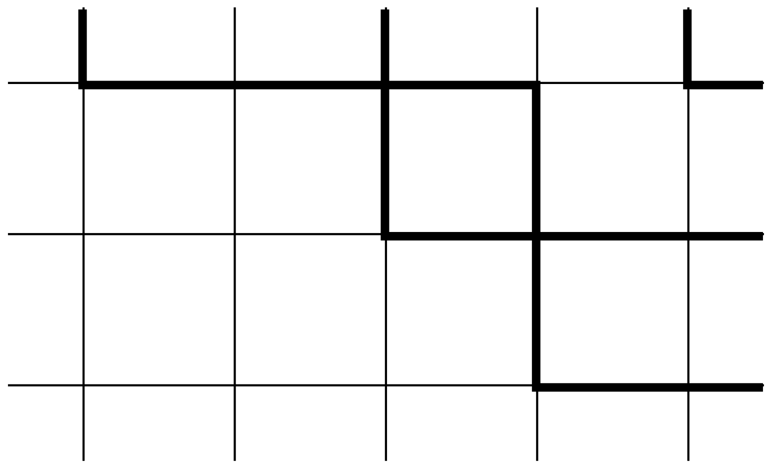
Uncolored vertex models

Schur polynomials and spherical Whittaker functions

First 5-vertex model

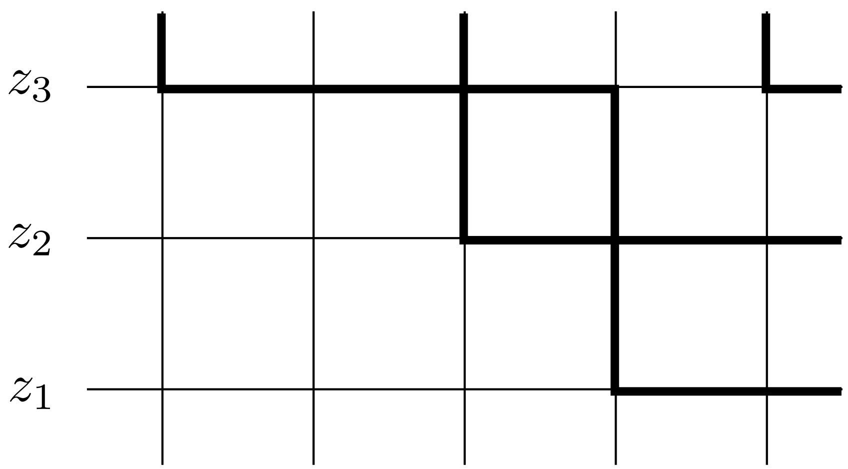
The lattice model consists of a two-dimensional grid with r rows, sufficiently many columns, and each vertex has four adjacent edges.

We will assign data to these edges according to certain rules, and in this first example the data is binary: the edge is **filled in**, or not. Later we will color the edges.



These edges will form paths on the grid, and for given boundary conditions there is a finite number of configurations called states.

First 5-vertex model



These edges will form paths on the grid, and for given boundary conditions there is a finite number of configurations called states.

A state $\mathfrak{s}$ is assigned a Boltzmann weight $\beta(\mathfrak{s}) \in \mathbb{C}[\mathbf{z}]$ depending on parameters $\mathbf{z} = (z_1, z_2, \ldots, z_r) \in \mathbb{C}^r$ (one for each row).

The partition function, given some fixed boundary conditions:

$$Z := \sum_{\substack{\text{state } \mathfrak{s} \\ \text{with given b.c.}}} \beta(\mathfrak{s})$$

Goal: any Schur polynomial in $\mathbf{z}$ = such a partition function.

Schur polynomials

Let $\lambda = (\lambda_1, \dots \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ be an integer partition. We define the Schur polynomial $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by

$$s_\lambda(\mathbf{z}) = \frac{\det(z_i^{(\lambda+\rho)_j})_{ij}}{\det(z_i^{\rho_j})_{ij}}$$

where $\mathbf{z} = (z_1, \dots, z_r)$ and $\rho = (r-1, r-2, \dots, 1, 0)$.

$r = 3:$ $\quad s_{1,1,1}(\mathbf{z}) = z_1 z_2 z_3 \quad\quad s_2(\mathbf{z}) = s_{2,0,0}(\mathbf{z}) = z_1^2 + z_1 z_2 + z_2^2 + z_2 z_3 + z_3^2 + z_1 z_3$

Combinatorial description using **S**emi-**S**tandard **Y**oung **T**ableaux of shape λ

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

$\lambda = (3,1,1)$ $\quad \mathrm{SSYT}(\lambda) \ni T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$ $\quad \mathrm{wt}(T) = (2,2,0,0,1)$

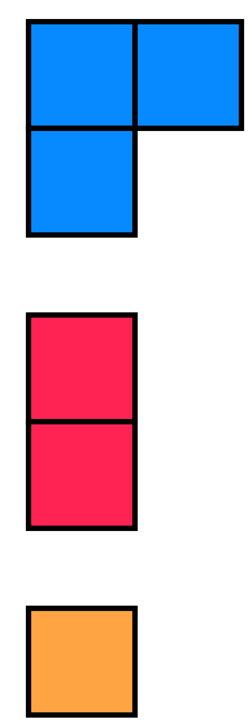
(#ones, #twos, #threes, ...)

Gelfand–Tsetlin patterns

$$\mathrm{SSYT}(\lambda) \overset{\sim}{\longleftrightarrow} \mathrm{GTP}(\lambda) = \{\text{Gelfand–Tsetlin patterns with top row } \lambda\}$$

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & \geq & a_{01} & & a_{02} & & \cdots & & a_{0\ell} \\ & \geq \; a_{11} \; \geq & & a_{12} & & & \cdots & a_{1\ell} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & & a_{\ell\ell} & & & & \end{array} \right\}$$

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\} = \left\{ \begin{array}{c} \lambda^{(3)} \\ \lambda^{(2)} \\ \lambda^{(1)} \end{array} \right\}$$



Gelfand–Tsetlin patterns

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$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & \geq & a_{01} & & a_{02} & & \cdots & & a_{0\ell} \\ & \geq \; a_{11} \; \leq & & a_{12} & & & \cdots & a_{1\ell} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & & a_{\ell\ell} & & & & \end{array} \right\}$$

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\} = \left\{ \begin{array}{c} \lambda^{(3)} \\ \lambda^{(2)} \\ \lambda^{(1)} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} = T$$

$$\mathrm{wt}(\mathfrak{T}) = (\mathrm{row}_\ell,\ \mathrm{row}_{\ell-1} - \mathrm{row}_\ell,\ \ldots,\ \mathrm{row}_0 - \mathrm{row}_1) \quad \text{where } \mathrm{row}_i = \sum_j a_{ij}$$

$$= \mathrm{wt}(T)$$

Gelfand–Tsetlin patterns

$$\mathrm{SSYT}(\lambda) \overset{\sim}{\longleftrightarrow} \mathrm{GTP}(\lambda) = \{\text{Gelfand–Tsetlin patterns with top row } \lambda\}$$

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & \geq & a_{01} & & a_{02} & & \cdots & & a_{0\ell} \\ & \rotatebox{-45}{$\geq$} \; a_{11} \; \rotatebox{45}{$\geq$} & & a_{12} & & & \cdots & a_{1\ell} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & & a_{\ell\ell} & & & & \end{array} \right\}$$

Left-strict: $a_{00} \geq a_{01}$, $a_{00} > a_{11}$, $a_{11} \geq a_{01}$

Right-strict: $a_{00} \geq a_{01}$, $a_{00} \geq a_{11}$, $a_{11} > a_{01}$

$$\mathrm{GTP}^{\mathrm{ls}}(\lambda+\rho) \overset{\sim}{\longleftrightarrow} \mathrm{GTP}(\lambda) \overset{\sim}{\longleftrightarrow} \mathrm{GTP}^{\mathrm{rs}}(\lambda+\rho)$$

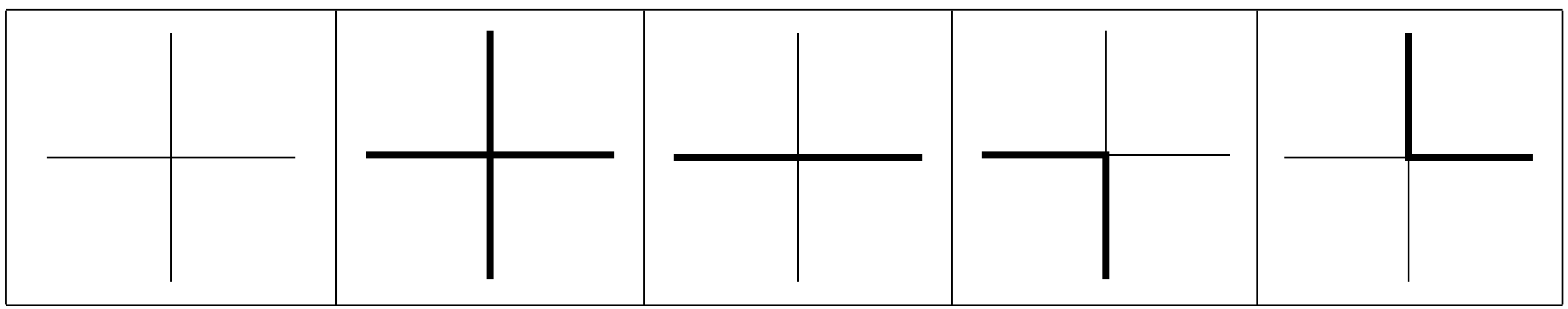
$$\mathfrak{T}+\rho_{\mathrm{ls}} \qquad\qquad \mathfrak{T} \qquad\qquad \mathfrak{T}+\rho_{\mathrm{rs}}$$

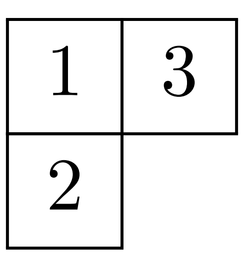
$$\rho_{\mathrm{ls}} = \left\{ \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{array} \right\} \qquad\qquad \rho_{\mathrm{rs}} = \left\{ \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{array} \right\}$$

5-vertex model

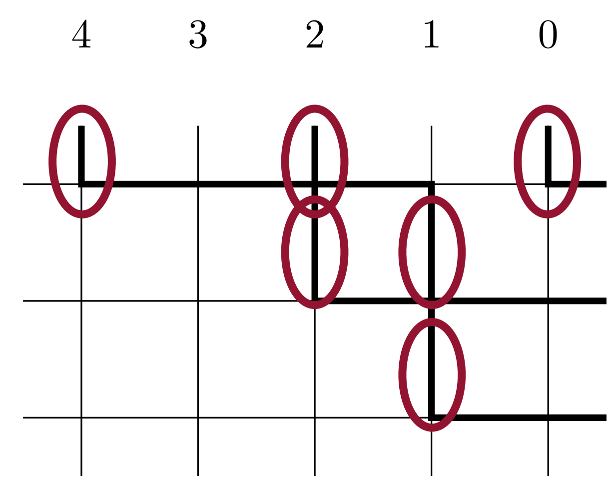
$$\mathrm{GTP}^{\mathrm{ls}}(\lambda+\rho) \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{l} \text{5-vertex model configurations with} \\ \text{top boundary edges at columns } \lambda+\rho \end{array} \right\}$$

Five different vertex configurations:



$T =$ 

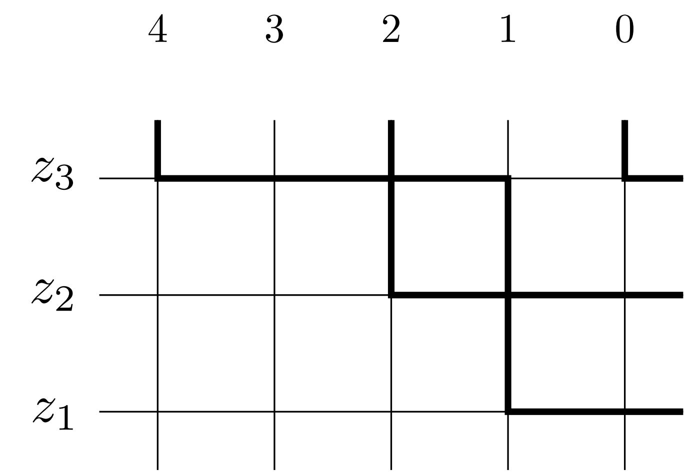
$$\mathfrak{T} + \rho_{\mathrm{ls}} = \left\{ \begin{array}{ccccc} 4 & & 2 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \right\}$$



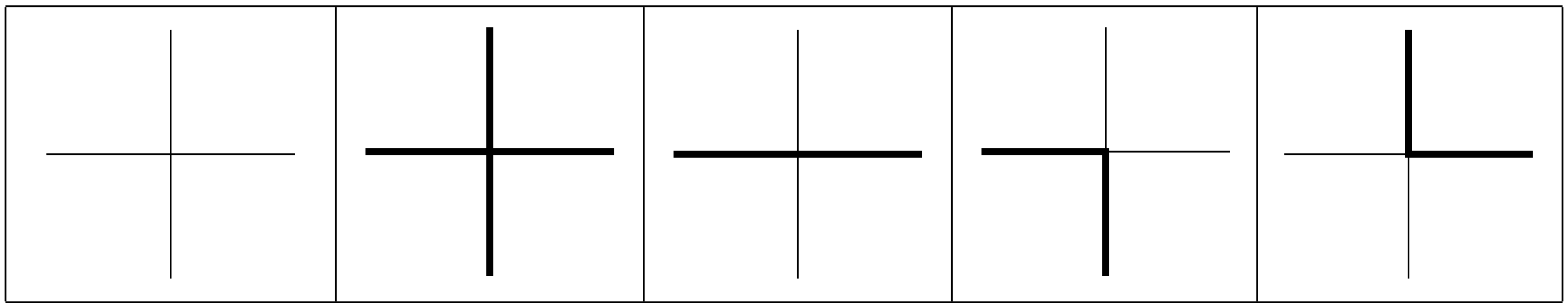
(right-moving)

5-vertex model

state $\mathfrak{s}$ →



Five different vertex configurations:



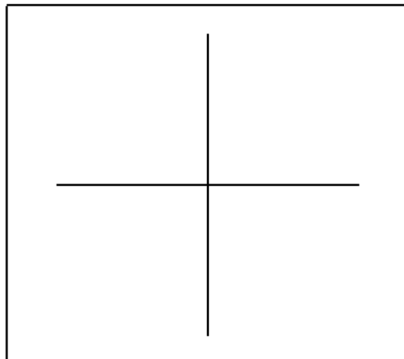
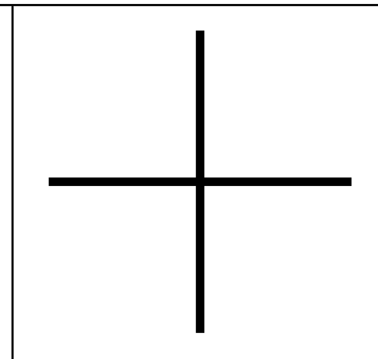
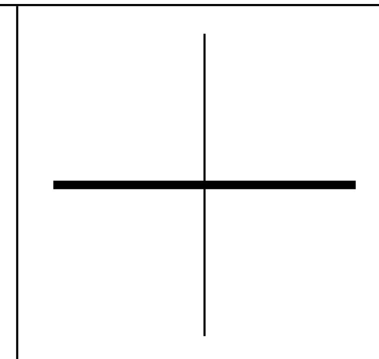
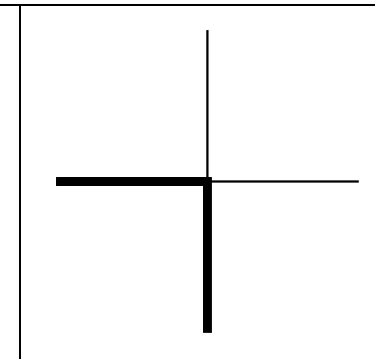
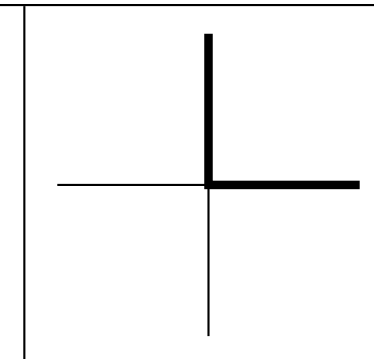
$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)} = \sum_{\mathfrak{T} \in \mathrm{GTP}(\lambda)} \mathbf{z}^{\mathrm{wt}(\mathfrak{T})} = \mathbf{z}^{-\rho} \sum_{\mathfrak{T} \in \mathrm{GTP}^{\mathrm{ls}}(\lambda+\rho)} \mathbf{z}^{\mathrm{wt}(\mathfrak{T})} = \mathbf{z}^{-\rho} \sum_{\mathfrak{s} \text{ with top } \lambda+\rho} \beta(\mathfrak{s})$$

Boltzmann weight $\beta(\mathfrak{s}) := \prod_{\text{vertex}}$ vertex weights

counting row differences = counting left-edges

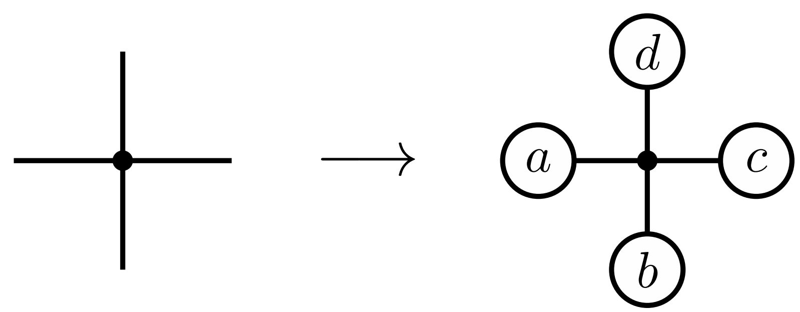
Partition function $Z(\lambda, \mathbf{z}) := \sum_{\mathfrak{s} \text{ with top } \lambda+\rho} \beta(\mathfrak{s}) \; = \mathbf{z}^{\rho} s_\lambda(\mathbf{z})$

5-vertex model

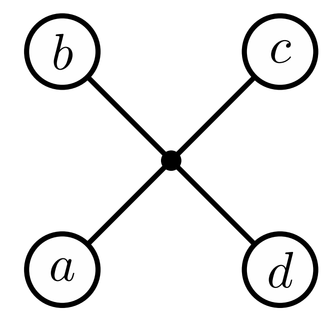
				
1	z_i	z_i	z_i	1

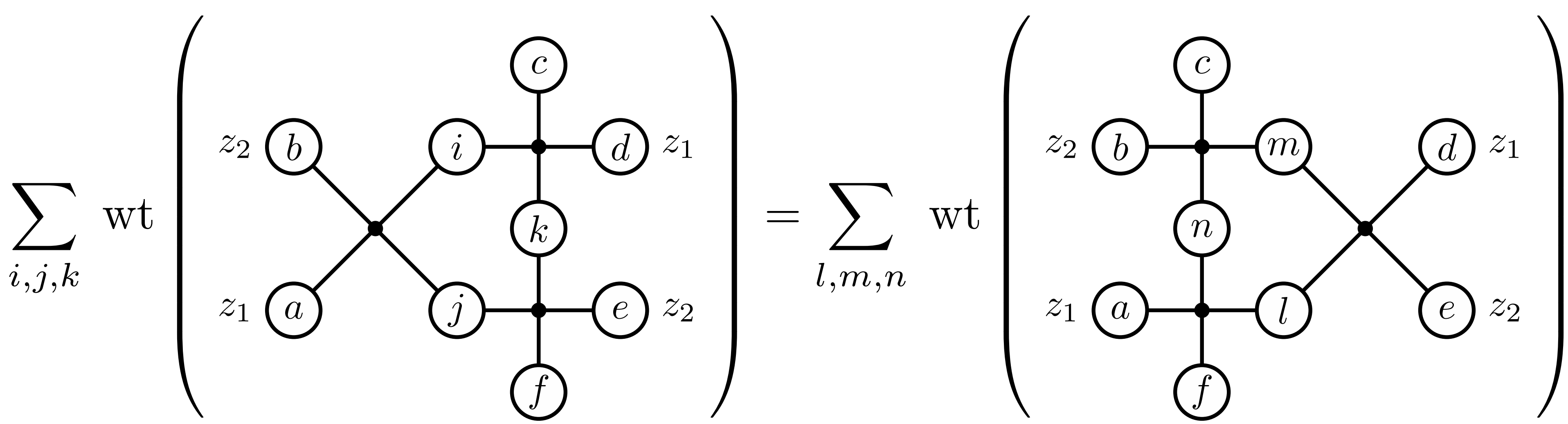
Satisfies a functional (Yang–Baxter) equation for swapping two rows

In this case: Schur polynomial is symmetric function.

Label edges: 

NEW

R-vertex: 



5-vertex model

Representation theory

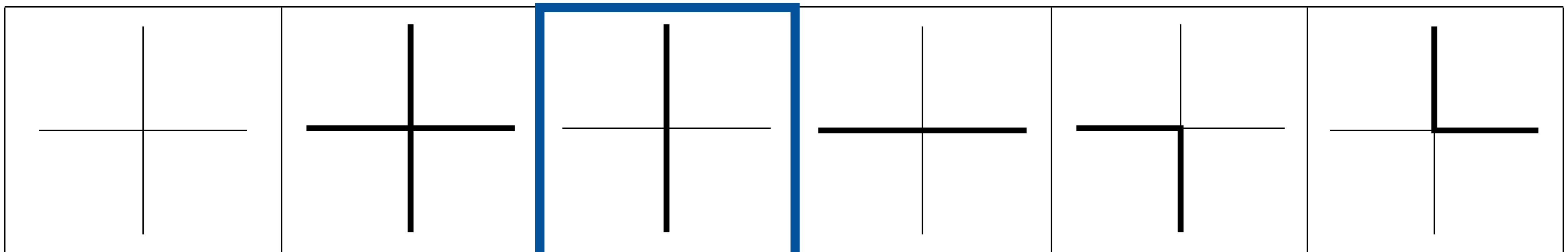
The Schur polynomial $s_\lambda(\mathbf{z})$ is the character $\chi_\lambda\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}\right)$ of the highest weight representation V_λ of $\mathrm{GL}_r(\mathbb{C})$.

We can see this by the Weyl character formula:

$$\chi_\lambda\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}\right) = \frac{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\rho)}} = \frac{\det\left(z_i^{(\lambda+\rho)_j}\right)_{ij}}{\det(z_i^{\rho_j})_{ij}} = s_\lambda(\mathbf{z})$$

Lattice model	**Associated object in representation theory**
5-vertex model (uncolored)	character (Schur polynomial)

6-vertex model



1	z	$-v$	z	$(1-v)z$	1

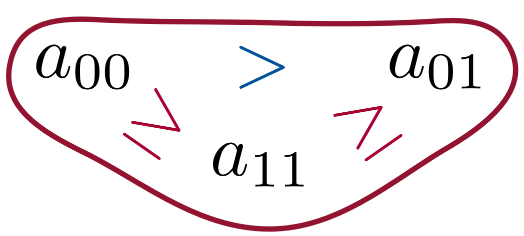
For Yang–Baxter equation

$$\mathrm{GTP}^{\mathrm{ls}}(\lambda+\rho) \overset{\sim}{\longleftrightarrow} \{\ \text{5-vertex configurations with top boundary } \lambda+\rho\ \}$$

$$\cap$$

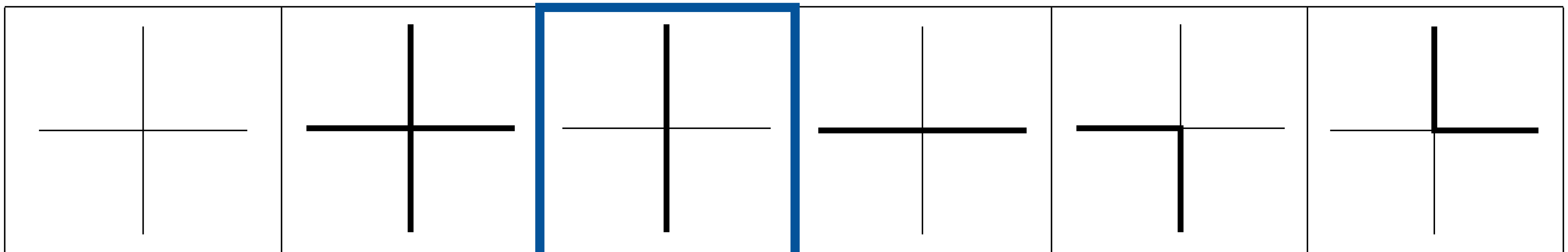
$$\mathrm{GTP}^{\mathrm{s}}(\lambda+\rho) \overset{\sim}{\longleftrightarrow} \{\ \text{6-vertex configurations with top boundary } \lambda+\rho\ \}$$

Strict:



[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

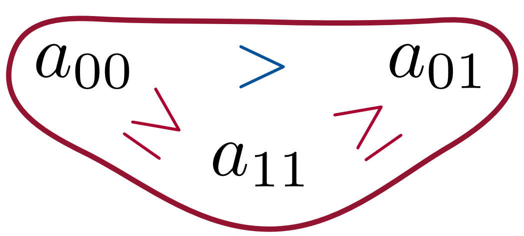
6-vertex model



1	z	$-v$	z	$(1-v)z$	1

$$\mathrm{GTP^s}(\lambda+\rho) \overset{\sim}{\longleftrightarrow} \{\ \text{6-vertex configurations with top boundary } \lambda+\rho\ \}$$

Strict:



The partition function in terms of Gelfand–Tsetlin patterns is one side of the Tokuyama formula which gives:

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^{\rho} \prod_{i<j} (1 - v\tfrac{z_j}{z_i}) s_\lambda(\mathbf{z})$$

[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

6-vertex model

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$$Z(\lambda;\mathbf{z}) = \mathbf{z}^{\rho} \prod_{i<j} \left(1 - v\frac{z_j}{z_i}\right) s_\lambda(\mathbf{z})$$

This is in fact a Whittaker function for a principal series representation $\pi_{\mathbf{z}}$ of $\mathrm{GL}_r(F)$ where F is a non-archimedean field (e.g. p-adic)

Let $\mathfrak{o}$ be the ring of integers, $\mathfrak{p} = \varpi\mathfrak{o}$ the unique prime ideal and $v^{-1} := |\mathfrak{o}/\mathfrak{p}|$.

$$\pi_{\mathbf{z}} = \left\{ \Phi_{\mathbf{z}} : \mathrm{GL}_r(F) \to \mathbb{C} \text{ such that } \Phi_{\mathbf{z}}(n\varpi^{\lambda} g) = \mathbf{z}^{\lambda}\Phi_{\mathbf{z}}(g) \right\}$$

(simplified for clarity)

$$\varpi^{\lambda} = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_r} \end{pmatrix} \qquad n \in \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} = N$$

[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

6-vertex model

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In particular, we are interested in the unique (normalized) spherical vector $\Phi_{\mathbf{z}}^{\circ} \in \pi_{\mathbf{z}}$ where spherical means that it is invariant under the maximal compact subgroup $K = \mathrm{GL}_r(\mathfrak{o})$, that is, $\Phi_{\mathbf{z}}^{\circ}(gk) = \Phi_{\mathbf{z}}^{\circ}(g)$ for $k \in K$.

The associated Whittaker function is similar to a Fourier integral:

character on F trivial on $\mathfrak{o}$

$$\phi^{\circ}(\mathbf{z}; g = \varpi^{-\lambda}) := \int_N \Phi_{\mathbf{z}}^{\circ}(w_0 n g)\mathbf{e}(n_{1,2} + n_{2,3} + \cdots + n_{r-1,r})^{-1} dn = \mathbf{z}^{-\rho} Z(\lambda; \mathbf{z})$$

long Weyl group element

Casselman–Shalika: character on Langlands dual group $\mathrm{GL}_r(\mathbb{C})$

See Valentin Buciumas' talk

[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

6-vertex model

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$$Z(\lambda; \mathbf{z}) = \mathbf{z}^{\rho} \prod_{i<j} (1 - v\frac{z_j}{z_i}) s_\lambda(\mathbf{z})$$

Lattice model	**Associated object in representation theory**
5-vertex model (uncolored)	character for $\mathrm{GL}_r(\mathbb{C})$, Schur polynomial
6-vertex model (uncolored)	spherical Whittaker function on $\mathrm{GL}_r(F)$

Spherical → Iwahori-~~spherical~~ fixed

Spherical: $\Phi^\circ_{\mathbf{z}}(gk) = \Phi^\circ_{\mathbf{z}}(g)$ for $k \in K = \mathrm{GL}_r(\mathfrak{o})$.

In representation theory it is more natural to consider a larger space of vectors invariant under $I \subset K$ called the Iwahori subgroup given by

$$I = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \vdots & \vdots & \ddots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} \cap K$$

Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwI$$

← Weyl group

Iwahori-spherical: $\Phi_{\mathbf{z}}(gk) = \Phi_{\mathbf{z}}(g)$ for $k \in I$.

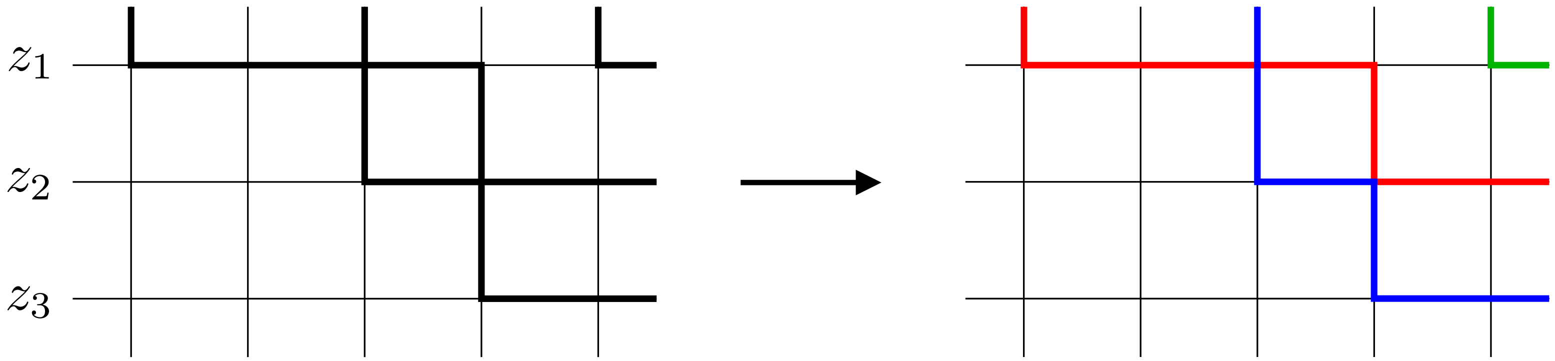
Basis $\{\Phi^w_{\mathbf{z}}\}_{w \in W}$ of Iwahori-spherical vectors each supported on only one cell BwI.

We get a refinement $\Phi^\circ_{\mathbf{z}} = \sum_{w \in W} \Phi^w_{\mathbf{z}}$ and similarly for the corresponding Whittaker functions $\phi^\circ(\mathbf{z}; g) = \sum_{w \in W} \phi^w(\mathbf{z}; g)$.

Spherical → Iwahori-spherical

We get a refinement $\Phi^\circ_{\mathbf{z}} = \sum_{w\in W} \Phi^w_{\mathbf{z}}$ and similarly for the corresponding Whittaker functions $\phi^\circ(\mathbf{z}; g) = \sum_{w\in W} \phi^w(\mathbf{z}; g)$.

The lattice model can also be refined to compute the values of each ϕ^w by assigning colors to the paths.



w is given by the relative orders of the colors on the boundaries

We will first however introduce colors to the 5-vertex model.

Schur polynomials → Demazure atoms which are similar constituents enumerated by Weyl elements.

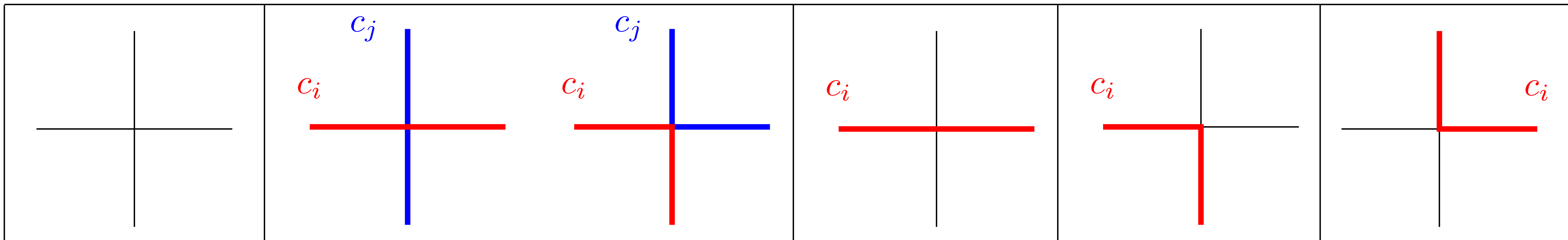
Colored vertex models

Demazure atoms and Iwahori Whittaker functions

Colored 5-vertex model

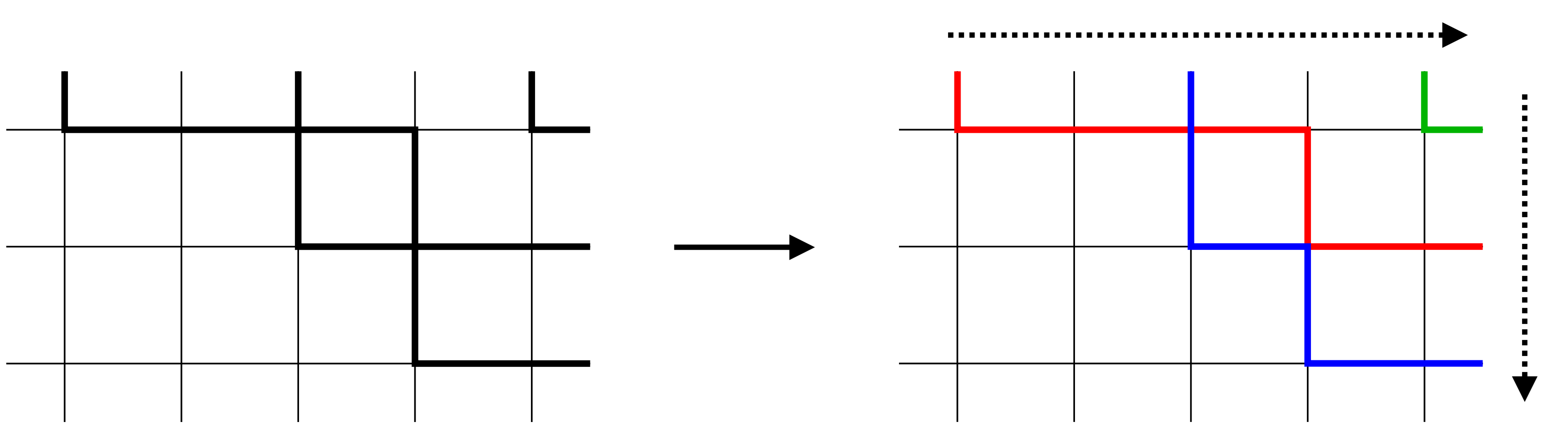
The set of paths is the same as before, but the paths are colored in a unique way.

Ordered palette: $c_1 < c_2 < c_3 < \cdots$



1	$\begin{cases} z & i \geq j \\ 0 & i < j \end{cases}$	$\begin{cases} 0 & i > j \\ z & i < j \end{cases}$	z	z	1

Fix color order on top boundary

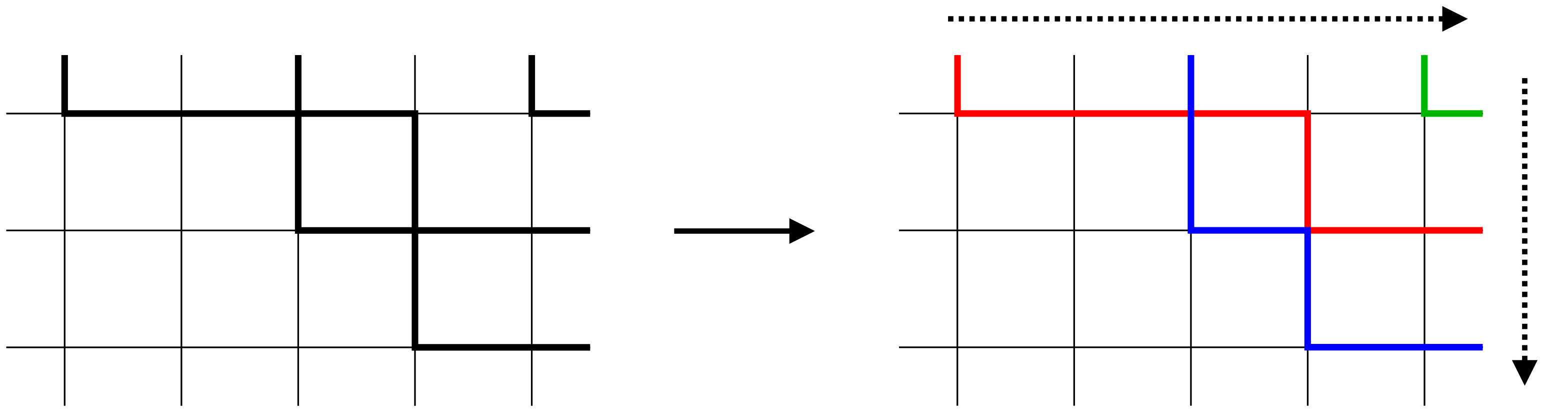


Read out $w = s_1 s_2$ from permutation

[Brubaker–Buciumas–Bump–HG 2021]

Colored 5-vertex model

Ordered palette: $c_1 < c_2 < c_3 < \cdots$



$s_\lambda(\mathbf{z}) = \sum_{w \in W} \mathcal{D}_w \mathbf{z}^\lambda$ ⟵ Demazure atoms

$\mathcal{D}_w = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_k}$ for $w = s_{i_1} \cdots s_{i_k}$ where $\mathcal{D}_i f(\mathbf{z}) = \dfrac{f(\mathbf{z}) - f(s_i \mathbf{z})}{z_i/z_{i+1} - 1}$

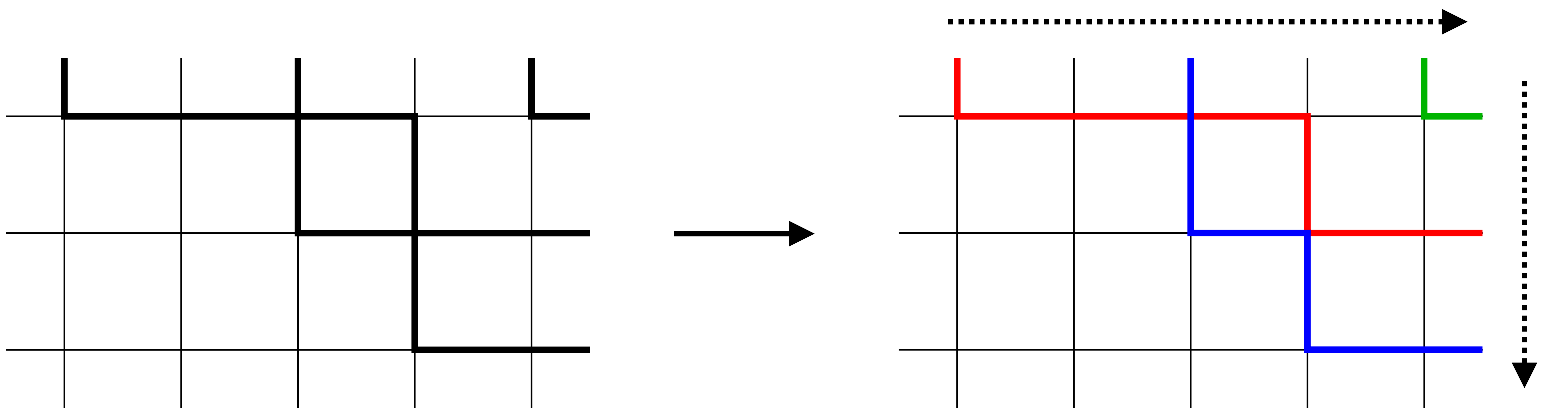
Demazure character: $\partial_w \mathbf{z}^\lambda = \sum_{y \leq w} \mathcal{D}_y \mathbf{z}^\lambda$ $\qquad \partial_i = \mathcal{D}_i + 1$

Partition function for colored model: $Z(\lambda, w; \mathbf{z}) = \mathbf{z}^\rho D_w \mathbf{z}^\lambda$

The recurrence relations for the Demazure atoms can be obtained from the Yang–Baxter equation using the train argument.

[Brubaker–Buciumas–Bump–HG 2021]

Colored 5-vertex model



Read out $w = s_1 s_2$ from permutation

$$s_\lambda(\mathbf{z}) = \sum_{w \in W} \mathcal{D}_w \mathbf{z}^\lambda$$ ⟵ Demazure atoms

Partition function for colored model: $Z(\lambda, w; \mathbf{z}) = \mathbf{z}^\rho D_w \mathbf{z}^\lambda$

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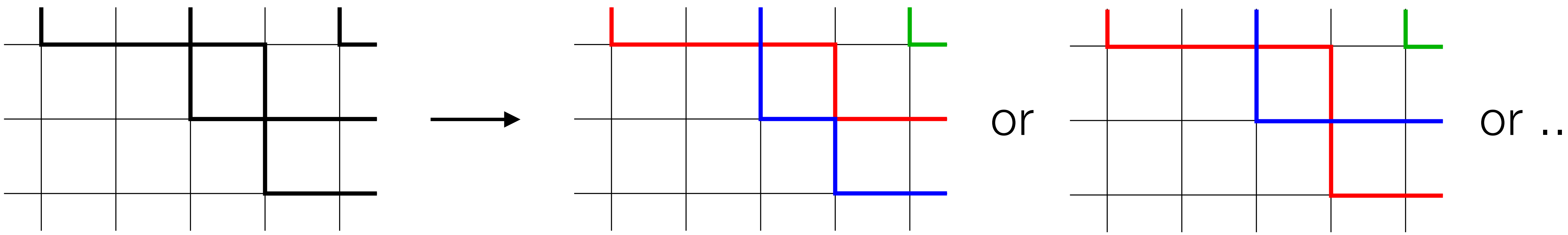
6-vertex model (uncolored)	spherical Whittaker function on $\mathrm{GL}_r(F)$
5-vertex model (colored)	Demazure atoms and characters for $\mathrm{GL}_r(\mathbb{C})$

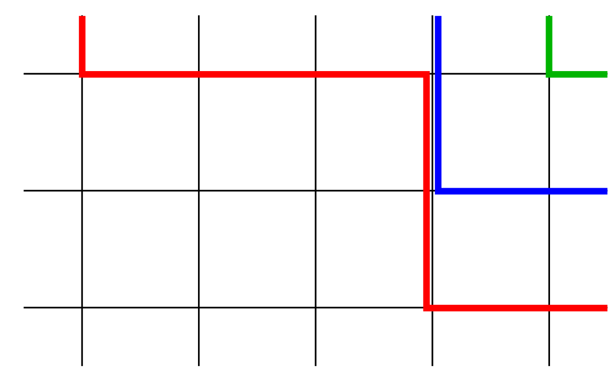
[Brubaker–Buciumas–Bump–HG 2021]

Colored 6-vertex model



To satisfy a Yang–Baxter equation, need to include multi-colored vertical edges with color-multiplicity at most one:

fermionic



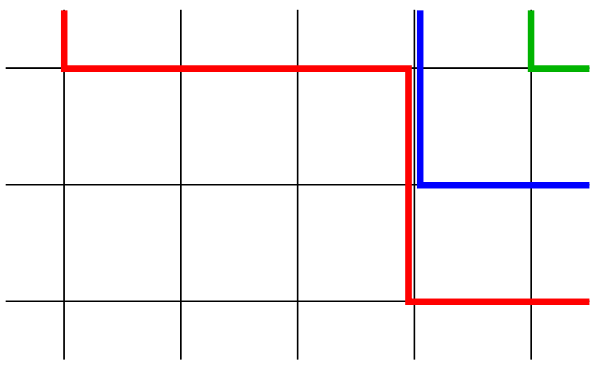
See Daniel Bump's talk about bosonic models with any multiplicity

We have extra paths compared to the uncolored 6-vertex model

[Brubaker–Buciumas–Bump–HG 2021]

Colored 6-vertex model

To satisfy a Yang–Baxter equation, need to include multi-colored vertical edges with color-multiplicity at most one:



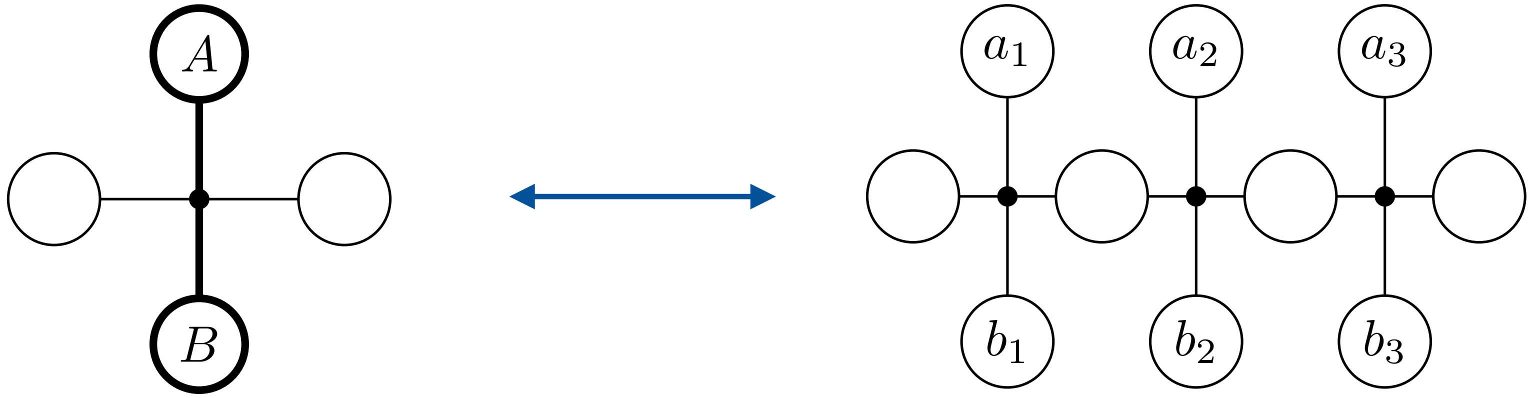
(Will be discussed further in Bump's talk.)

To be able to describe the Boltzmann weights we use fusion.

Each column can only carry a specific color

$A, B \subseteq \{c_1, c_2, c_3\}$ $\qquad$ $c_1 < c_2 < c_3$

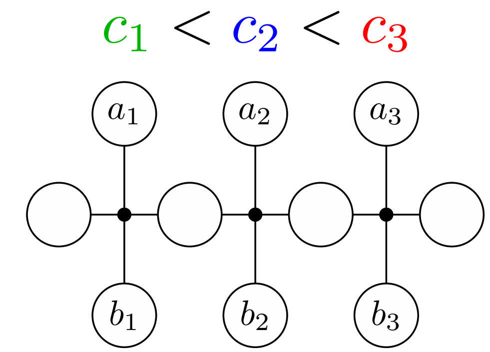
Note that horizontal edges can carry any **single** color



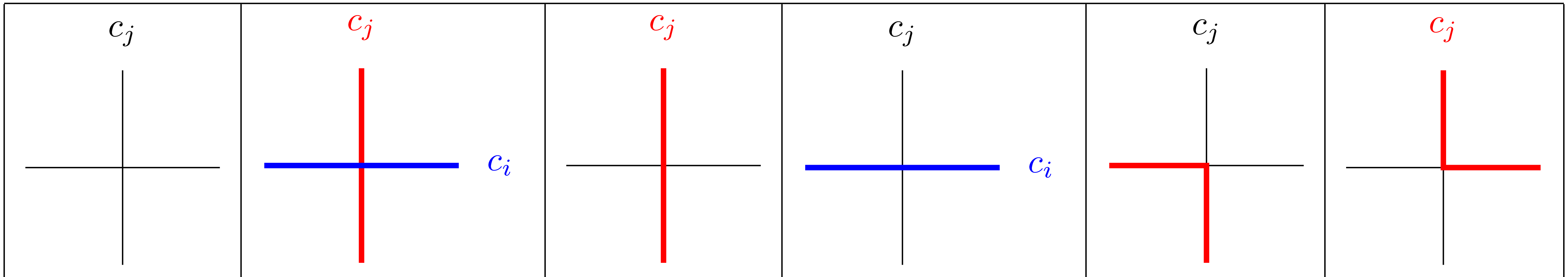
$a_i, b_i \in \{c_i, \emptyset\}$

Sometimes denoted $\oplus$

[Brubaker–Buciumas–Bump–HG arXiv:1906.04140 to appear in Selecta]



Colored 6-vertex model



1	$\begin{cases} z & i=j \\ v & i<j \\ 1 & i>j \end{cases}$	$-v$	$\begin{cases} z & i=j \\ 1 & i \neq j \end{cases}$	$(1-v)z$	1

Partition function: $Z(\lambda, w; \mathbf{z}) = \mathbf{z}^{\rho} \phi^{w}(\mathbf{z}; g = \varpi^{-\lambda})$ Iwahori Whittaker function

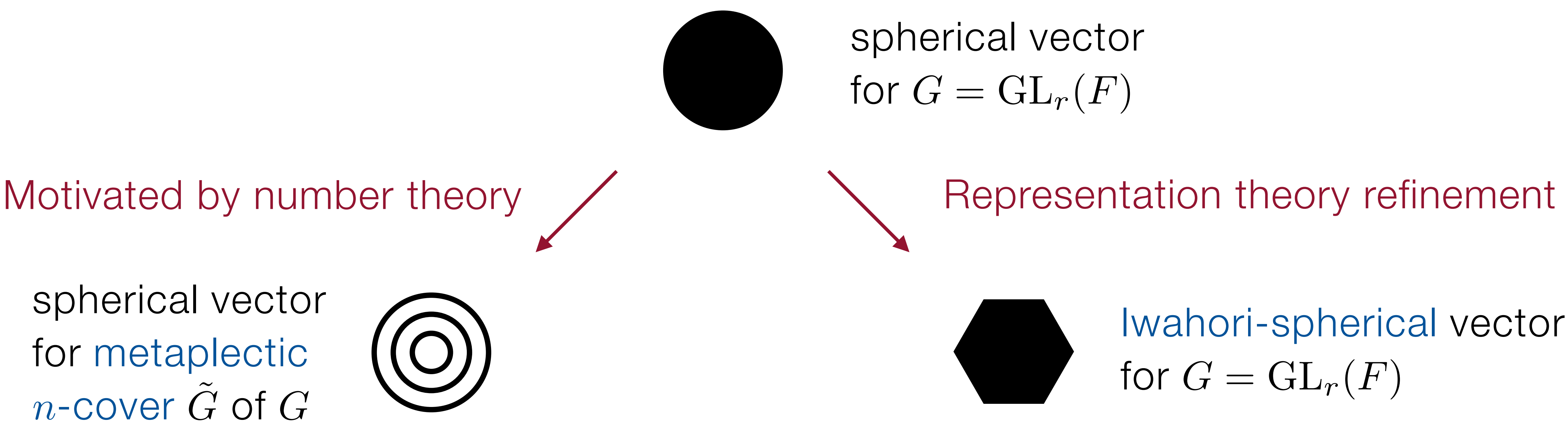
Lattice model	**Associated object in representation theory**
5-vertex model (uncolored)	character for $\mathrm{GL}_r(\mathbb{C})$, Schur polynomial
6-vertex model (uncolored)	spherical Whittaker function on $\mathrm{GL}_r(F)$
5-vertex model (colored)	Demazure atoms and characters for $\mathrm{GL}_r(\mathbb{C})$
6-vertex model (colored)	Iwahori Whittaker function on $\mathrm{GL}_r(F)$

[Brubaker–Buciumas–Bump–HG arXiv:1906.04140 to appear in Selecta]

Metaplectic covers

Two different generalizations

We consider Whittaker functions for certain vectors of the principal series representation:



An important motivation for introducing metaplectic covers comes from the study of multiple Dirichlet series. See Manish Patnaik's talk.

Metaplectic Whittaker functions

The metaplectic n-cover $\tilde{G}$ of G is a central extension:

$$1 \longrightarrow \langle e^{2\pi i/n} \rangle \longrightarrow \tilde{G} \xrightarrow{\text{proj}} G \longrightarrow 1$$

$\tilde{T} := \text{proj}^{-1}(T)$ not abelian

group of n-th roots of unity

In this talk we pick a particular metaplectic cover giving particularly nice commutation relations for elements in $\tilde{T}$.

See Claire Frechette's talk for more general metaplectic covers

The principal series representation $\pi_{\mathbf{z}}$ with $\mathbf{z} \in (\mathbb{C}^\times)^r$ is constructed similarly, but is now vector-valued of dimension n^r.

See for example [Savin 04, McNamara 16]

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \subset G$$

abelian ⟶ its irreps are 1-dimensional

$\tilde{T}/\text{max abelian} \cong \Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^r$

weight lattice

[Matsumoto 1969, Kazhdan–Patterson 1984, Brylinski–Deligne 2001, McNamara 2012]

Metaplectic Whittaker functions

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The Whittaker integrals need to be projected to $\mathbb{C}$, with respect to a component $\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$.

Thus, we get a basis of n^r metaplectic spherical Whittaker functions.

Often, (e.g. [Chinta–Gunnells–Puskás 2017, Patnaik–Puskás 2017, McNamara 2016, Sahi–Stokman–Venkateswaran 2022]) the σ-average is considered.

Metaplectic Whittaker functions

Lattice models for metaplectic spherical Whittaker functions were constructed in [Brubaker–Bump–Chinta–Friedberg–Gunnells 2012, Brubaker–Buciumas–Bump–Gray 2017 & 2019]

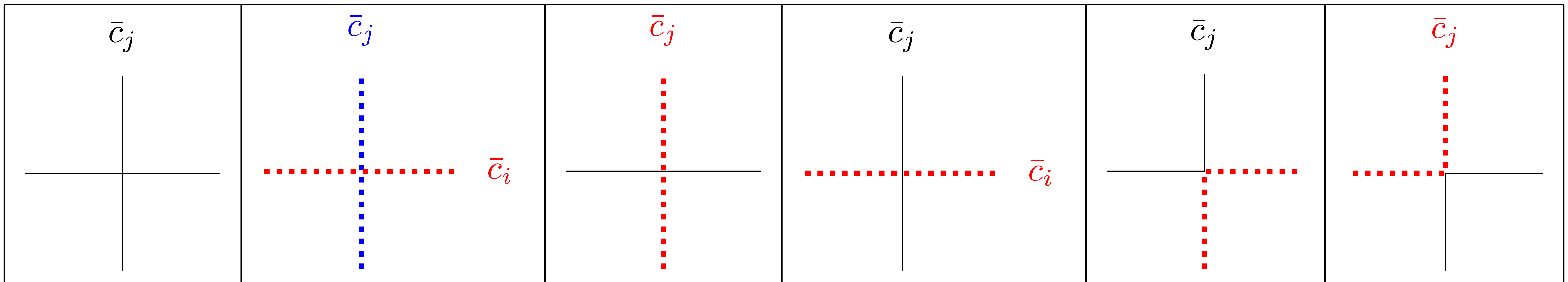
Instead of colors, their edges were assigned charges in $(\mathbb{Z}/n\mathbb{Z})^r$ increasing along the paths.

In [Brubaker–Buciumas–Bump–HG arXiv:2012.15778] we rewrote these in terms of another palette of colors, called supercolors which we draw with dashed lines.

The name supercolors comes from the fact that they are associated to the odd part of a supersymmetric quantum group.

Metaplectic 6-vertex model

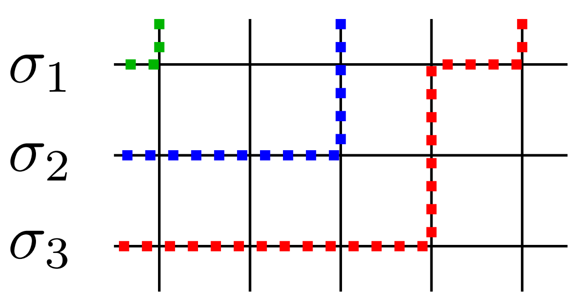
Ordered palette in a fusion block: $\cdots > \bar{c}_2 > \bar{c}_1 > \bar{c}_0$



z	$g(j-i)$	z	1	$(1-v)z$	1

↳ Gauss sum

Projection-component $\sigma = (\sigma_1, \ldots, \sigma_r) \in (\mathbb{Z}/n\mathbb{Z})^r$



Partition function = spherical metaplectic Whittaker function

$$Z(\lambda, \sigma; \mathbf{z}) = \mathbf{z}^\rho \tilde{\phi}^\circ_\sigma(\mathbf{z}; g = \varpi^{-\lambda})$$

Metaplectic 6-vertex model

Partition function = metaplectic spherical Whittaker function

$Z(\lambda, \sigma; \mathbf{z}) = \mathbf{z}^{\rho} \tilde{\phi}^{\circ}_{\sigma}(\mathbf{z}; g = \varpi^{-\lambda})$

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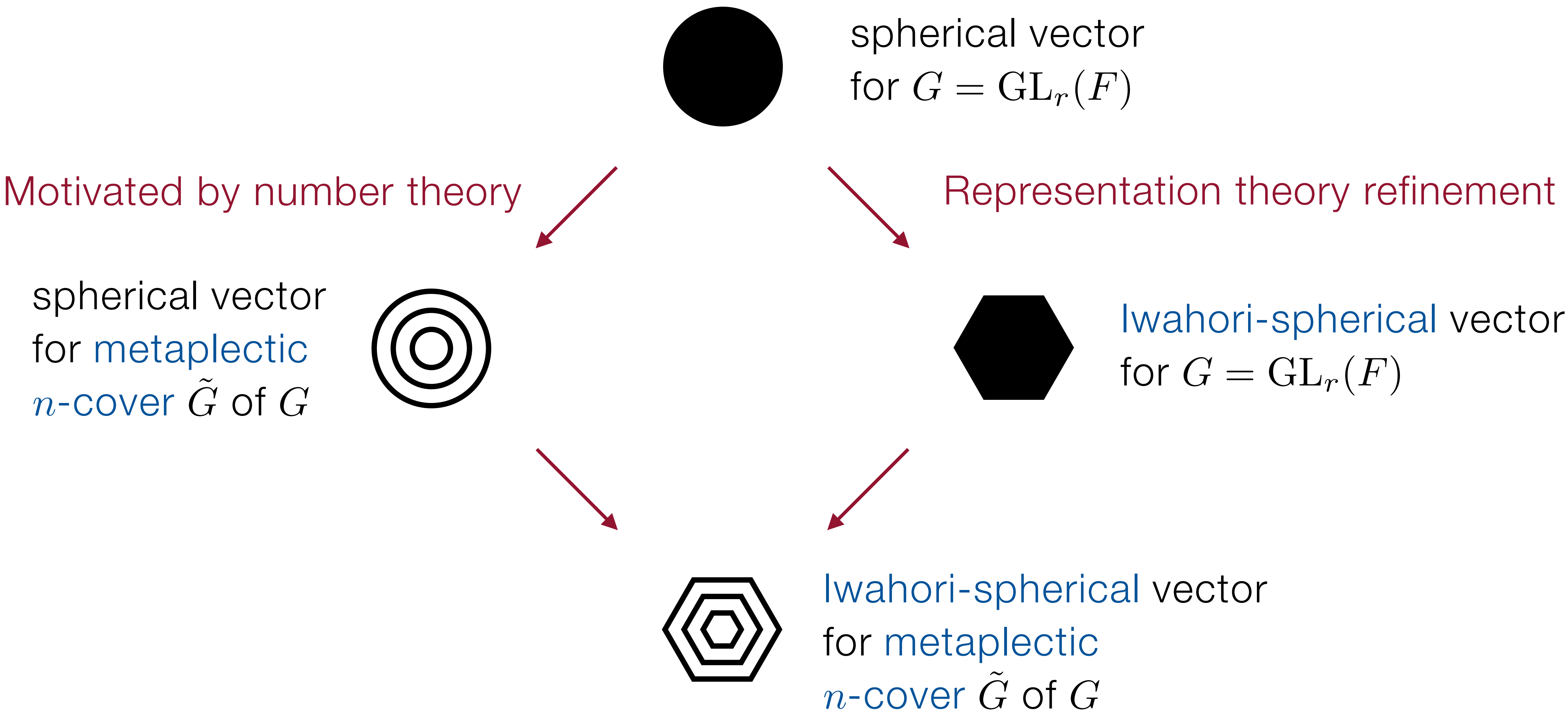
6-vertex model (supercolored)	Metaplectic spherical Whittaker function on $\widetilde{\mathrm{GL}}_r(F)$

Two different generalizations

We consider Whittaker functions for certain vectors of the principal series representation:



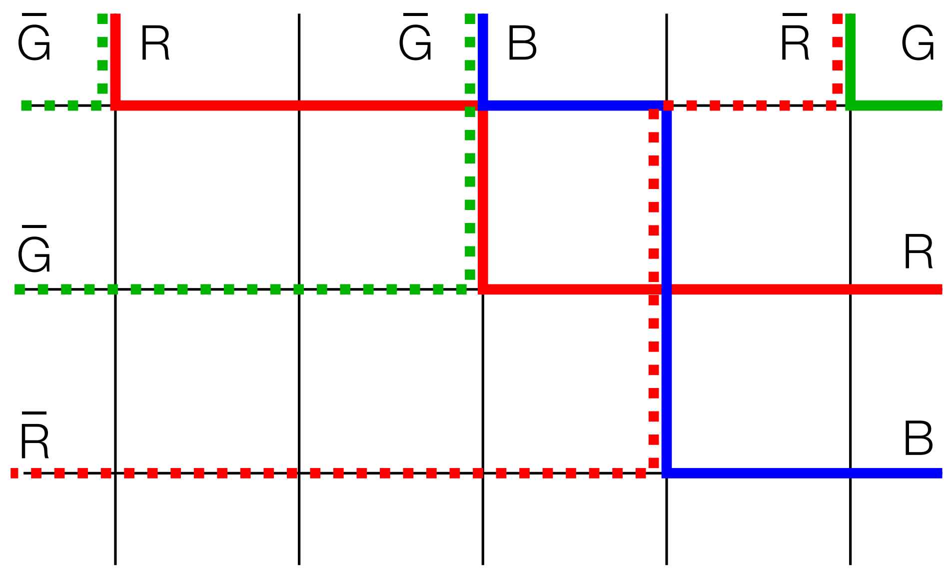
Metaplectic Iwahori lattice model

Two sets of r paths assigned colors from two different palettes:

- m (or r) colors   distinct south-east moving paths
- n supercolors   south-west moving paths

Boundary data: group argument $g = \varpi^{-\lambda}$

Whittaker projection
$\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$



Iwahori basis
$w \in S_r$

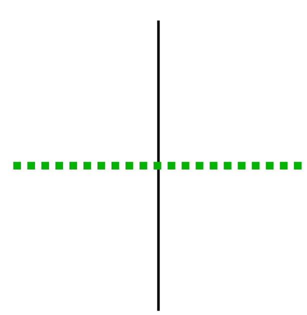
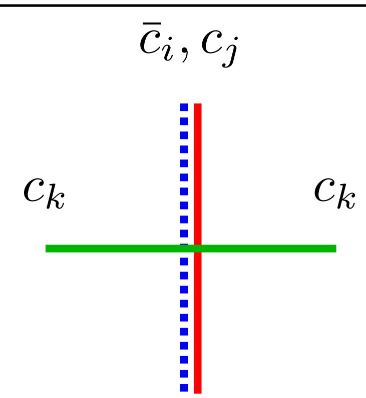
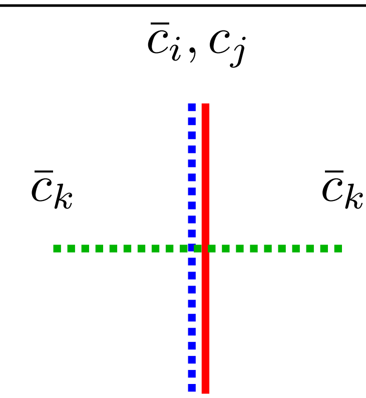
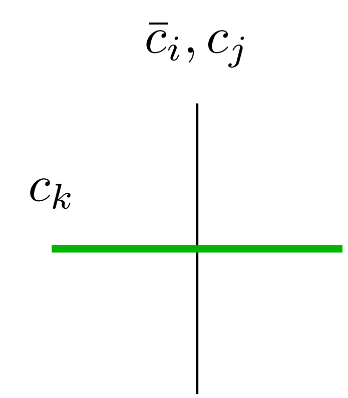
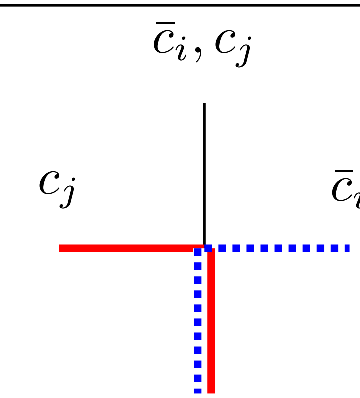
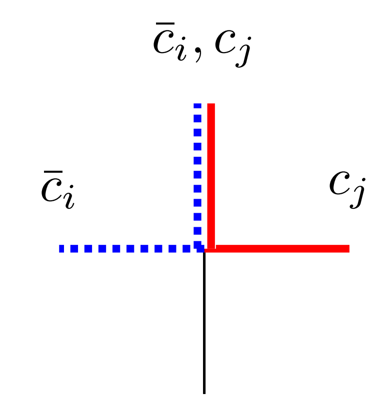
$$Z(\lambda, \sigma, w; \mathbf{z}) = \mathbf{z}^\rho \tilde{\phi}_\sigma^w(\mathbf{z}; g = \varpi^{-\lambda})$$ Metaplectic Iwahori Whittaker function

[Brubaker–Buciumas–Bump–HG arXiv:2012.15778]

Metaplectic Iwahori lattice model

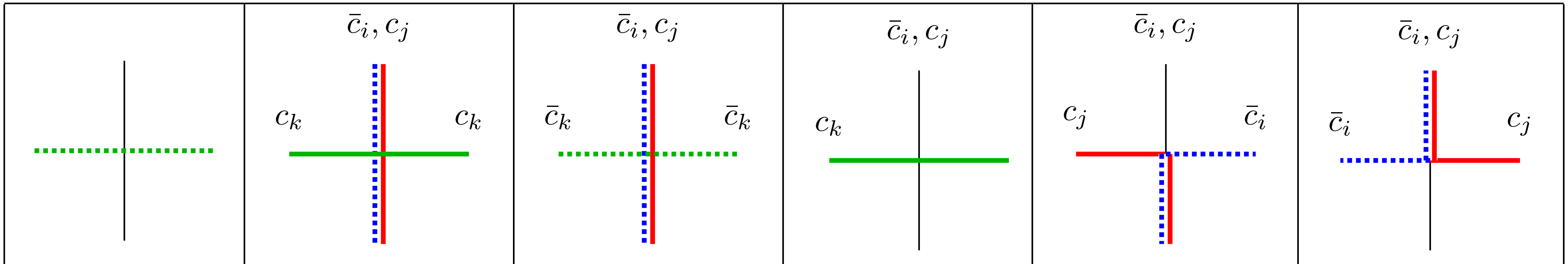
Boltzmann weights are described by applying fusion twice: once for colors and once for supercolors.

In the fully expanded description, each column is assigned a color and a supercolor. If a vertical edge is occupied, it carries both. Horizontal edges carry either a color or a supercolor.

					
1	$\begin{cases} v & \text{if } j > k \\ z & \text{if } j = k \\ 1 & \text{if } j < k \end{cases}$	$g(k-i)$	$\begin{cases} z & \text{if } k = j, \\ 1 & \text{otherwise.} \end{cases}$	$(1-v)z$	1

[Brubaker–Buciumas–Bump–HG arXiv:2012.15778]

Metaplectic Iwahori lattice model



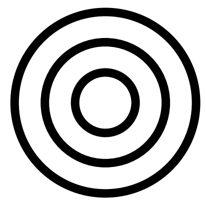
1	$\begin{cases} v & \text{if } j > k \\ z & \text{if } j = k \\ 1 & \text{if } j < k \end{cases}$	$g(k-i)$	$\begin{cases} z & \text{if } k = j, \\ 1 & \text{otherwise.} \end{cases}$	$(1-v)z$	1

	Lattice model		**Associated object in representation theory**
$v \to 0$	5-vertex model (uncolored)	$m = n = 1$	character for $\mathrm{GL}_r(\mathbb{C})$, Schur polynomial
	6-vertex model (uncolored)	$m = n = 1$	spherical Whittaker function on $\mathrm{GL}_r(F)$
$v \to 0$	5-vertex model (colored)	$n = 1$	Demazure atoms and characters for $\mathrm{GL}_r(\mathbb{C})$
	6-vertex model (colored)	$n = 1$	Iwahori Whittaker function on $\mathrm{GL}_r(F)$
	6-vertex model (supercolored)	$m = 1$	Metaplectic spherical Whittaker function on $\widetilde{\mathrm{GL}}_r(F)$
	6-vertex model ($m = r$ colors + n supercolors)		Metaplectic Iwahori Whittaker function on $\widetilde{\mathrm{GL}}_r(F)$

Dualities

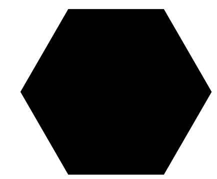
Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G



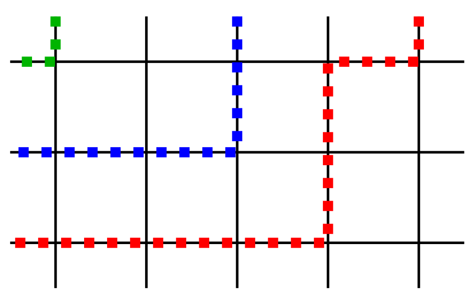
duality

a priori very different

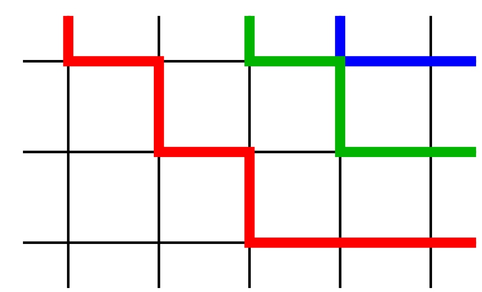


Iwahori-spherical vector for $G = \mathrm{GL}_r(F)$

Lattice models are similar:



metaplectic
spherical

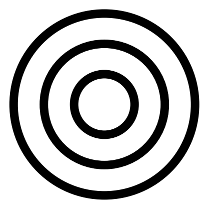


non-metaplectic
Iwahori

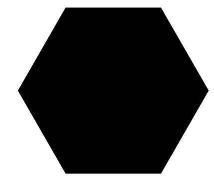
[Brubaker–Buciumas–Bump–HG 2024]

Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G



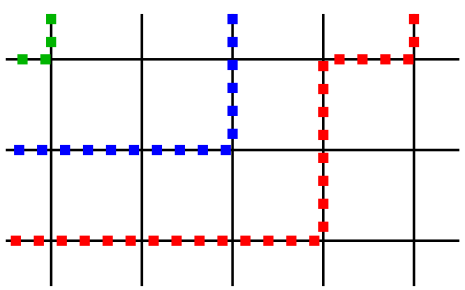
duality



Iwahori-spherical vector for $G = \mathrm{GL}_r(F)$

Lattice models are similar:

family of lattice models

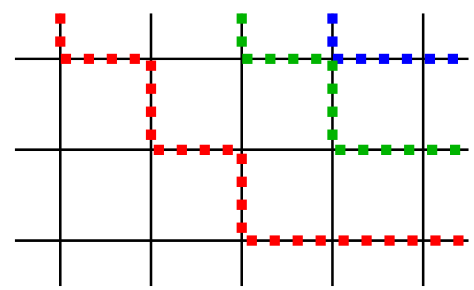


metaplectic spherical

Γ-Δ duality

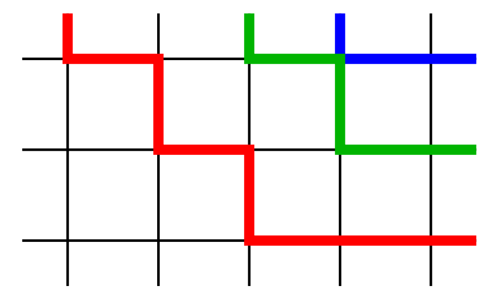
equality for partition functions (not for individual states)

[Brubaker–Bump–Friedberg 2011, Brubaker–Buciumas–Bump 2019]



Drinfeld twist

equality for states (changes Boltzmann weights and partition function)

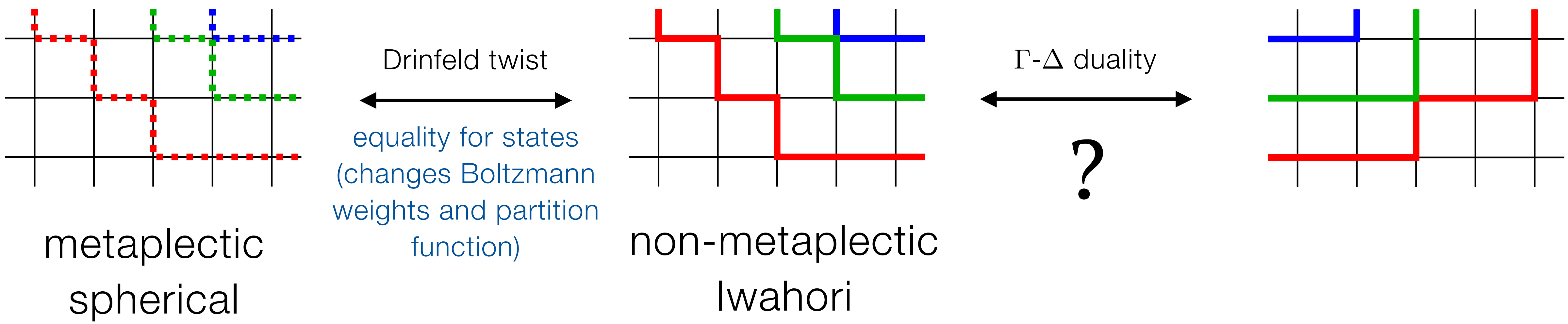


non-metaplectic Iwahori

Proved equality for distinct supercolors in [Brubaker–Buciumas–Bump–HG 2024].

See Vidya Venkateswaran's talk for further results using other methods.

WIP with Carl Westerlund



We have proved Γ-Δ duality for the whole family of lattice models above.

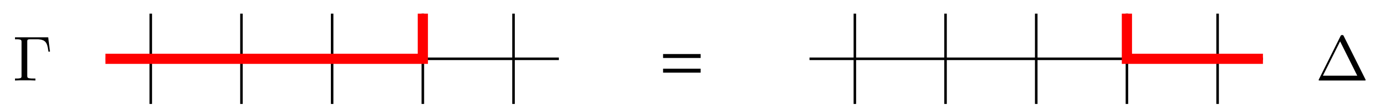
(Including the metaplectic and the Iwahori lattice models)

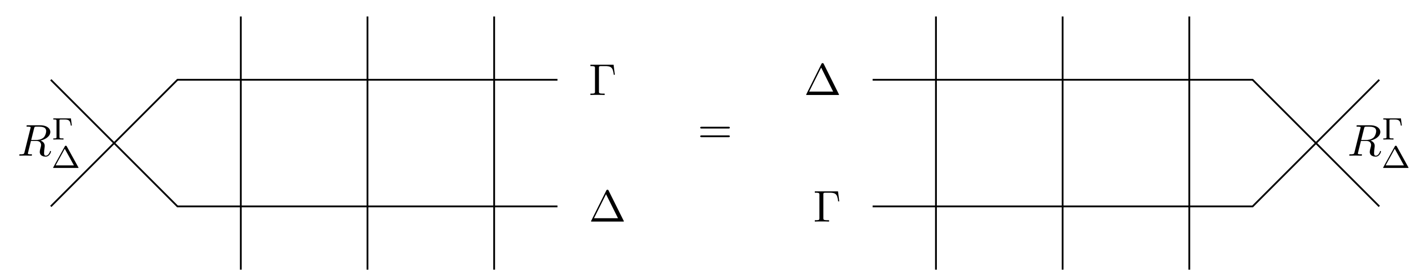
We have obtained a family of mixed R-matrices: R_{Γ}^{Γ}, R_{Δ}^{Δ}, R_{Δ}^{Γ}, R_{Γ}^{Δ}

WIP with Carl Westerlund

We have obtained a family of mixed R-matrices: R^{Γ}_{Γ}, R^{Δ}_{Δ}, R^{Γ}_{Δ}, R^{Δ}_{Γ}

Proof idea from [Brubaker–Buciumas–Bump and Gray 2019] which proved the case when the family is specialized to the metaplectic model:

- Bottom row: 

- Yang–Baxter equation: 

Procedure: Change bottom Γ-row to Δ-row.

Move next Γ-row to the bottom using Yang–Baxter equations. Repeat.

We get an equality on the level of partition functions (not individual states) because the Yang–Baxter equation is on the level of partition functions.

WIP with Carl Westerlund

We get an equality on the level of partition functions (not individual states) because the Yang–Baxter equation is on the level of partition functions.

There is also a conjectured equality on the level of states (or rather some small, yet-to-be-defined packets of states) in the metaplectic case in [Brubaker–Bump–Friedberg 2011]

The Γ- and Δ-states are related via their associated Gelfand–Tsetlin patterns by the Schützenberger involution. (Defined in the next slides)

I and Carl propose that the packets should be the same for the whole family of lattice models. It is much easier to work with the Iwahori case.

In the Iwahori case we may also take the simplifying limit $v \to 0$ to obtain a 5-vertex model.

Note however that the difficulty for the 6-vertex model lies in matching the v's.

WIP with Carl Westerlund

The Γ- and Δ-states are related via their associated Gelfand–Tsetlin patterns by the Schützenberger involution.

I and Carl propose that the packets should be the same for the whole family of lattice models. It is much easier to work with the Iwahori case.

In the Iwahori case we may also take the simplifying limit $v \to 0$ to obtain a 5-vertex model.

In this limit, we claim that the packets each contain just a single state.

That is, we get a Γ-Δ duality on the level of states:

$$\mathrm{wt}_\Delta(\mathfrak{s})(\mathbf{z}) = \mathrm{wt}_\Gamma(\mathrm{Sch}(\mathfrak{s}))(w_0\mathbf{z})$$

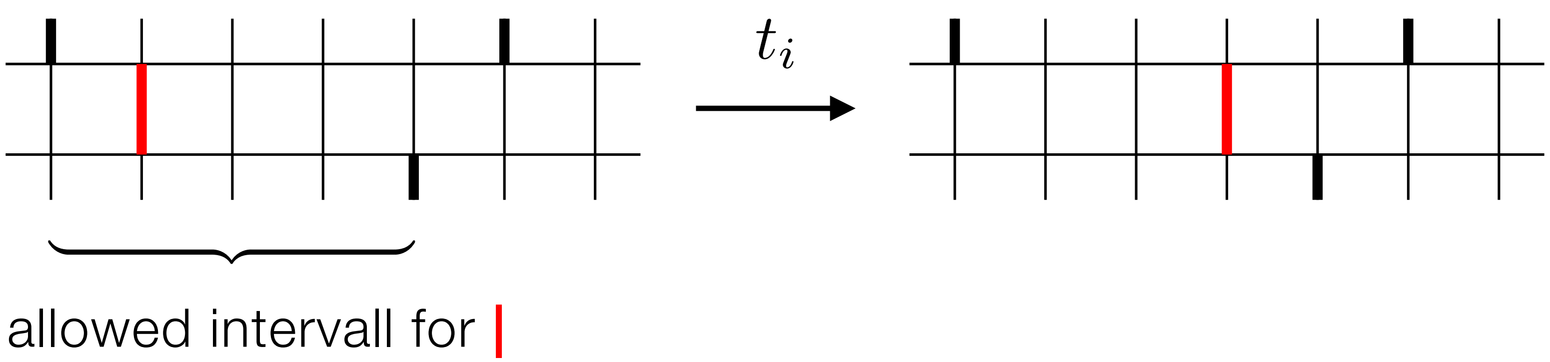
$$\mathrm{GTP}^{\mathrm{ls}}(\lambda+\rho) \qquad\qquad \mathrm{GTP}^{\mathrm{rs}}(\lambda+\rho)$$

Let us do an example.

WIP with Carl Westerlund

The Schützenberger involution Sch on Gelfand–Tsetlin patterns is a product of Berenstein–Kirillov operators acting on a single row.

The operator reflects all vertical edges within their allowed intervalls governed by the adjacent rows:



$$\mathrm{Sch} = \underbrace{(t_{r-1} \cdots t_1)}_{\text{move 1st row to the bottom}} \cdot \underbrace{(t_{r-1} \cdots t_2)}_{\text{move 2nd row to the bottom}} \cdots (t_{r-1} t_{r-2}) \cdot t_{r-1}$$

move 1st row to the bottom

move 2nd row to the bottom

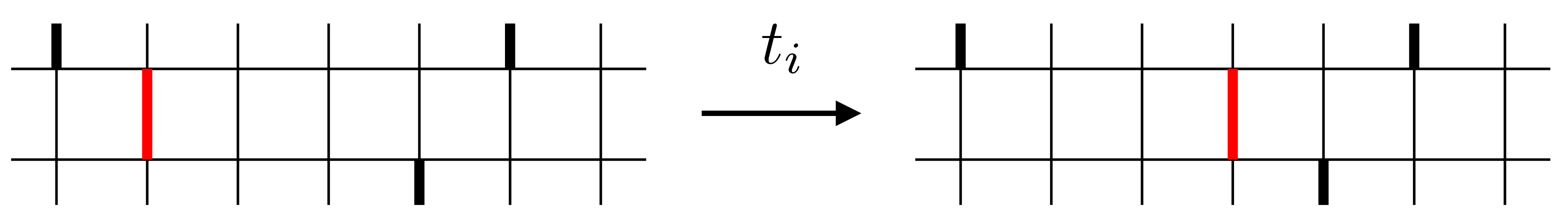
...

move next-to-last row to the bottom

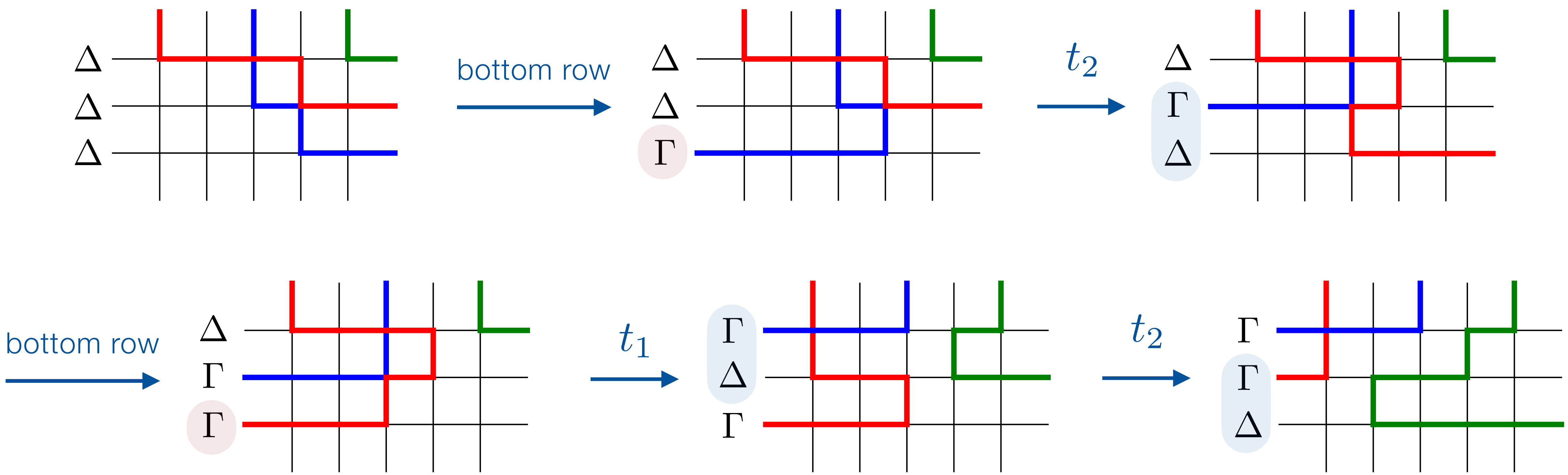
These are exactly the row swaps we did with the R-matrices

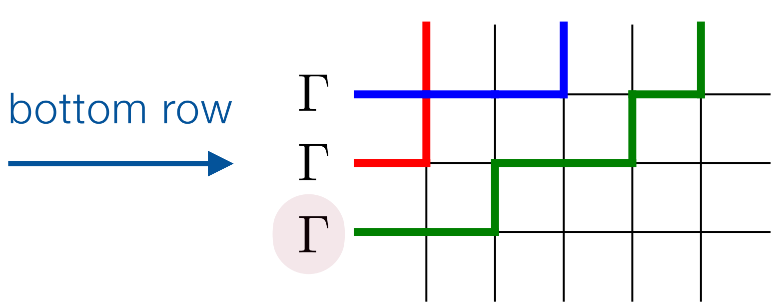
[Berenstein–Kirillov 1995, Brubaker–Bump–Friedberg 2011]

WIP with Carl Westerlund



We can follow each step using models with mixed Γ and Δ rows:





Changing the bottom row from Δ to Γ leaves the weight invariant while t_i acts on the weight by $s_i \in S_r$

WIP with Carl Westerlund

What about type B/C lattice models?

- Built out of alternating rows of Γ and Δ with connecting caps at one end
- Lattice models for metaplectic spherical Whittaker functions constructed by [Gray 2017]
- A 5-vertex Demazure lattice model was constructed by [Buciumas–Scrimshaw 2022].
- The latter had an open question about the solvability of the model due to one missing, mixed R-matrix (connected to an issue with color loops).
- We have both Γ and Δ weights and all four R-matrices for the whole family, including metaplectic and Iwahori case.
- When taking the 5-vertex model limit $v \to 0$ we recover the same Γ and Δ weights as [Buciumas–Scrimshaw 2022] and can explain what happens to the missing R-matrix and how (at least some) color loops are resolved by telescopic sums (WIP).

Two-year postdoc scholarship within project about lattice models led by me

Link to application will soon appear at:

https://hgustafsson.se/postdoc/

Deadline September 15

Thank you!

Slides are available at

https://hgustafsson.se