Eulerianity of Fourier coefficients of automorphic forms

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Papers

Joint work with Dmitry Gourevitch, Axel Kleinschmidt, Daniel Persson and Siddhartha Sahi

A reduction principle for Fourier coefficients of automorphic forms
arXiv:1811.05966

Fourier coefficients of minimal and next-to-minimal automorphic representations of simply-laced groups
arXiv:1908.08296

Eulerianity of Fourier coefficients of automorphic forms
arXiv:2004.14244

Will be referred to as [GGKPS 18, 19, and 20]

Transfer of Eulerianity

Under certain conditions:

Given: Fourier coefficient $\mathcal{F}_A$ factorizes as an Euler product

Implies: Fourier coefficient $\mathcal{F}_B$ factorizes as an Euler product

Outline

- Different types of Fourier coefficients
- Vanishing properties and automorphic representations
- Transfer theorem for Eulerianity
- Applications to small automorphic representations
- Proof of transfer theorem

Motivation

A standard example of Eulerian Fourier coefficients are so called Whittaker coefficients.

Used to study L-functions, both via the Langlands–Shahidi and Rankin–Selberg method where Eulerianity is a key ingredient.

In these global integral representations of L-functions the Whittaker coefficients of automorphic forms appear either directly or after a so called unfolding of the integral.

Can also study global integrals where other Fourier coefficients of automorphic forms in various representations appear.

Motivation

To show that such an integral has the properties of an L-function it is not only important to know which Fourier coefficients vanish for which representations, but also which are Eulerian.

For a program in this direction see for example [Ginzburg 06, 14, 16, Ginzburg–Hundley 13]

[GGKPS] started with a different initial motivation in mind:

Study Fourier coefficients of automorphic forms inspired by questions from string theory concerning scattering amplitudes of gravitons and non-perturbative effects such as instantons and black holes.

The mathematical questions we study and the methods we use are closely intertwined.

Setup

$\mathbb{K}$ number field, $\mathbb{A} = \mathbb{A}_{\mathbb{K}}$

$\mathbf{G}$ reductive algebraic group $/\mathbb{K}$ (split) $\mathfrak{g}$ Lie algebra of $\mathbf{G}(\mathbb{K})$

$\mathcal{A}(\mathbf{G}(\mathbb{A}))$ space of automorphic forms on $\mathbf{G}(\mathbb{A})$

Whittaker coefficients

Fix a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{N}$

$\psi_{\mathbf{N}} : \mathbf{N}(\mathbb{A}) \to \mathbb{C}^\times$ unitary character trivial on $\mathbf{N}(\mathbb{K})$ generic

$\eta \in \mathcal{A}(\mathbf{G}(\mathbb{A})), \quad g \in \mathbf{G}(\mathbb{A})$ $\quad g = \prod_\nu g_\nu, \quad g_\nu \in \mathbf{G}(\mathbb{K}_\nu)$

$$\mathcal{W}_{\psi_{\mathbf{N}}}[\eta](g) := \int\limits_{\mathbf{N}(\mathbb{K})\backslash\mathbf{N}(\mathbb{A})} \eta(ng)\psi_{\mathbf{N}}^{-1}(n)\, dn = \prod_\nu \mathcal{W}_{\psi_{\mathbf{N}},\nu}[\eta](g_\nu)$$

Uniqueness of Whittaker models [Gelfand–Kajdan 71/75, Shalika 74, Rodier 73, Kostant 78]

Fourier coefficients

(Different authors include different notions)

Unipotent subgroup $\mathbf{U} \subset \mathbf{G}$

$\psi_{\mathbf{U}} : \mathbf{U}(\mathbb{A}) \to \mathbb{C}^\times$ unitary character trivial on $\mathbf{U}(\mathbb{K})$

$$\mathcal{F}_{\psi_{\mathbf{U}}}[\eta](g) := \int_{\mathbf{U}(\mathbb{K})\backslash\mathbf{U}(\mathbb{A})} \eta(ug)\psi_{\mathbf{U}}^{-1}(u)\, du$$

unipotent period integral

In general not Eulerian

Called a Whittaker coefficient if $\mathbf{U}$ is maximal, e.g. $\mathbf{N}$

Called a parabolic Fourier coefficient if $\mathbf{U}$ is the unipotent radical of a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$

Whittaker pairs

$$(\mathbf{U}, \psi_{\mathbf{U}}) \longrightarrow (S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$$

S semisimple $\mathrm{ad}(S)$ has eigenvalues in $\mathbb{Q}$ $\mathrm{ad}^*(S)\varphi = -2\varphi$

$\mathfrak{g}^S_\lambda :=$ λ-eigenspace of $\mathrm{ad}(S)$ in $\mathfrak{g}$

$\mathfrak{g}_\varphi :=$ centralizer of φ in $\mathfrak{g}$ under the coadjoint action

$$\mathbf{U}_{S,\varphi} := \mathrm{Exp}(\mathfrak{u}_{S,\varphi}) \qquad \mathfrak{u}_{S,\varphi} := \mathfrak{g}^S_{>1} \oplus \mathfrak{g}^S_1 \cap \mathfrak{g}_\varphi$$

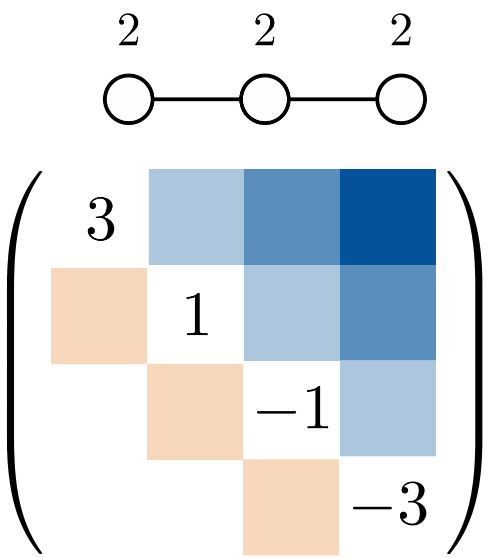
Fix a non-trivial unitary, additive character $\chi : \mathbb{A} \to \mathbb{C}^\times$ trivial on $\mathbb{K}$

$\psi_{\mathbf{U}} : u \mapsto \chi\big(\varphi(\log u)\big)$

$$\mathcal{F}_{S,\varphi}[\eta](g) := \int_{\mathbf{U}_{S,\varphi}(\mathbb{K})\backslash \mathbf{U}_{S,\varphi}(\mathbb{A})} \eta(ug)\chi\big(\varphi(\log u)\big)^{-1}\, du$$

Mœglin–Waldspurger 87, Gomez–Gourevitch–Sahi 17

Whittaker pairs

$S =$ 

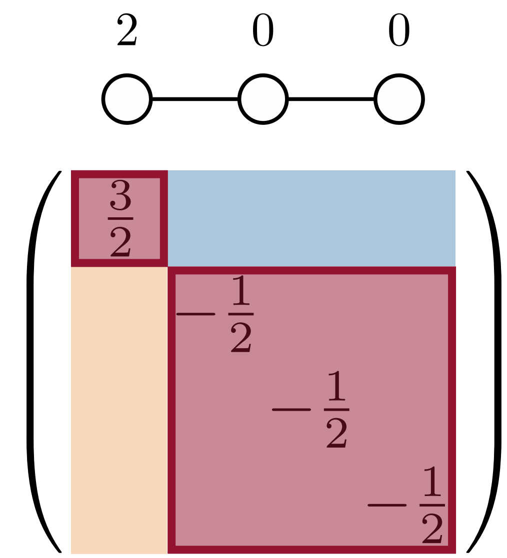
■ $= \mathfrak{g}^S_6$

■ $= \mathfrak{g}^S_4$

■ $= \mathfrak{g}^S_2$

$\mathfrak{u}_{S,\varphi} := \mathfrak{g}^S_{>1} \oplus \mathfrak{g}^S_1 \cap \mathfrak{g}_\varphi$

■ $= \mathfrak{g}^S_{-2} \leftrightarrow (\mathfrak{g}^*)^S_{-2} \ni \varphi$



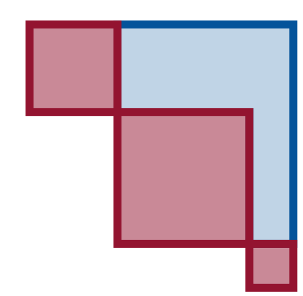
Generic = supported on all root spaces

(standard)
parabolic subgroup $P = L\ U$ ⟵ Dynkin weights $\in \{0, 2\}$

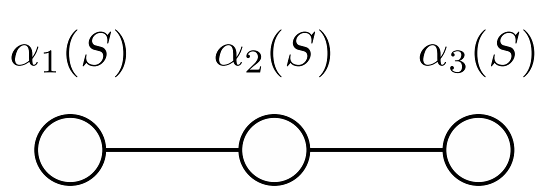
Levi Unipotent

weights 0

For GL_n, L is block-diagonal and P is block-upper-triangular.



Describe Cartan element using weighted Dynkin diagram



$$\begin{pmatrix} * & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \\ & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} \\ & & * & \mathfrak{g}_{\alpha_3} \\ & & & * \end{pmatrix}$$

Whittaker pairs

Conjugation

Let $\gamma \in \mathbf{G}(\mathbb{K})$ and $(S', \varphi') := (\mathrm{Ad}(\gamma)S, \mathrm{Ad}^*(\gamma)\varphi)$.

Then $\mathcal{F}_{S,\varphi}[\eta](g) = \mathcal{F}_{S',\varphi'}[\eta](\gamma g)$

Neutral pairs

Exists unique nilpotent element $f \in \mathfrak{g}$ such that φ is the Killing form pairing with f

A Whittaker pair (h, φ) that can be completed to an $\mathfrak{sl}_2$-triple (f, h, e) is called neutral

Conjugacy classes of $\mathfrak{sl}_2$-triples $\simeq$ nilpotent orbits in $\mathfrak{g}$

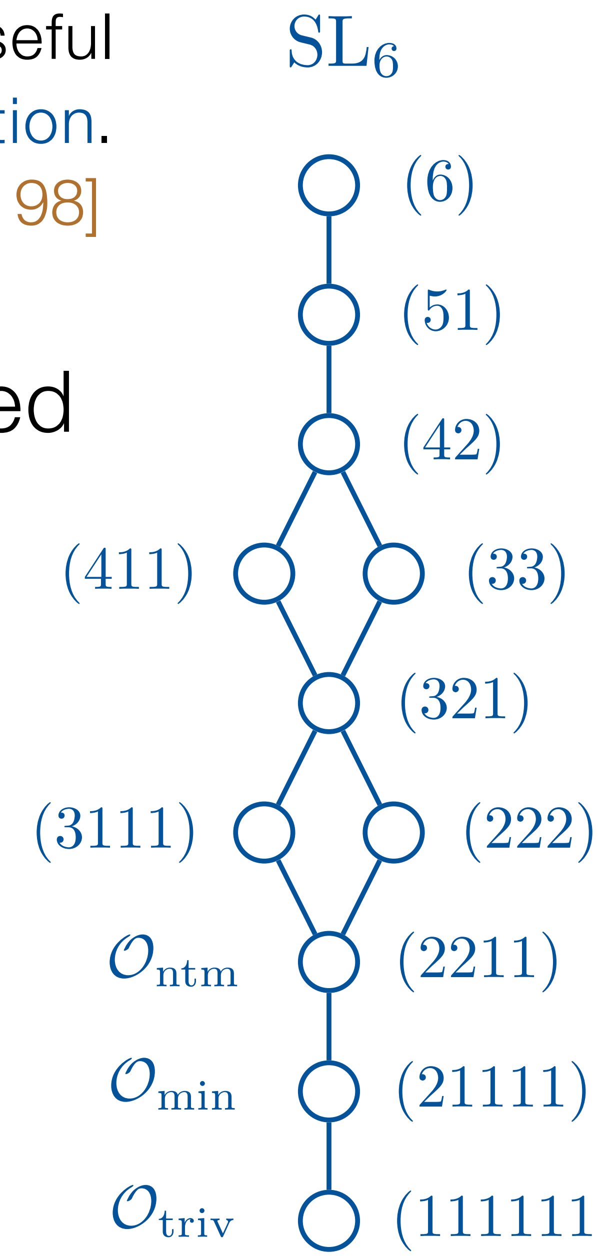
These neutral Fourier coefficients are exactly the orbit coefficients studied in [Ginzburg 06]

$$[h, f] = -2f$$
$$[h, e] = 2e$$
$$[e, f] = h$$

[Jacobson, Morozov, Kostant; see Collingwood–McGovern 93]

Nilpotent orbits

Although the orbits on the previous slide are $\mathbf{G}(\mathbb{K})$-orbits, it is useful to embed them in complex orbits for which we have a classification. Do not depend on the complex embedding $\mathbb{K} \hookrightarrow \mathbb{C}$. [Đoković 98]



For classical groups, complex orbits are classified by integer partitions.

SL_n: all partitions of n.
SO_{2n+1}: partitions of $2n+1$ where even parts have even multiplicities.

Partial order:

$$(\lambda_1, \ldots, \lambda_N) \leq (\mu_1, \ldots, \mu_N) \iff \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \text{ for } 1 \leq k \leq N$$

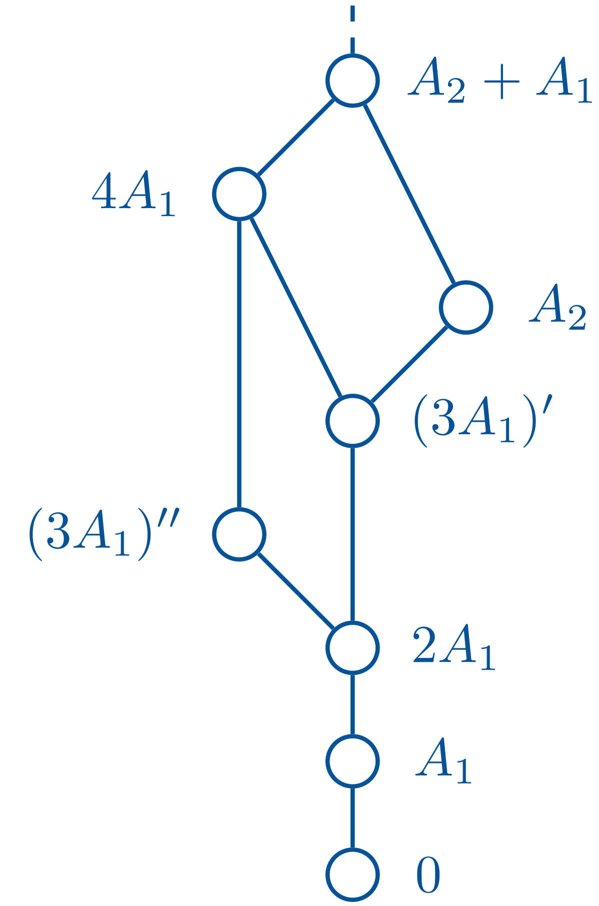
There is a unique minimal orbit (aside from the trivial), but there can exist more than one next-to-minimal orbit.

Nilpotent orbits

In general, classified by Bala–Carter label determined by the Cartan type of the unique conjugacy class of minimal Levi subalgebras that intersect $\mathcal{O}$.

Correction: E7

Partial order from inclusion under Zariski closure.



The behavior of degenerate Whittaker coefficients of Eisenstein series is very much decided by the Bala–Carter label of the character orbit.

(Reduction formula in later slides)

Global wave-front set

Automorphic form $\eta \in \mathcal{A}(\mathbf{G}(\mathbb{A}))$ Automorphic representation π

$$\underset{\pi}{\mathrm{WF}(\eta)} := \left\{ \begin{array}{l} \text{nilpotent orbits } \mathcal{O} \text{ such that} \\ \exists \text{ neutral pair } (h, \varphi) \text{ with } \varphi \in \mathcal{O} \text{ and } \mathcal{F}_{h,\varphi}[\eta] \neq 0 \end{array} \right\}$$

for some $\eta \in \pi$

Define the Whittaker support WS to be the set of maximal orbits in WF for a given $\eta \in \mathcal{A}(\mathbf{G}(\mathbb{A}))$ or $\pi \subset \mathcal{A}(\mathbf{G}(\mathbb{A}))$.

If $\mathrm{WS}(\pi)$ consists of minimal orbits we say that π is a minimal automorphic reperesentation.

Similarly for next-to-minimal representations.

Global wave-front set

Theorem [Gomez–Gourevitch–Sahi 17]

Let (S, φ) be any Whittaker pair with $\mathbf{G}(\mathbb{K})\varphi \notin \mathrm{WF}(\eta)$.

Then $\mathcal{F}_{S,\varphi}[\eta] = 0$.

Small representation have few non-vanishing Fourier coefficients

Similar local statements: Mœglin–Waldspurger 87, Matumoto 87

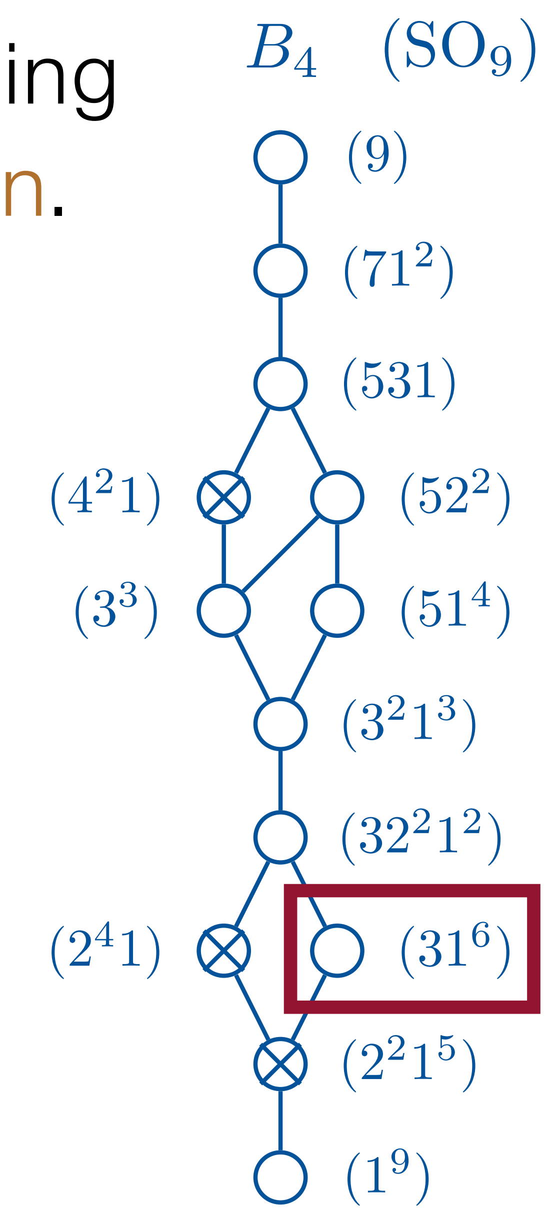
Global wave-front set

For split, classical groups (types A to D), the Whittaker support contains only special orbits. [Jiang–Liu–Savin 16]

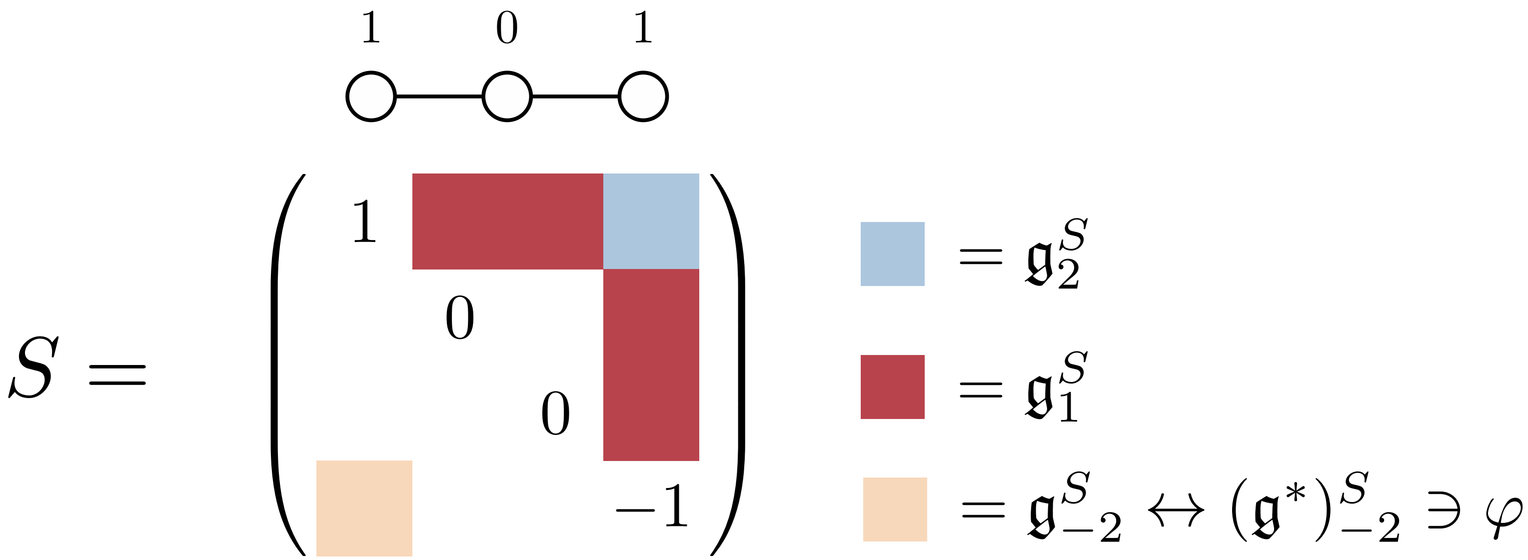
For classical groups, special orbits are defined using order-reversing maps on partitions by Spaltenstein.

For type A, all orbits are special but this is not true in general.
In fact, B_n has, for example, no representation associated to its minimal orbit for any n.

More generally, (including exceptional groups) the Whittaker support contains only so-called quasi-admissible orbits. [Gomez–Gourevitch–Sahi]



Maximal isotropic Fourier coefficients



$$\mathfrak{u}_{S,\varphi} := \mathfrak{g}_{>1}^S \oplus \mathfrak{g}_1^S \cap \mathfrak{g}_\varphi$$

$\mathfrak{g}_2^S$ $\{0\}$

Define antisymmetric form $\omega_\varphi : \mathfrak{g}_{\geqslant 1}^S \times \mathfrak{g}_{\geqslant 1}^S \to \mathbb{K}$, $(X, Y) \mapsto \varphi([X, Y])$

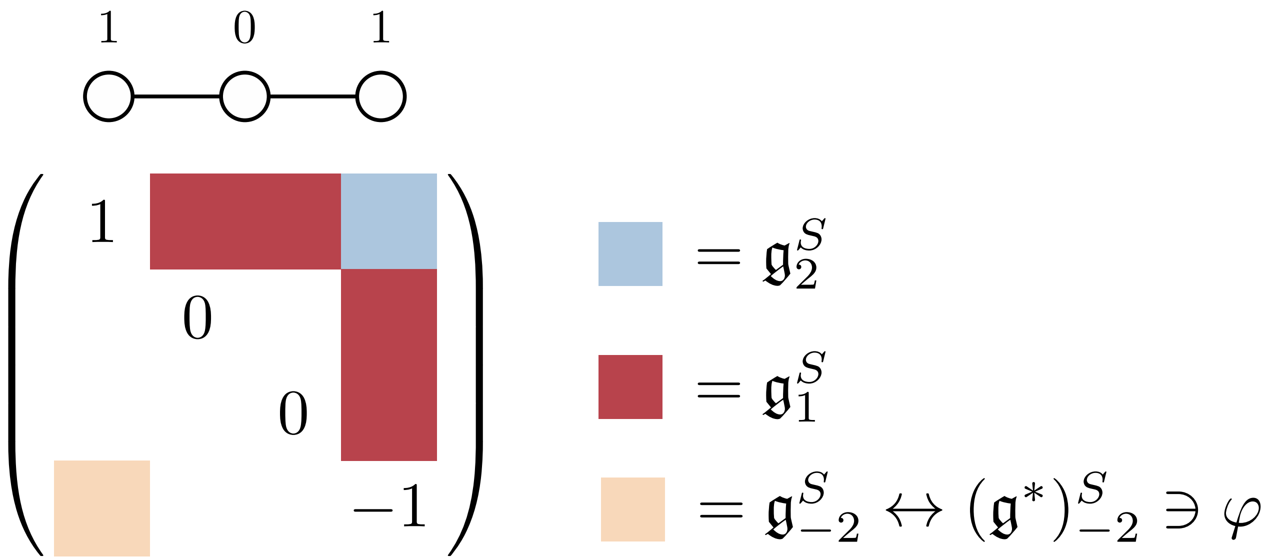
Then $\mathfrak{u}_{S,\varphi}$ is the radical of ω_φ $= \{X \in \mathfrak{g}_{\geqslant 1}^S \, : \, \omega(X,Y) = 0 \quad \forall Y \in \mathfrak{g}_{\geqslant 1}^S\}$

Let $\mathfrak{i} \subset \mathfrak{g}_{\geqslant 1}^S$ be an isotropic space w.r.t. ω_φ $\omega_\varphi(\mathfrak{i}, \mathfrak{i}) = 0$

which is maximal w.r.t. inclusion. Let $\mathbf{I} := \mathrm{Exp}(\mathfrak{i}) \subset \mathbf{G}$.

Maximal isotropic Fourier coefficients

$S =$



$\mathbf{I} = \begin{pmatrix} 1 & * & * & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\mathbf{I}' = \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$

Define antisymmetric form $\omega_\varphi : \mathfrak{g}^S_{\geqslant 1} \times \mathfrak{g}^S_{\geqslant 1} \to \mathbb{K}$, $(X, Y) \mapsto \varphi([X, Y])$

Then $\mathfrak{u}_{S,\varphi}$ is the radical of ω_φ $\quad = \{X \in \mathfrak{g}^S_{\geqslant 1} \; : \; \omega(X, Y) = 0 \quad \forall Y \in \mathfrak{g}^S_{\geqslant 1}\}$

Let $\mathfrak{i} \subset \mathfrak{g}^S_{\geqslant 1}$ be an isotropic space w.r.t. ω_φ $\qquad \omega_\varphi(\mathfrak{i}, \mathfrak{i}) = 0$

which is maximal w.r.t. inclusion. Let $\mathbf{I} := \operatorname{Exp}(\mathfrak{i}) \subset \mathbf{G}$.

$$\mathcal{F}^{\mathbf{I}}_{S,\varphi}[\eta](g) := \int_{\mathbf{I}(\mathbb{K})\backslash \mathbf{I}(\mathbb{A})} \eta(ug) \chi_\varphi(u)^{-1} \, du$$

Maximal isotropic Fourier coefficients

$$\mathcal{F}^{\mathbf{I}}_{S,\varphi}[\eta](g) := \int\limits_{\mathbf{I}(\mathbb{K})\backslash\mathbf{I}(\mathbb{A})} \eta(ug)\chi_\varphi(u)^{-1}\,du$$

$$\mathbf{I} = \begin{pmatrix} 0 & * & * & * \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

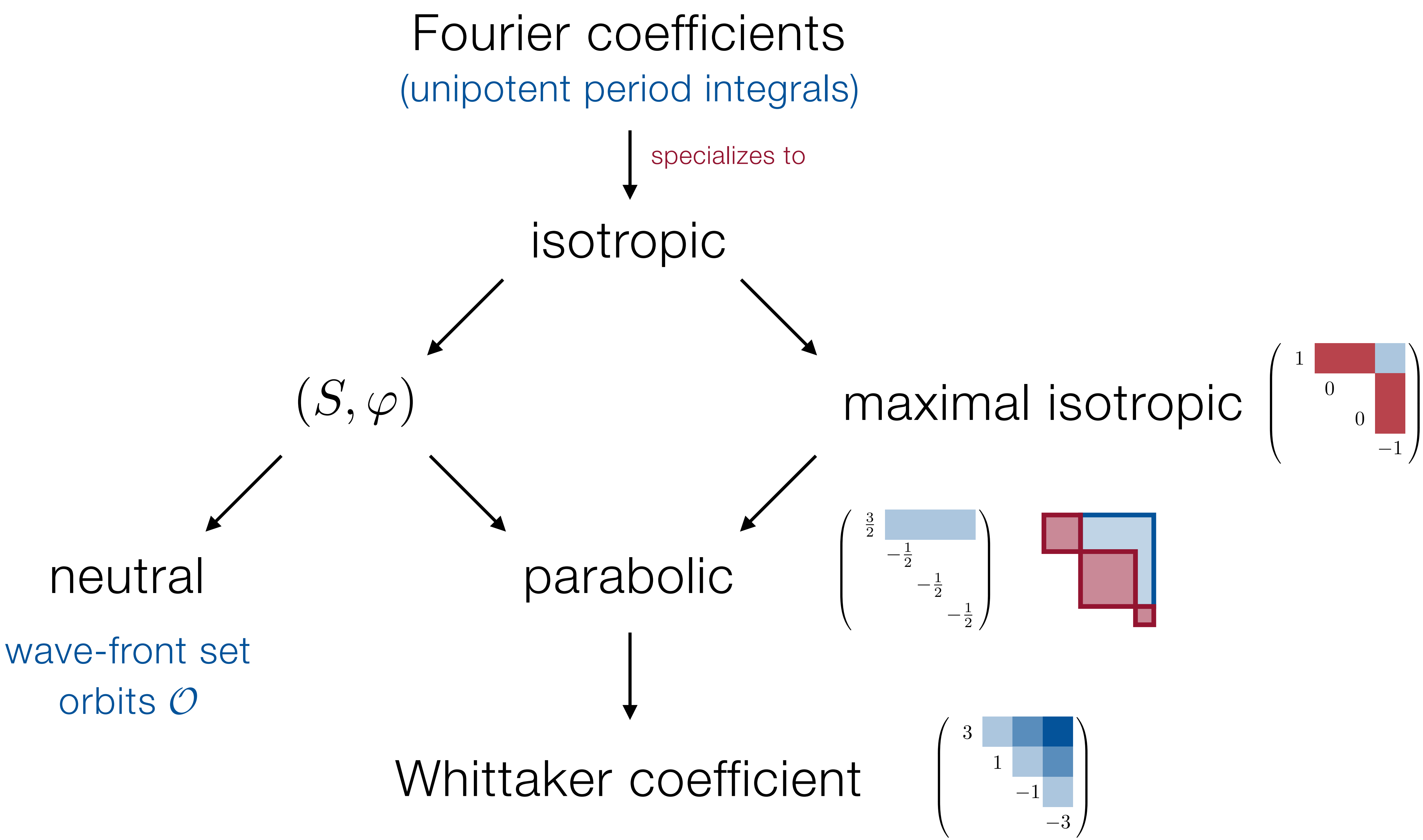
$$\mathbf{I}' = \begin{pmatrix} 0 & & & * \\ & 0 & & * \\ & & 0 & * \\ & & & 0 \end{pmatrix}$$

Parabolic Fourier coefficients are maximal isotropic coefficients

In fact, $\mathfrak{g}_1^S = \{0\} \implies \mathfrak{u}_{S,\varphi}$ is maximal isotropic and $\mathcal{F}^{\mathbf{I}}_{S,\varphi} = \mathcal{F}_{S,\varphi}$

In general, $\mathcal{F}^{\mathbf{I}}_{S,\varphi}$ is a further period integral of $\mathcal{F}_{S,\varphi}$.

Different types of Fourier coefficients



Main results

Theorem (Transfer of Eulerianity) [Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Let $\mathbf{G}$ be a reductive algebraic group over $\mathbb{K}$ and let η be an automorphic form on $\mathbf{G}(\mathbb{A})$.

Let (S, φ) and (H, ψ) be two Whittaker pairs such that $\mathbf{G}(\mathbb{K})\varphi = \mathbf{G}(\mathbb{K})\psi \in \mathrm{WS}(\eta)$. ← Whittaker support (maximal orbits in global wave-front set)

Suppose that a maximal isotropic Fourier coefficient $\mathcal{F}^{\mathbf{I}}_{S,\varphi}[\eta]$ is Eulerian.

Then *any* maximal isotropic Fourier coefficient $\mathcal{F}^{\mathbf{I}'}_{H,\psi}[\eta]$ is Eulerian.

Special case: replace maximal isotropic with parabolic

Remarks

- The proof also details how to write one coefficient as an Eulerian integral of the other.

- For more general statements on how to relate different Fourier coefficients, although not necessarily preserving Eulerianity, see [GGKPS 18 and 19]

- Related transfer of local uniqueness for non-archimedean degenerate Whittaker models [Mœglin–Waldspurger 87]
  - **Only** consider non-archimedean places
  - Local wave-front set can vary over places of $\mathbb{K}$ and be bigger than the global one

Applications

Find automorphic forms for which some Fourier coefficient is known to be Eulerian and transfer to other coefficients.

Generic Whittaker coefficients are Eulerian but cannot be transferred to any new type of Fourier coefficients. [GGKPS 18]

- Minimal representations
- Next-to-minimal representations
- Eisenstein realizations

Will state general results and from where they are transferred

Applications

For the following autormorphic representations π and characters φ we show that all corresponding parabolic and other maximal isotropic Fourier coefficients are Eulerian:

π is a unitary minimal representation of $\mathbf{G}$, split of type D_n or E_7 and $\varphi \neq 0$

Transferred from maximal parabolic Fourier coefficient with abelian unipotent radical which we first showed to be Eulerian using local uniqueness results from [Loke–Savin 06, Kobayashi–Savin 15]

π is a next-to-minimal* representation of $\mathbf{G}$, split of type B_n or D_n and $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\pi)$ corresponding to $\mathcal{O}_{(31\ldots1)}$

Transferred from maximal parabolic Fourier coefficient which we first showed to be Eulerian by proving a hidden invariance and using local uniqueness of Bessel models from [Gan–Gross–Prasad 12, Jiang–Sun–Zhu 10]

Applications

A Whittaker coefficient with degenerate (not generic) character $\psi_{\mathbf{N}}$ is, in general, not Eulerian.

However, for Eisenstein series E_λ, these degenerate Whittaker coefficients can be computed directly via a reduction formula and shown to be Eulerian for certain weights λ.

Setup: $\mathbf{G}$ split simply-laced. $\mathbb{K} = \mathbb{Q}$. Weyl vector ρ.
Iwasawa decomposition $\mathbf{G}(\mathbb{A}) = \mathbf{N}(\mathbb{A})\mathbf{A}(\mathbb{A})K_{\mathbb{A}}$.

Spherical Borel Eisenstein series

$$E_\lambda(g) = \sum_{\gamma \in \mathbf{B}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g)\rangle}$$

$H : \mathbf{G}(\mathbb{A}) \to \mathfrak{h}$ by $e^{\langle \lambda, H(nak)\rangle} = |\lambda(a)|_{\mathbb{A}}$ $\forall \lambda \in X^*(\mathbf{A}) := \mathrm{Hom}(\mathbf{A}, \mathbb{G}_m)$. $\mathfrak{h}^* := X^*(\mathbf{A}) \otimes \mathbb{R}$

See for example [Bernstein–Lapid 19] for notation

Applications

Reduction formula

Bala–Carter label

Levi subgroup on which $\psi_{\mathbf{N}}$ is generic

Schematically: $$\mathcal{W}^{(\mathbf{G})}_{\psi_{\mathbf{N}}}[E_\lambda] = \sum_w M(w,\lambda)\ \mathcal{W}^{(\mathbf{G}')}_{\psi_{\mathbf{N}}}[E_{w\lambda}]$$

degenerate

intertwiner

[Fleig–Kleinschmidt–Persson 14]

Sum of generic Whittaker coefficients, each of which is Eulerian

We show that for certain λ and $\psi_{\mathbf{N}}$ only one term remains which leaves $\mathcal{W}^{(\mathbf{G})}_{\psi_{\mathbf{N}}}[E_\lambda]$ Eulerian

Applications

$\lambda = 2s_i\Lambda_i - \rho$ for some i and s_i given by the following table

Motivated by string theory

Group	π_{min}	π_{ntm}	$\cdots$
SL_n	generic s_1 **or** generic s_{n-1}	generic s_2 **or** generic s_{n-2}	
$\mathrm{SO}_{n,n}$	$s_1 = \frac{n-2}{2}$ **or** $s_n = 1$ **or** $s_{n-1} = 1$	$\begin{cases} \text{generic } s_1 & (2A_1)' \\ s_{n-1} = 2 \textbf{ or } s_n = 2 & (2A_1)'' \end{cases}$	
$\mathrm{E}_{6(6)}$	$s_1 = \frac{3}{2}$ **or** $s_6 = \frac{3}{2}$	generic s_1 **or** generic s_6 **or** $s_5 = 1$	
$\mathrm{E}_{7(7)}$	$s_1 = \frac{3}{2}$ **or** $s_7 = 2$	$s_1 = \frac{5}{2}$ **or** $s_6 = \frac{3}{2}$ **or** $s_7 = 4$	
$\mathrm{E}_{8(8)}$	$s_1 = \frac{3}{2}$ **or** $s_8 = \frac{5}{2}$	$s_1 = \frac{5}{2}$ **or** $s_7 = 2$ **or** $s_8 = \frac{9}{2}$	
φ	$\mathcal{O}_{\text{min}} = \mathcal{O}_{A_1}$	$\mathcal{O}_{\text{ntm}} = \mathcal{O}_{2A_1}$	$\cdots$

Transfer theorem implies that any parabolic or other maximal isotropic Fourier coefficient $\mathcal{F}^{\mathbf{I}}_{S,\varphi}[E_\lambda]$ with these data is Eulerian

Maximal rank

For all these Eulerian coefficients the character has been in the largest possible orbit. (Of maximal rank)

If (S, φ) is neutral one can show that the dimension of $\mathbf{I}$ is half the dimension of the orbit of φ.

This means that if $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$ then $\mathcal{F}^{\mathbf{I}}_{S,\varphi}[\eta]$ satisfies Ginzburg's dimension formula which is a rule of thumb for when one can expect certain global integrals to be Eulerian.

Using the transfer theorem one would then expect any maximal isotropic Fourier coefficient, not necessarily neutral, with character in $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$ to be Eulerian.

Maximal rank

Indeed, we show, using different methods:

Theorem [Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Let π be an irreducible admissible automorphic representation in the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ and let (S, φ) be a Whittaker pair with $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\pi)$.

Then any maximal isotropic Fourier coefficient $\mathcal{F}^{\mathbf{I}}_{S,\varphi}$ is Eulerian on π.

We expect this to hold in wider generality

Towards a proof of the transfer theorem

- Quasi order on Whittaker pairs: "dominates"

- Relate maximal isotropic Fourier coefficients for Whittaker pairs where one is dominating the other

Based on [GGKPS 18]

- Use neutral pairs and conjugation of Whittaker pairs to relax this condition and glue everything together.

Dominates

Definition

Let (H, φ) and (S, φ) be two Whittaker pairs. We say that (H, φ) dominates (S, φ) if H and S commute and $\mathfrak{g}_\varphi \cap \mathfrak{g}^H_{\geq 1} \subseteq \mathfrak{g}^{S-H}_{\geq 0}$.

(quasi order)

(example on next slide)

Then $\mathcal{F}_{S,\varphi}$ is naturally and linearly determined by $\mathcal{F}_{H,\varphi}$ and $\dim(\mathfrak{u}_{H,\varphi}) \leqslant \dim(\mathfrak{u}_{S,\varphi})$. [GGKPS 18]

Lemma [Gomez–Gourevitch–Sahi 17, GGKPS 18]

Let (S, φ) be a Whittaker pair. Then there exists a neutral pair (h, φ) which dominates (S, φ).

sharper — dominates — coarser

neutral coefficients $\longrightarrow \cdots \longrightarrow$ Whittaker coefficients $\subset$ Levi-distinguished coefficients

Dominates

Example

$$H = \begin{pmatrix} \frac{3}{2} & & & \\ & -\frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{1}{2} \end{pmatrix} \qquad S = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix} \qquad 0 \neq \varphi \in (\mathfrak{g}_{-\alpha_1})^*$$

Then (H, φ) dominates (S, φ) and $\mathcal{F}_{S,\varphi}[\eta](g) = \int\limits_{(\mathbb{K}\backslash\mathbb{A})^3} \mathcal{F}_{H,\varphi}[\eta]\left(\begin{pmatrix} 1 & & & \\ & 1 & x_1 & x_2 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} g\right) d^3x$

In the other direction we would, in general, need a sum of coefficients

$$\mathcal{F}_{H,\varphi}[\eta](g) = \mathcal{F}_{S,\varphi}[\eta](g) + \sum_{\substack{\varphi' \in (\mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3})^* \\ \varphi' \neq 0}} \sum_{\gamma \in \Gamma_{\varphi'}} \mathcal{F}_{S,\varphi+\varphi'}[\eta](\gamma g) \qquad \Gamma_{\varphi'} \subset \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix} \right\} \cap \mathrm{SL}_4(\mathbb{K})$$

[Ahlén–HG–Kleinschmidt–Liu–Persson 17]

However, if η is in a minimal representation then $\mathrm{WF}(\eta)$ consists of the trivial and minimal orbits, and $\mathbf{G}(\mathbb{K})(\varphi + \varphi') \in \mathrm{WF}(\eta)$ only for $\varphi' = 0$.

Then, $\mathcal{F}_{S,\varphi+\varphi'}[\eta] \not\equiv 0$ only for $\varphi' = 0$.

For Eisenstein series we computed the Whittaker coefficient on the RHS using the reduction formula to show that they are Eulerian for a minimal representation.

Deformation of Whittaker pairs

Let (H, φ) dominate (S, φ). For non-negative $t \in \mathbb{Q}$ let $H_t := H + t(S - H)$.

We say that t is regular if $\mathfrak{g}^{H_t}_{\geqslant 1} = \mathfrak{g}^{H_{t+\varepsilon}}_{\geqslant 1}$ for any small enough $\varepsilon \in \mathbb{Q}$ and otherwise we call it critical.

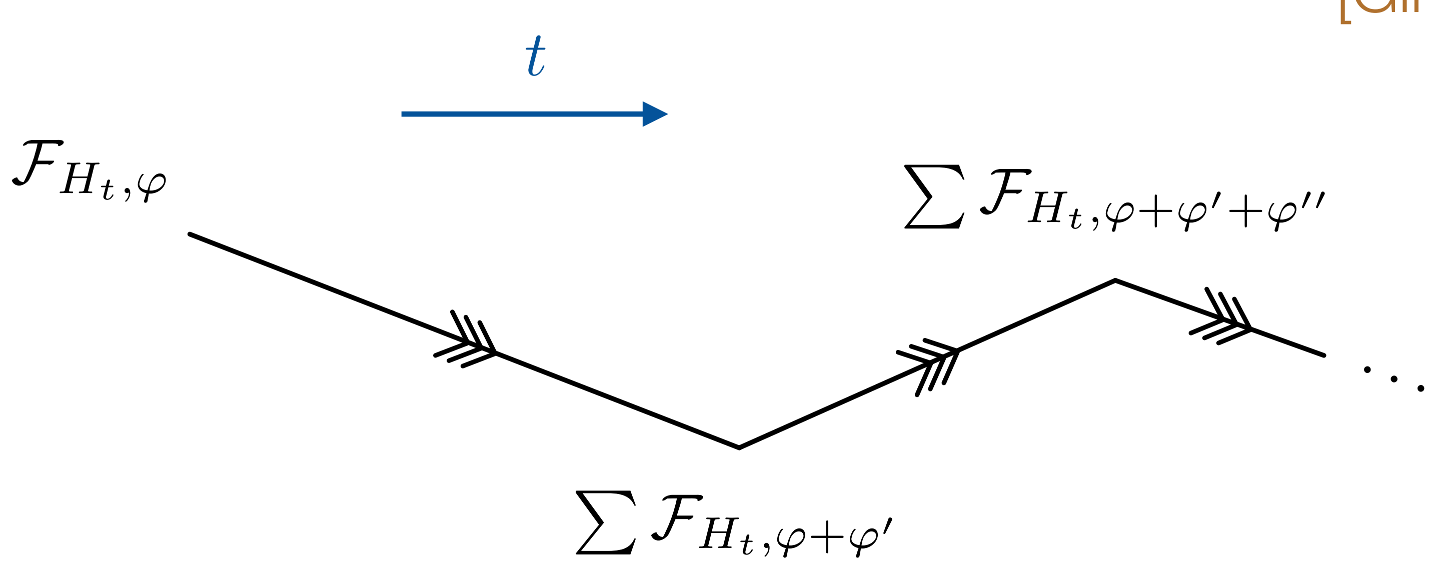
At a critical point t the integration domain $\mathbf{U}_{H_t,\varphi}$ may change. We want to relate $\mathcal{F}_{H_{t-\varepsilon},\varphi}$ and $\mathcal{F}_{H_{t+\varepsilon},\varphi}$.

As in the example it is easier to write the coarser $\mathcal{F}_{H_{t+\varepsilon},\varphi}$ in terms of the sharper $\mathcal{F}_{H_{t-\varepsilon},\varphi}$. In the other direction we may need to sum over characters $\varphi + \varphi'$.

Deformation of Whittaker pairs

Operations relating the two during deformation:

- ~~Compact (period) integration~~
- ✓ Fourier expansion (character sum) — Provided only one term survives, e.g. by constraining Whittaker support
- ✓ Root exchange (non-compact integration) — Similar to lemma of [Ginzburg–Rallis–Soudry 11]



Which preserve Eulerianity?

[GGKPS 18]

Relating maximal isotropic coefficients

Setup: $\mathbf{G}$ reductive algebraic group over $\mathbb{K}$. η an automorphic form on $\mathbf{G}(\mathbb{A})$.

Proposition ($\ast$) [Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Let (H, φ) and (S, φ) be Whittaker pairs such that (H, φ) dominates (S, φ) and $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$.

Let $\mathfrak{i} \subset \mathfrak{g}^{H}_{\geqslant 1}$ and $\mathfrak{i}' \subset \mathfrak{g}^{S}_{\geqslant 1}$ be maximal isotropic subspaces, $\mathfrak{v} := \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{i}')$, $\mathfrak{v}' := \mathfrak{i}'/(\mathfrak{i} \cap \mathfrak{i}')$ and let $\mathbf{I}$, $\mathbf{I}'$, $\mathbf{V}$ and $\mathbf{V}'$ be the corresponding subgroups of $\mathbf{G}$

Then

$$\mathcal{F}^{\mathbf{I}}_{H,\varphi}[\eta](g) = \int\limits_{\mathbf{V}(\mathbb{A})} \mathcal{F}^{\mathbf{I}'}_{S,\varphi}[\eta](vg)dv \qquad \mathcal{F}^{\mathbf{I}'}_{S,\varphi}[\eta](g) = \int\limits_{\mathbf{V}'(\mathbb{A})} \mathcal{F}^{\mathbf{I}}_{H,\varphi}[\eta](vg)dv$$

Note that the integrals preserve Eulerianity

[GGKPS 18 and 20]

Proof of transfer theorem

We start with two Whittaker pairs (S,φ) and (H,ψ) such that $\psi \in \mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$ and two corresponding maximal isotropic subspaces $\mathfrak{i}$ and $\mathfrak{i}'$.

There exists neutral pairs (s,φ) and (h,ψ) which dominate (S,φ) and (H,ψ) respectively.

By the theory of $\mathfrak{sl}_2$-triples there exists $\gamma \in \mathbf{G}(\mathbb{K})$ such that $\big(\mathrm{Ad}(\gamma)h, \mathrm{Ad}^*(\gamma)\psi\big) = (s,\varphi)$.

[Bourbaki, Groupes et algèbres de Lie, ch 7–8, §11]

Let $\mathfrak{r}$ be any maximal isotropic subspace for (s,φ). Then $\mathfrak{r}' = \mathrm{Ad}(\gamma^{-1})\mathfrak{r}$ is a maximal isotropic subspace for (h,ψ).

We can now relate the maximal isotropic Fourier coefficients by the following Eulerianity-preserving transformations

$$\mathcal{F}^{\mathbf{I}'}_{H,\psi}[\eta](g) \overset{(*)}{\longleftrightarrow} \mathcal{F}^{\mathbf{R}'}_{h,\psi}[\eta](g) = \mathcal{F}^{\mathbf{R}}_{s,\varphi}[\eta](\gamma g) \overset{(*)}{\longleftrightarrow} \mathcal{F}^{\mathbf{I}}_{S,\varphi}[\eta](\gamma g)$$

■

Levi-distinguished coefficients

In [GGKPS 18] we show that the minimal coefficients under the (transitive closure of) the domination quasi order are so-called Levi-distinguished coefficients. These include the class of Whittaker coefficients.

For GL_n all Levi-distinguished Fourier coefficients are Whittaker coefficients, and by a generalization of the Piatetski-Shapiro–Shalika formula for non-cusp forms, we have that an automorphic form is completely determined by its Whittaker coefficients.

This is not true in general. There are, for other groups, (non-generic) cusp forms for which all Whittaker coefficients vanish.

We prove that, for any reductive group, an automorphic form is completely determined by its Levi-distinguished coefficients, and these are the most coarse (in the meaning of the above quasi order) to do so.

Furthermore, we show that any Fourier coefficient $\mathcal{F}_{S,\varphi}$ is completely determined by Levi-distinguished coefficient with characters in orbits which are equal to or bigger than $\mathbf{G}(\mathbb{K})\varphi$.

Piatetski-Shapiro–Shalika for E8

Minimal representations

$$\eta_{\min}(g) = \mathcal{F}_{S_{\alpha_8},0}[\eta_{\min}](g) + \sum_{\gamma\in\Gamma_7}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) + \sum_{\omega\in\Omega_8}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}} \mathcal{W}_\varphi[\eta_{\min}](\omega\gamma_8 g)$$

Next-to-minimal representations

$$\begin{aligned}\eta_{\mathrm{ntm}}(g) = {}& \mathcal{F}_{S_{\alpha_8},0}(g) + \sum_{\gamma\in\Gamma_7}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}} \mathcal{W}_\varphi(\gamma g) + \sum_{j=1}^{6}\sum_{\gamma'\in\Gamma_7}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}}\sum_{\gamma\in\Gamma_{j-1}}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_j}} \mathcal{W}_{\varphi+\psi}(\gamma\gamma' g)\\
&+ \frac{1}{2}\sum_{\tilde\gamma\in\Lambda_{\alpha_8}}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}}\sum_{\psi\in z\mathfrak{g}^\times_{-\delta_8}}\int_{V_{g_8}} \mathcal{W}_{\mathrm{Ad}^*(g_8)(\varphi+\psi)}(v g_8\tilde\gamma g)\,dv + \sum_{\omega\in\Omega_8}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}} \mathcal{W}_\varphi(\omega\gamma_8 g)\\
&+ \sum_{\omega\in\Omega_8}\sum_{\tilde\gamma\in\mathcal{M}_{\alpha_8}}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}}\sum_{\psi\in\mathfrak{g}^\times_{-\delta_8}}\int_{V_{g_8}} \mathcal{W}_{\mathrm{Ad}^*(g_8)(\varphi+\psi)}(v g_8\tilde\gamma\omega\gamma_8 g)\,dv\\
&+ \sum_{j=1}^{6}\sum_{\omega\in\Omega_8}\sum_{\varphi\in\mathfrak{g}^\times_{-\alpha_8}}\sum_{\gamma\in\Gamma'_{j-1}}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_j}} \mathcal{W}_{\varphi+\psi}(\gamma\omega\gamma_8 g)\end{aligned}$$

[GGKPS 19]

Hidden invariance

Setup: $\mathbf{G}$ reductive algebraic group over $\mathbb{K}$. η an automorphic form on $\mathbf{G}(\mathbb{A})$.

A Fourier coefficient $\mathcal{F}_{H,\varphi}[\eta](g)$ is invariant under left-translations of its argument g by an element in $\mathbf{U}_{H,\varphi}(\mathbb{A})$ as can be seen by a change of integration variable.

By the conjugation rule of Whittaker pairs we have seen before, we have that if $\gamma \in \mathbf{G}(\mathbb{K})$ centralizes (H, φ) then $\mathcal{F}_{H,\varphi}[\eta](g)$ is left-invariant under γ.

These are both natural symmetries, but there is another, hidden symmetry when $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$.

Theorem (Hidden invariance) [Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Let (H, φ) be a Whittaker pair with $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$.
Then any unipotent element of the centralizer of the pair (H, φ) in $\mathbf{G}(\mathbb{A})$ acts trivially on the Fourier coefficient $\mathcal{F}_{H,\varphi}[\eta]$ using the left regular action.

The proof follows by a combination of carefully chosen Whittaker pair deformations and the above natural symmetries.

Summary

- Transfer theorem for Eulerianity of Fourier coefficients of automorphic forms
- Applications for:
    - unitary minimal representations of D_n and E_7
    - next-to-minimal* representations of B_n and D_n with $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\pi)$ corresponding to $\mathcal{O}_{(31\ldots1)}$
    - Eisenstein series for simply laced groups
- Proof by deformation and conjugation of Whittaker pairs

Thank you!

Slides will be made available at

https://hgustafsson.se