CHALMERS
UNIVERSITY OF TECHNOLOGY

Automorphic string amplitudes

Henrik Gustafsson

String Theory Seminar
Oxford, Jan 2017

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
arXiv:1412.5625 [math.NT] [GKP14]
Journal of Number Theory 166 (Sep, 2016) 344–399

Eisenstein series and automorphic representations
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Cambridge University Press (2017)

Upcoming work with
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$SL(n)$ E_6, E_7, E_8

Outline

Compute Fourier coefficients of automorphic forms to capture information about non-perturbative effects such as instantons and black holes

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- Scattering amplitudes

  4-graviton | Derivative expansion | U-duality | SUSY constraints

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  Eisenstein series | Extracting physical information

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  Eisenstein series | Extracting physical information
- **Fourier coefficients**
  Parabolic subgroups | Limits of string theory | Adelic framework

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- Outlook

Motivation

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- Hecke eigenvalues
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- Langlands program
  L-functions | The Langlands–Shahidi method

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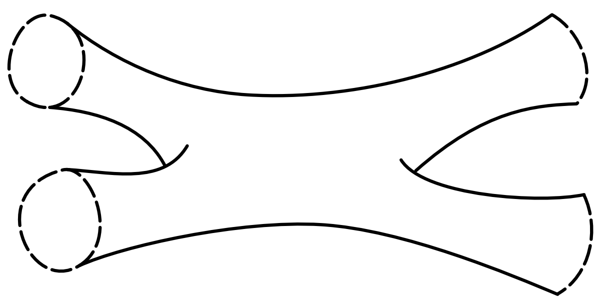
String theory

Toroidal compactifications of type IIB string theory

4-graviton scattering amplitudes



String theory

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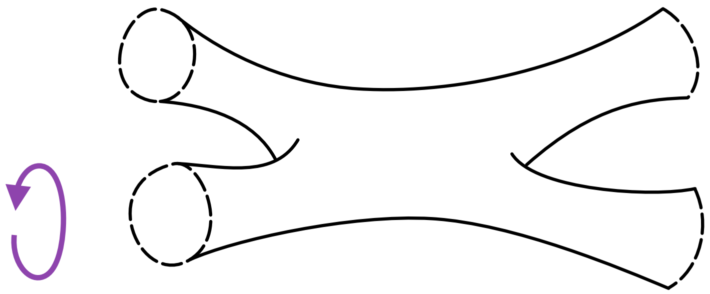
$\alpha' = \ell_{\mathrm{s}}^2$



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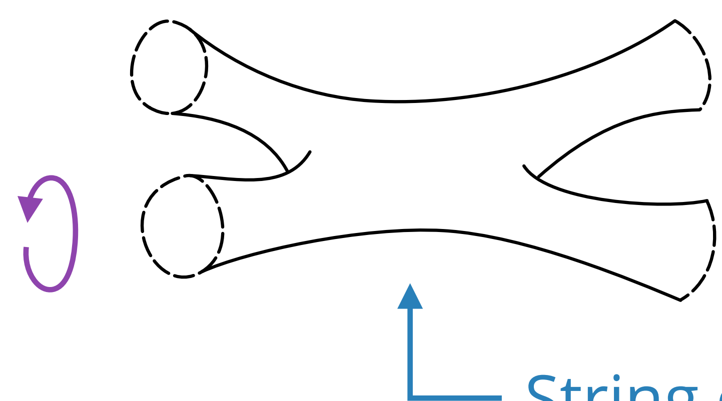

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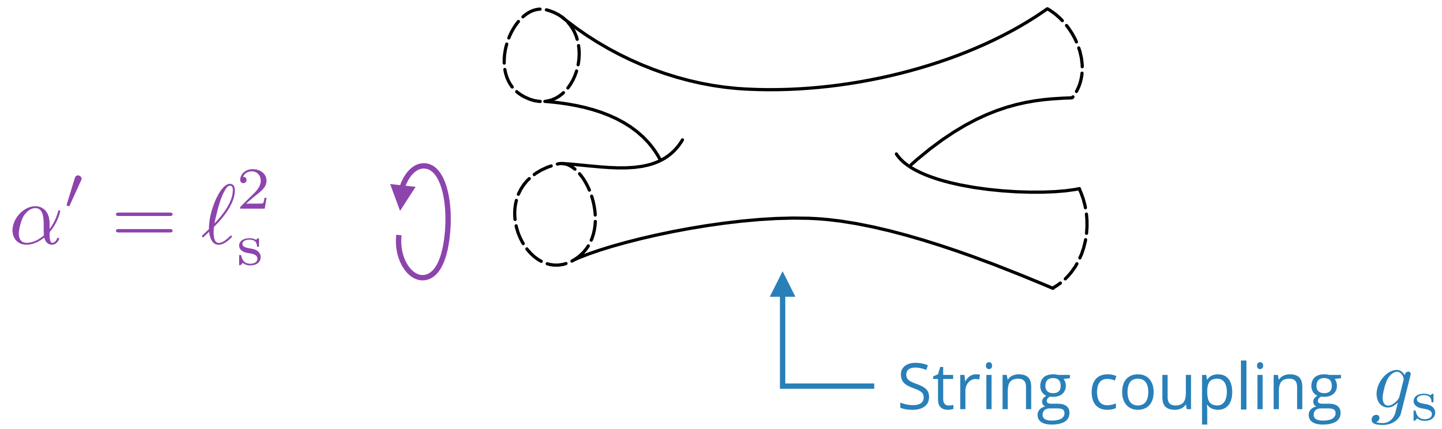
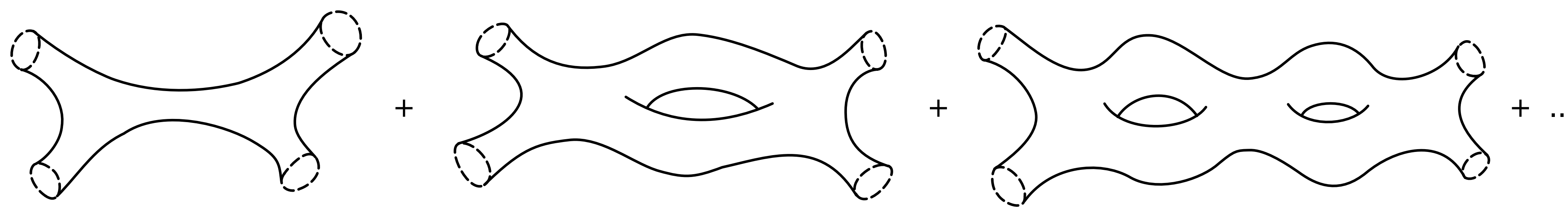


$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2 \qquad t = -\frac{\alpha'}{4}(k_1 + k_3)^2 \qquad u = -\frac{\alpha'}{4}(k_1 + k_4)^2$$

Interactions

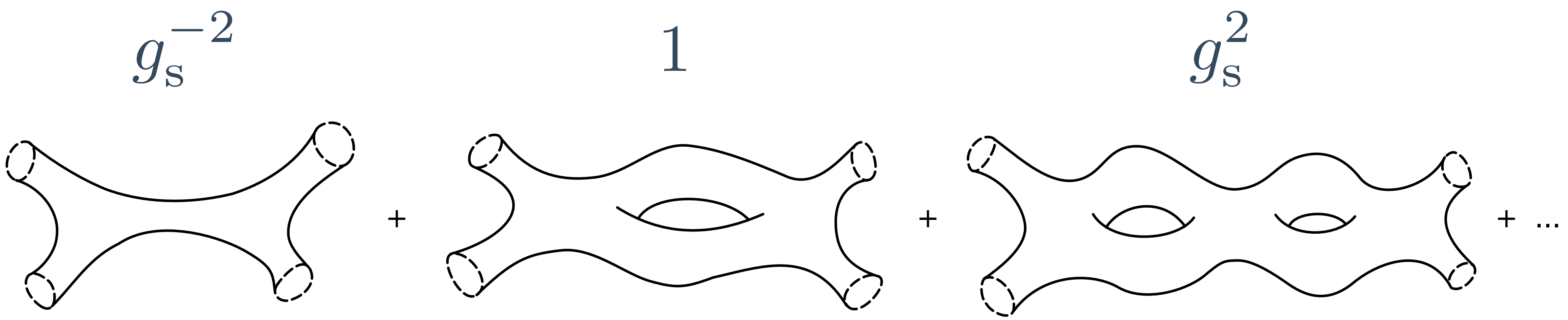
g_s^{-2} $\qquad$ 1 $\qquad$ g_s^{2}



4-graviton amplitude in 10 dimensions:

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions



4-graviton amplitude in 10 dimensions:

$$\mathcal{A} = \left(g_s^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \right) \mathcal{R}^4$$

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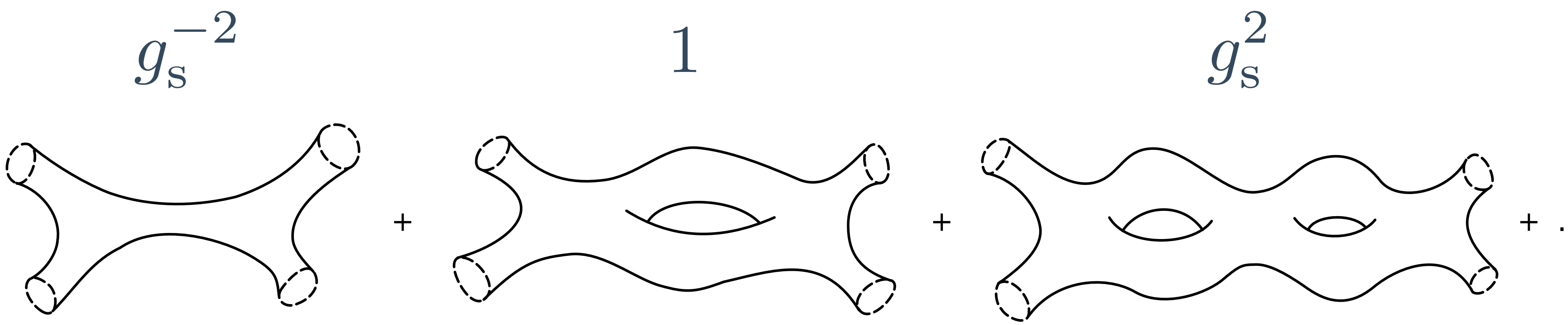

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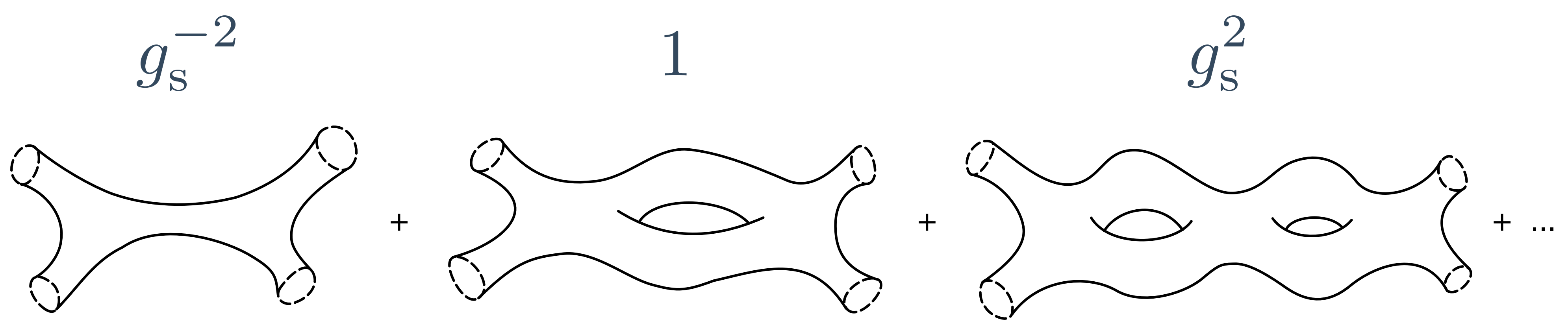
Contraction of 4 linearized Riemann tensors and standard rank 8 tensors $t_8 t_8 \mathcal{R}^4$

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions



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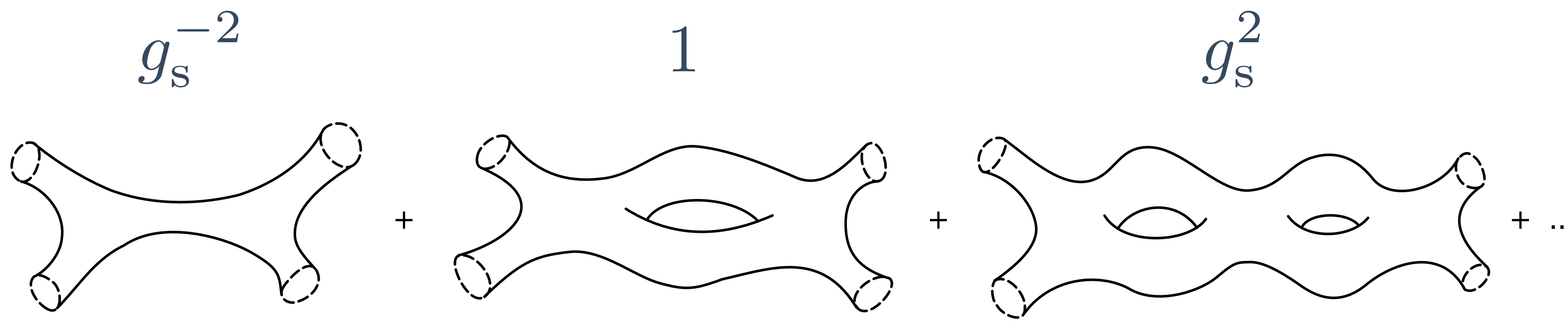
$$\mathcal{A} = \left(g_s^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + 2\pi \int_{\mathscr{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \mathcal{B}_1(s,t,u;\tau) \right) \mathcal{R}^4$$

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions



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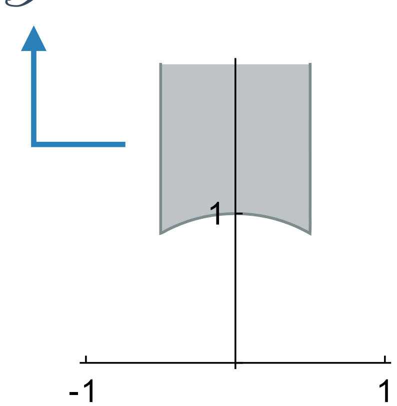

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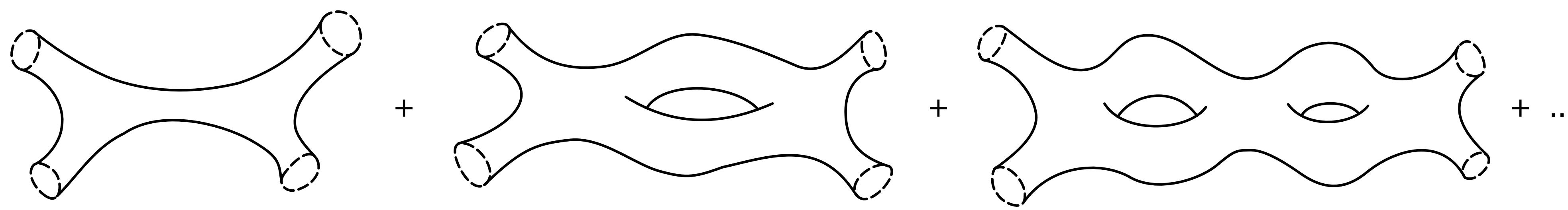
Interactions

$$g_{\mathrm{s}}^{-2} \qquad\qquad 1 \qquad\qquad g_{\mathrm{s}}^{2}$$



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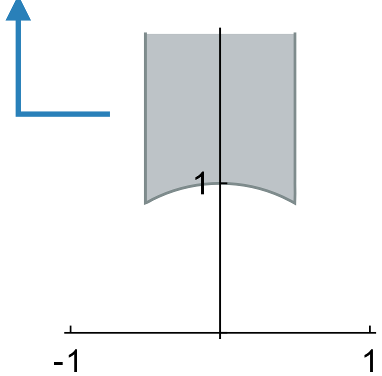
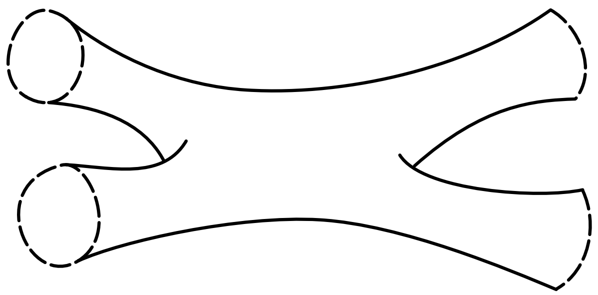


Modular invariant function

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

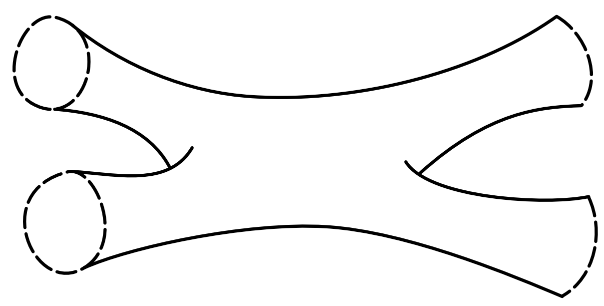
Interactions

String theory



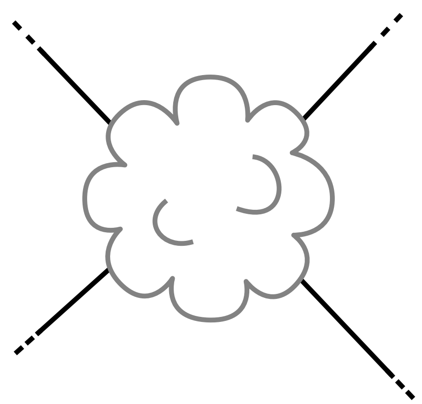
Interactions

String theory



Effective field theory

Supergravity + $\mathcal{O}(\alpha')$



Interactions

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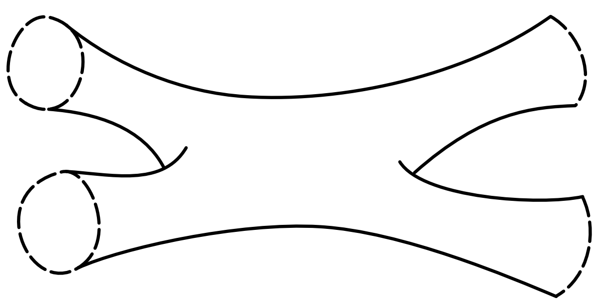

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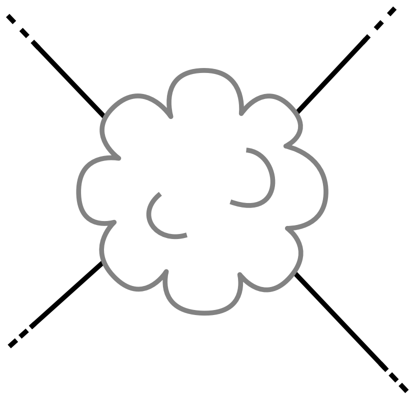
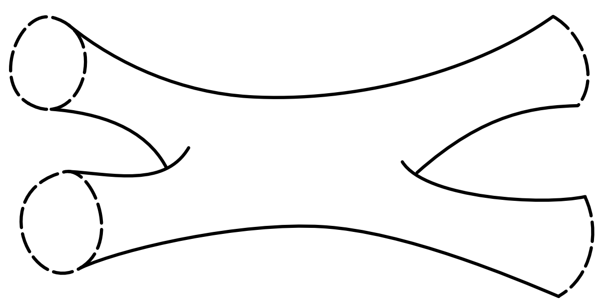


$s, t, u = \mathcal{O}(\alpha')$ $\quad p \longleftrightarrow \partial \implies \alpha'$-expansion = ∂-expansion

Interactions

String theory



Effective field theory $\longrightarrow$

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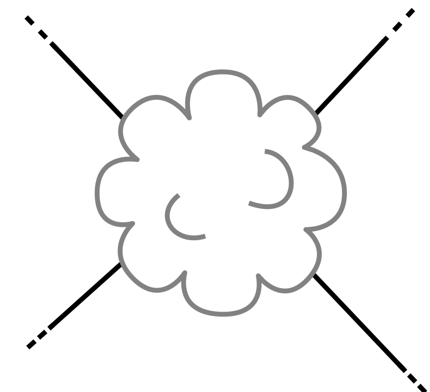

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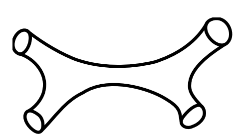
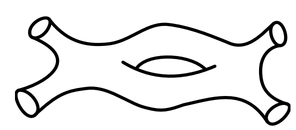


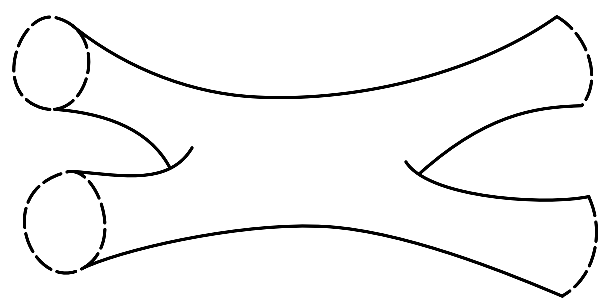


(Einstein frame)

$$\mathcal{L} \propto R + (\alpha')^3 \left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} \qquad + \ \dots \right) \quad R^4 + \dots$$

Interactions

String theory



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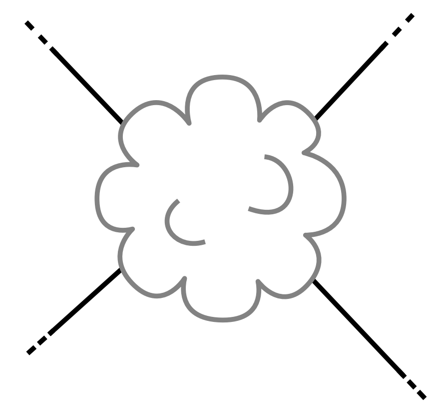
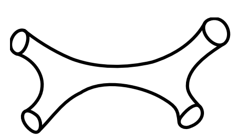
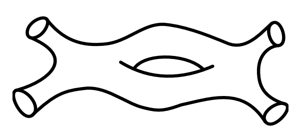


$$s, t, u = \mathcal{O}(\alpha') \qquad p \longleftrightarrow \partial \implies \alpha'\text{-expansion} = \partial\text{-expansion}$$





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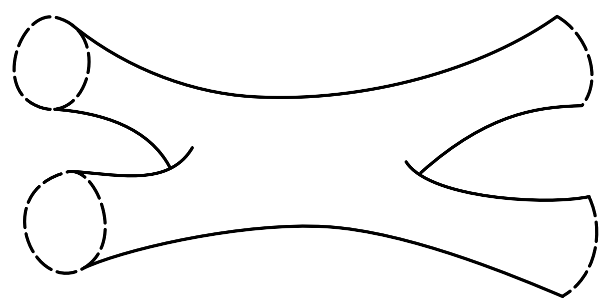
$$\mathcal{L} \propto R + (\alpha')^3 \left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots \right) R^4 + \ldots$$

Contraction of 4 Riemann tensors

Interactions

String theory



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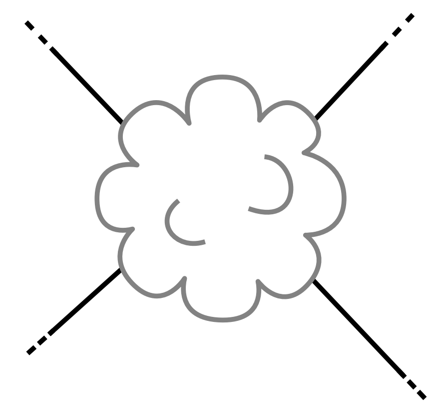
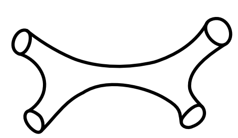
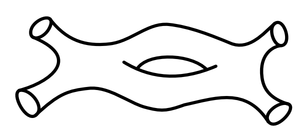


$s, t, u = \mathcal{O}(\alpha')$ $\quad p \longleftrightarrow \partial \implies$ α'-expansion = ∂-expansion





(Einstein frame)

$$\begin{aligned}\mathcal{L} \propto R + {} & (\alpha')^3\Big(2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + \dots\Big) R^4 + \\ & (\alpha')^5\Big(\zeta(5)g_s^{-5/2} + \dots\Big) D^4R^4 + \\ & (\alpha')^6\Big(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3)g_s^{-1} + \dots\Big) D^6R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

Interactions

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Interactions

$$\mathcal{L} \propto R + (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \dots \Big) R^4 +$$
$$(\alpha')^5 \Big(\zeta(5) g_s^{-5/2} + \dots \Big) D^4 R^4 +$$
$$(\alpha')^6 \Big(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3) g_s^{-1} + \dots \Big) D^6 R^4 + \mathcal{O}((\alpha')^7)$$

$$\mathcal{L} \propto R + (\alpha')^3 \mathcal{E}_0(\tau) R^4 + (\alpha')^5 \mathcal{E}_4(\tau) D^4 R^4 + (\alpha')^6 \mathcal{E}_6(\tau) D^6 R^4 + \dots$$

$$\tau = \chi + i g_s^{-1}$$

Interactions

$$\begin{aligned}\mathcal{L} \propto R + \ & (\alpha')^3 \left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots \right) R^4 + \\ & (\alpha')^5 \left(\zeta(5) g_s^{-5/2} + \ldots \right) D^4 R^4 + \\ & (\alpha')^6 \left(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3) g_s^{-1} + \ldots \right) D^6 R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

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$$\mathcal{L} \propto R + (\alpha')^3 \underbrace{\left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \dots\right)}_{} R^4 +$$

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$$(\alpha')^6 \left(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} + \tfrac{4}{3}\zeta(2)\zeta(3) g_s^{-1} + \dots\right) D^6 R^4 + \mathcal{O}((\alpha')^7)$$

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Interactions

$$\mathcal{L} \propto R + (\alpha')^3 \Big(\underbrace{2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots}\Big) R^4 +$$

$$(\alpha')^5 \Big(\underbrace{\zeta(5) g_s^{-5/2} + \ldots}\Big) D^4 R^4 +$$

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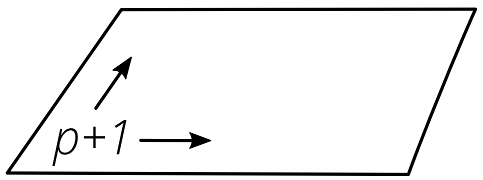
Ahlén, Bao, Basu, Bossard, Cederwall, Fleig, Green, Gubay, Gutperle, HG, Kazhdan, Kiritsis, Kleinschmidt, Lambert, Miller, Nilsson, Obers, Persson, Pioline, Russo, Sethi, Vanhove, Verschinin, Waldron, West, ...

Non-perturbative effects

[Green, Polchinski]

Non-perturbative effects

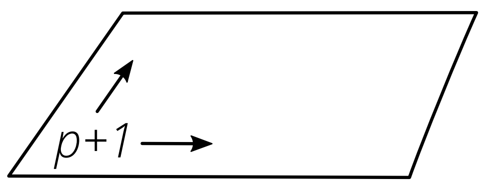


D*p*-brane

p space directions
1 time direction

[Green, Polchinski]

Non-perturbative effects



D*p*-brane

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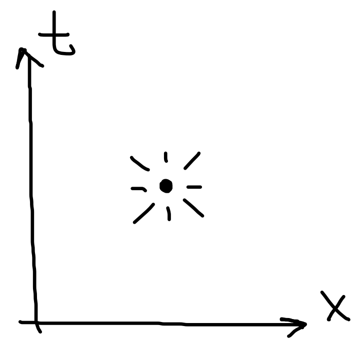
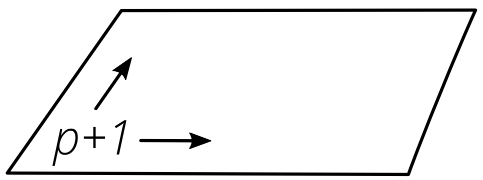


D-instanton

$p = -1$

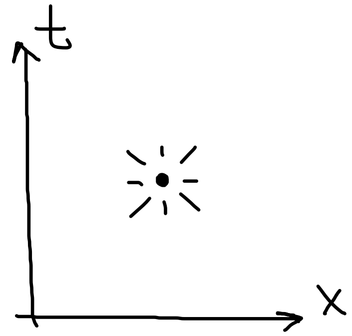
[Green, Polchinski]

Non-perturbative effects



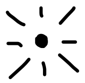
Dp-brane

p space directions
1 time direction



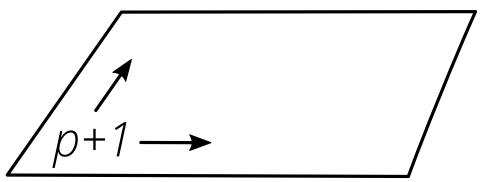
D-instanton

$p = -1$



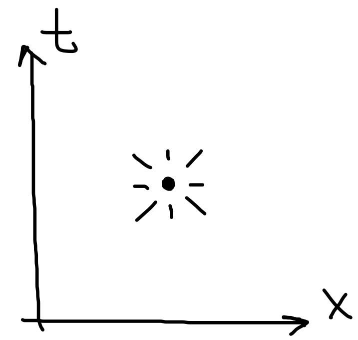
[Green, Polchinski]

Non-perturbative effects



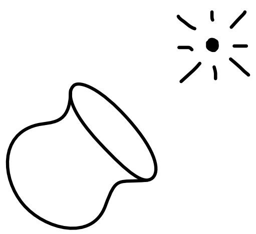
Dp-brane

p space directions
1 time direction



D-instanton

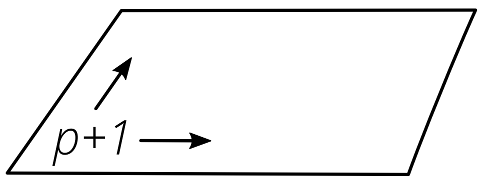
$p = -1$



$$\frac{1}{g_s}$$

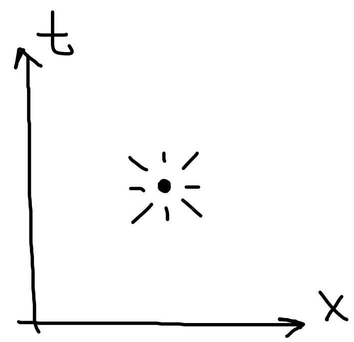
[Green, Polchinski]

Non-perturbative effects



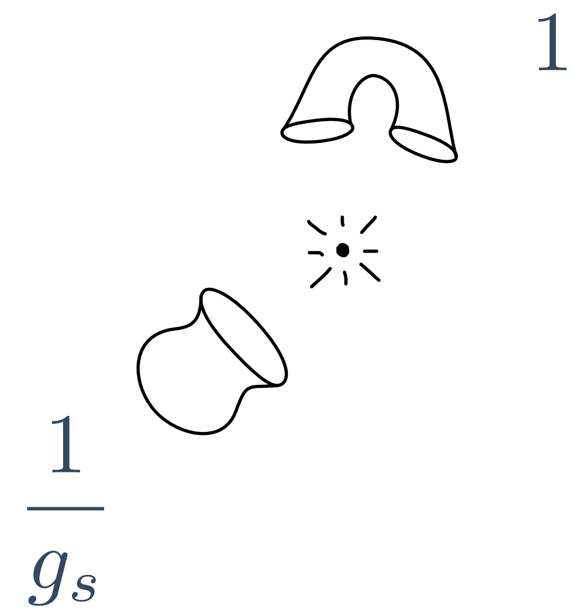
Dp-brane

p space directions
1 time direction



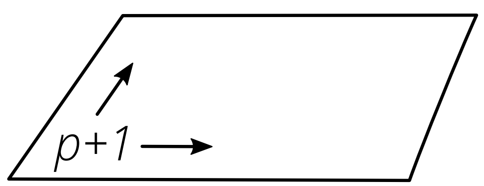
D-instanton

p = -1



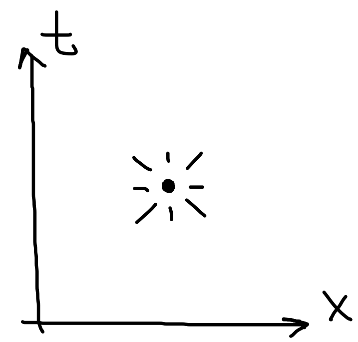
[Green, Polchinski]

Non-perturbative effects



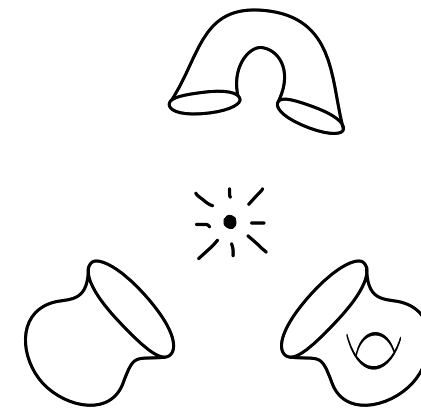
Dp-brane

p space directions
1 time direction



D-instanton

p = -1



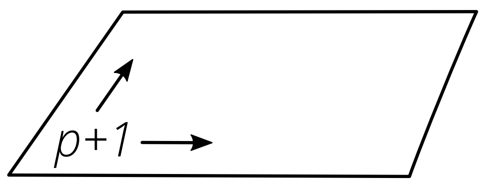
1

$\frac{1}{g_s}$

g_s

[Green, Polchinski]

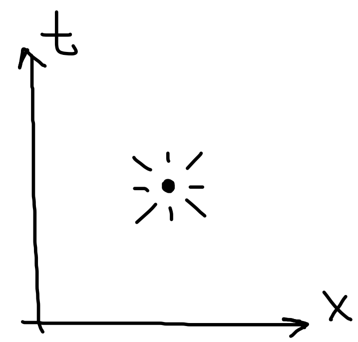
Non-perturbative effects



Dp-brane

p space directions

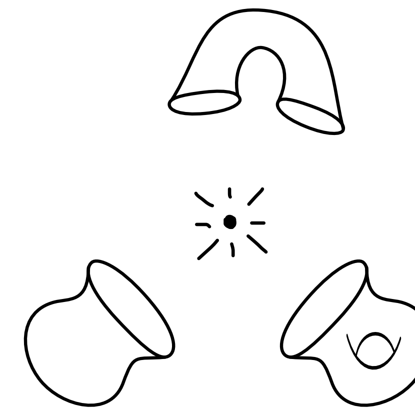
1 time direction



D-instanton

p = -1

1



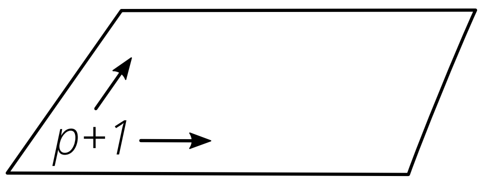
$\frac{1}{g_s}$

g_s

1

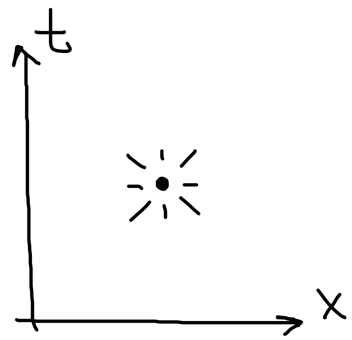
[Green, Polchinski]

Non-perturbative effects



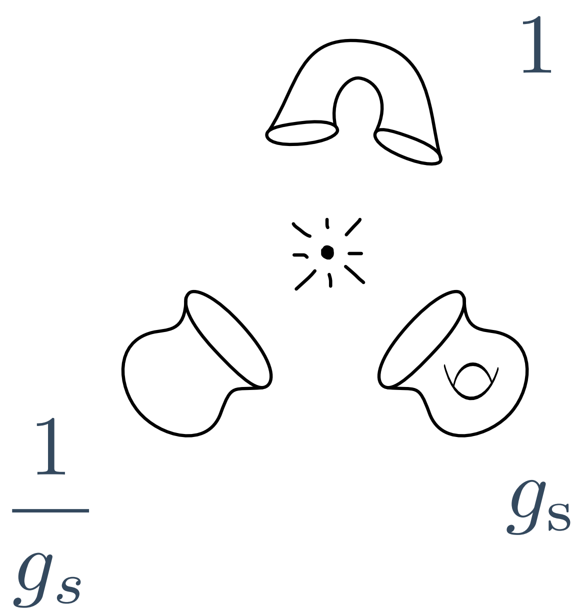
Dp-brane

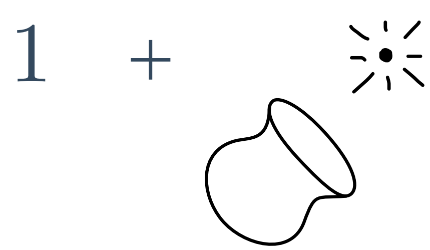
p space directions
1 time direction



D-instanton

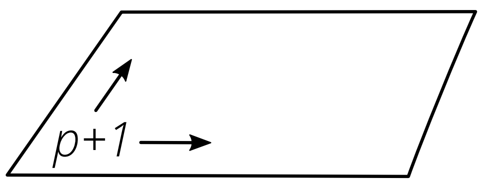
p = -1





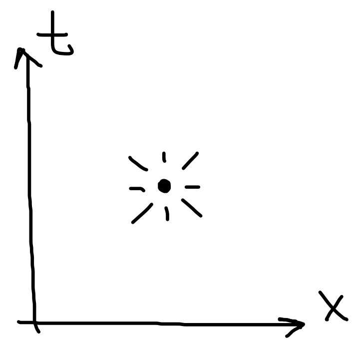
[Green, Polchinski]

Non-perturbative effects



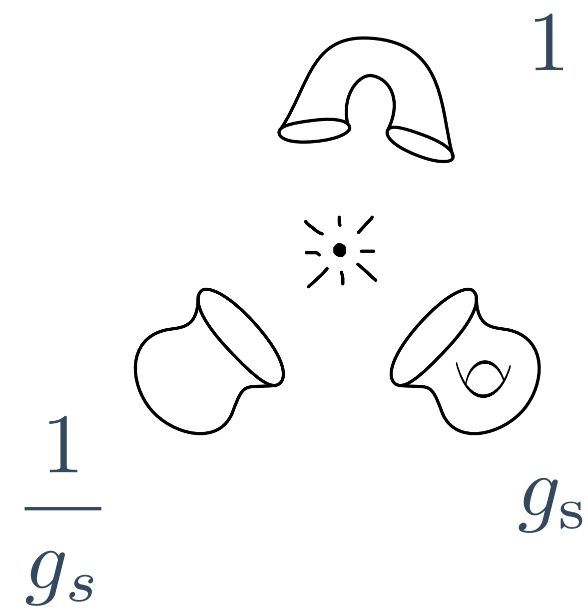
Dp-brane

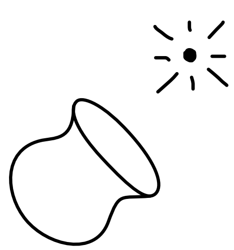
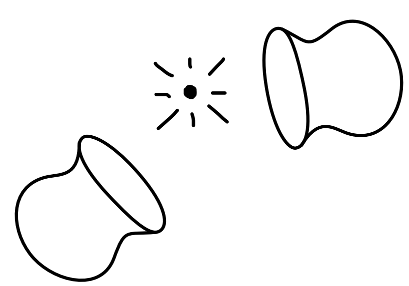
p space directions
1 time direction



D-instanton

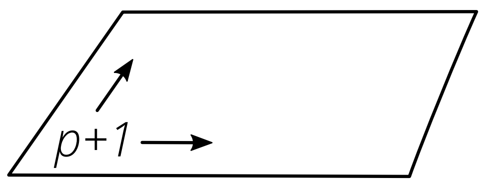
$p = -1$



$1 \quad + $  $\quad + \quad \frac{1}{2!}$ 

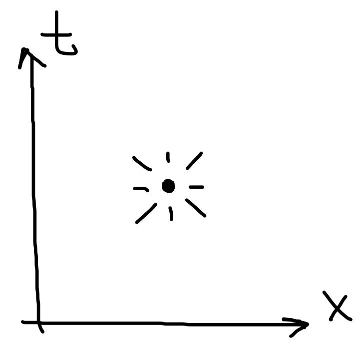
[Green, Polchinski]

Non-perturbative effects



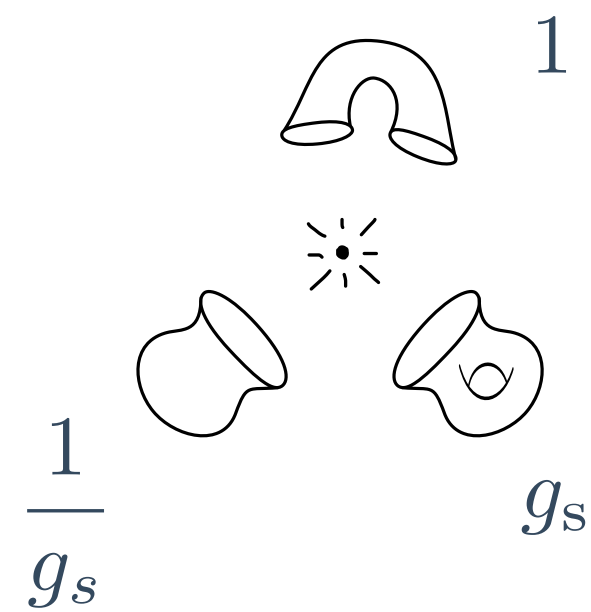
Dp-brane

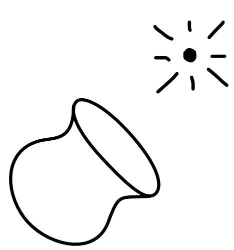
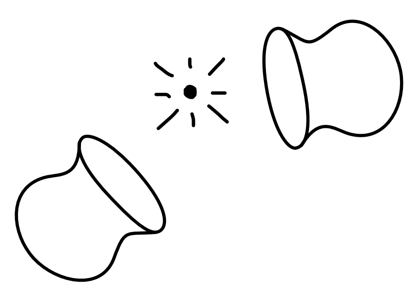
p space directions
1 time direction



D-instanton

p = -1

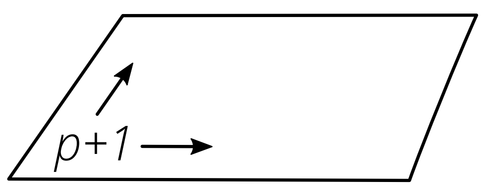


$1 \quad +$  $+ \quad \frac{1}{2!}$ 

Symmetry factor for identical disks

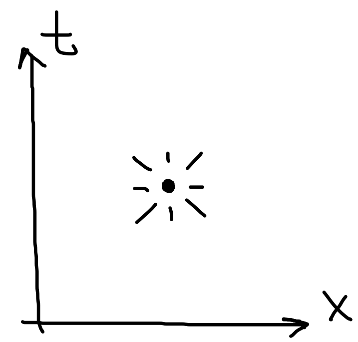
[Green, Polchinski]

Non-perturbative effects



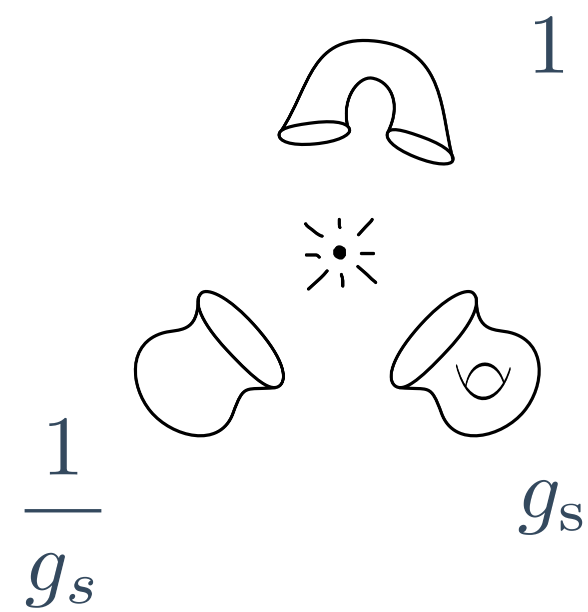
D*p*-brane

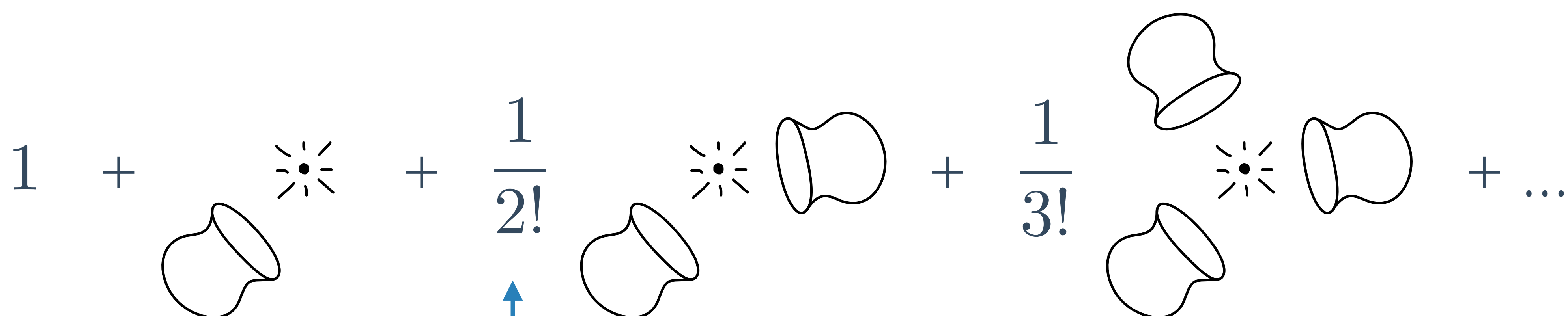
p space directions
1 time direction



D-instanton

p = -1

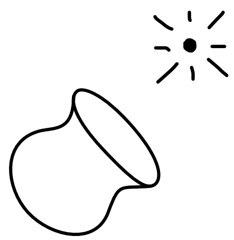
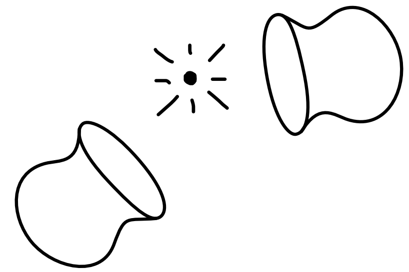
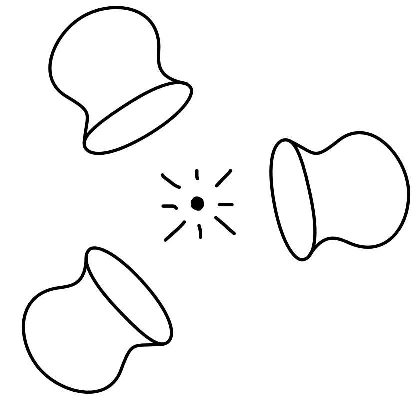




Symmetry factor for identical disks

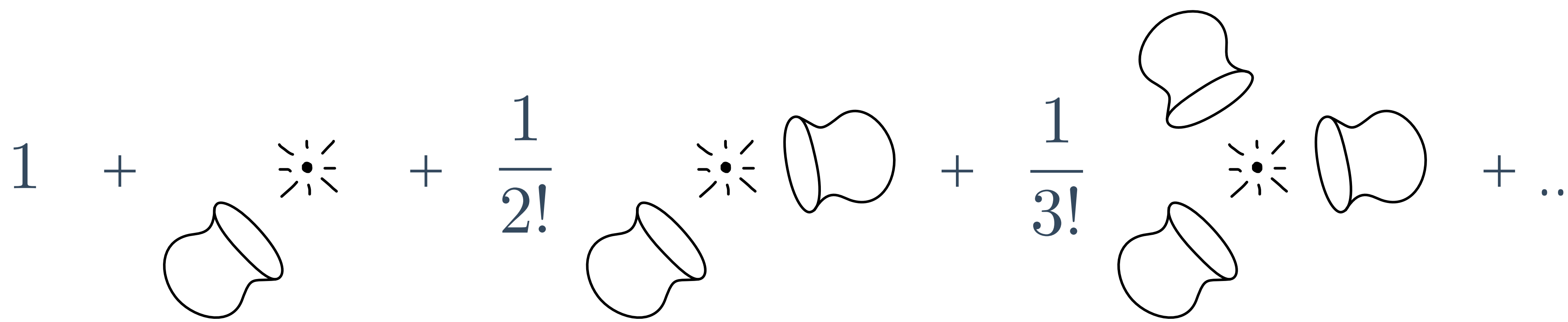
[Green, Polchinski]

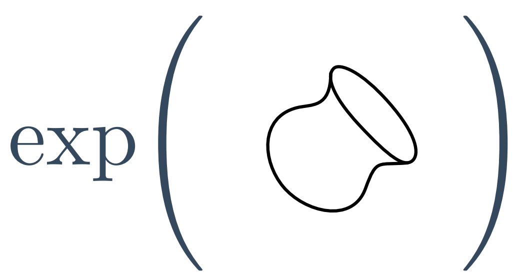
Non-perturbative effects

$1 \quad + $  $\quad + \quad \frac{1}{2!}$  $\quad + \quad \frac{1}{3!}$  $\quad + \ \ldots$

[Green-Gutperle]

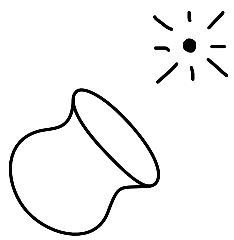
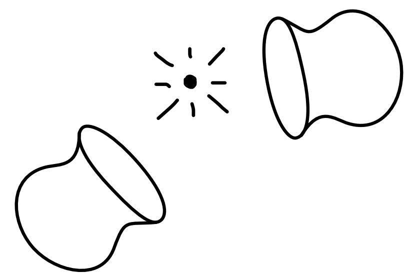
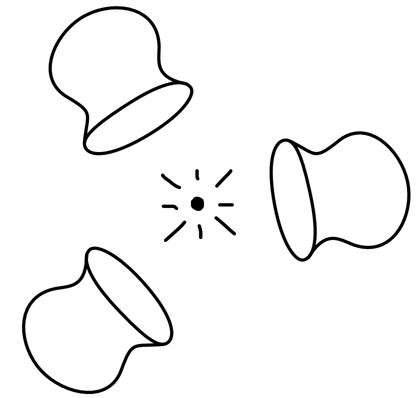
Non-perturbative effects

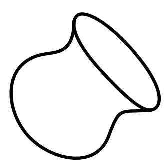




[Green-Gutperle]

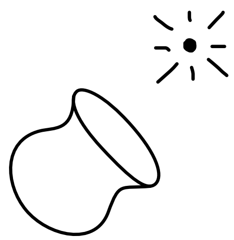
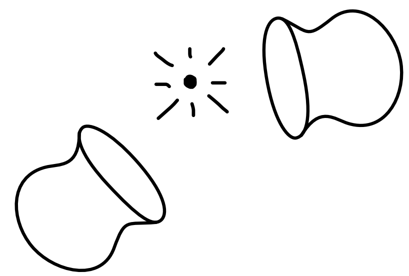
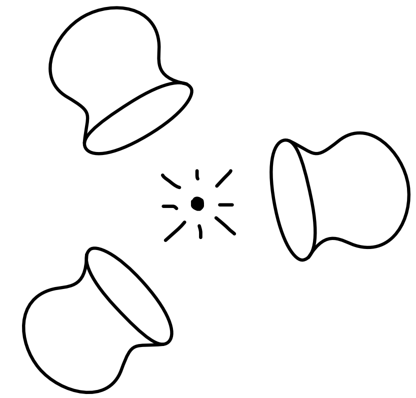
Non-perturbative effects

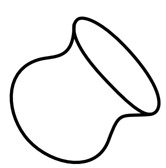
$$1 \quad + $$

$$+ \quad \frac{1}{2!}$$

$$+ \quad \frac{1}{3!}$$

$$+ \dots$$

$$\exp\Bigg($$

$$\Bigg) \sim \exp\left(-\frac{\mathrm{const}}{g_{\mathrm{s}}}\right)$$

[Green-Gutperle]

Non-perturbative effects

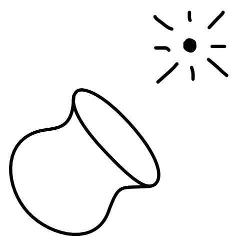
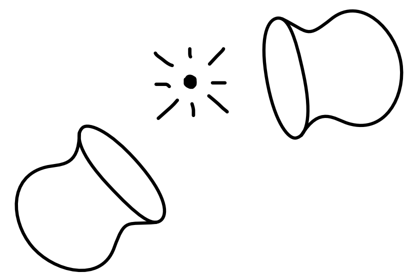
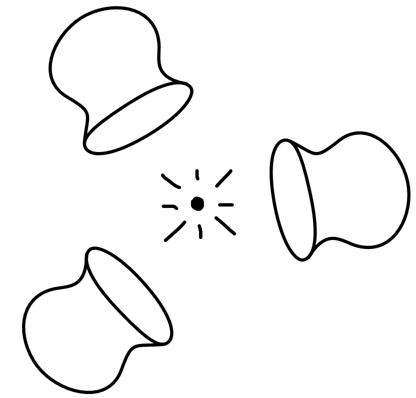
$1 +$  $+ \frac{1}{2!}$  $+ \frac{1}{3!}$  $+ \dots$

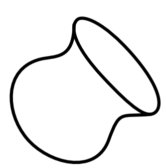
$\exp\Big($  $\Big) \sim \exp\left(-\frac{\text{const}}{g_{\text{s}}}\right)$

Non-perturbative in g_{s}

[Green-Gutperle]

Non-perturbative effects

1 +  + $\frac{1}{2!}$  + $\frac{1}{3!}$  + ...

$\exp\Big($  $\Big) \sim \exp\left(-\frac{\text{const}}{g_{\mathrm{s}}}\right)$

Non-perturbative in g_{s}

$$\mathcal{E}_0(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots + Ce^{2\pi i\tau} + \ldots$$

$$\tau = \tau_1 + i\tau_2 = \chi + ig_{\mathrm{s}}^{-1}$$

[Green-Gutperle]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \ldots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

[Cremmer-Julia]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \dots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

D	$G(\mathbb{R})$	K
10	$SL(2,\mathbb{R})$	$SO(2)$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SO(5)$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$

[Cremmer-Julia]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \dots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

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7	$SL(5,\mathbb{R})$	$SO(5)$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$
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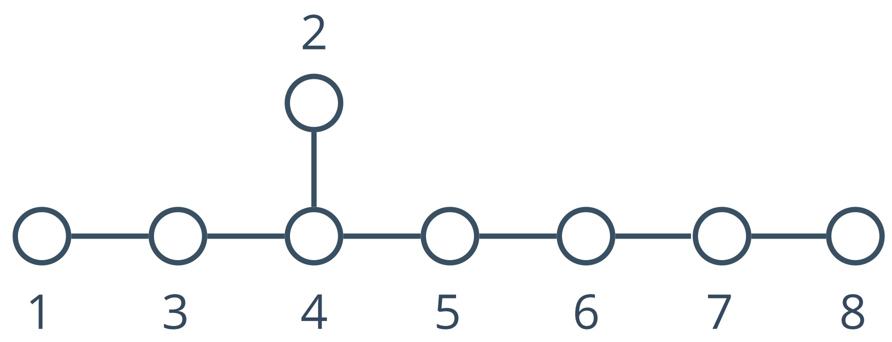

[Cremmer-Julia]

Moduli space

10 dimensions:

Moduli space

10 dimensions:

$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

Moduli space

10 dimensions:

$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

No similar structure for lower dimensions

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

U-duality

$G(\mathbb{R}) \circlearrowleft \mathcal{M}_{\text{classical}}$ classical symmetry

Quantization of charges

[Hull-Townsend]

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

[Hull-Townsend]

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

$G(\mathbb{R})$ Chevalley group $G(\mathbb{Z})$

Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

[Hull-Townsend]

U-duality

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Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
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[Hull-Townsend]

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Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

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10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

U-duality

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{R}$$

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Supersymmetry constraints

bosons	SUSY	fermions
dilaton		dilatino
graviton		gravitino
...		...

[Green-Sethi]

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Linearized SUSY: $$f_{w+1}(\tau) = i\left(\tau_2\frac{\partial}{\partial\tau} - i\frac{w}{2}\right)f_w(\tau)$$

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$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)}$$

$$\delta\Psi = \delta^{(0)}\Psi$$

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$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^5 S^{(5)} + \dots$$

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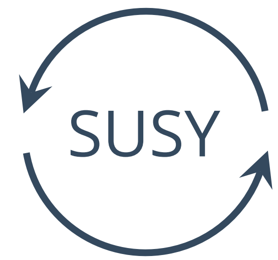
Laplacian on Poincaré UHP

[Green-Sethi]

Supersymmetry constraints



10 dimensions: $\Delta = 4\tau_2^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}$ Laplacian on Poincaré UHP

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Supersymmetry constraints

SUSY

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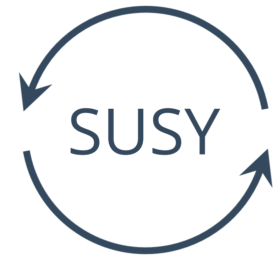

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(C) ✓

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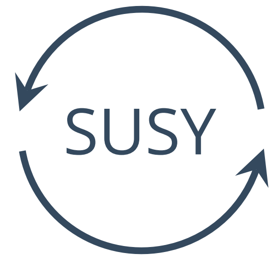

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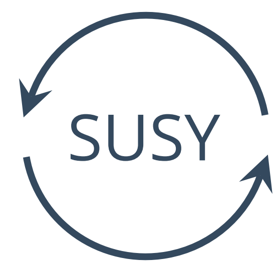

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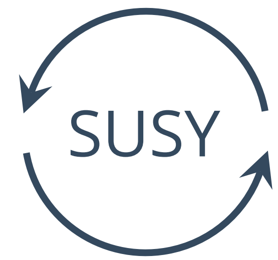

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Not an automorphic form in a strict sense

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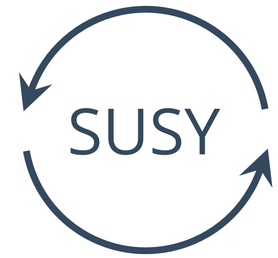

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Similarly for lower dimensions

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$E(s;\tau) =$ $\qquad s \in \mathbb{C}$

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$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL(2,\mathbb{Z}) \right\}$$

Borel subgroup

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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$$\tau \mapsto \mathrm{Im}(\tau)^s$$

Multiplicative character trivially extended to $G(\mathbb{R})$

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Fourier expansion $\quad \tau = \tau_1 + i\tau_2$

$$E(s;\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\tau_2^{1/2}}{\xi(2s)} \sum_{m \neq 0} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi |m| \tau_2) e^{2\pi i m \tau_1}$$

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Completed Riemann zeta function

$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$

Eisenstein series

$$E(s;\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi(\gamma(\tau))$$

$$\big(\Delta - s(s-1)\big)E(s;\tau) = 0$$

$$E(s;\gamma(\tau)) = E(s;\tau) \qquad E(s;\tau+1) = E(s;\tau) \qquad \gamma = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$$

Fourier expansion $\tau = \tau_1 + i\tau_2$

$$E(s;\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\tau_2^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2}\,\sigma_{1-2s}(m) K_{s-1/2}(2\pi\,|m|\,\tau_2) e^{2\pi i m\tau_1}$$

Completed Riemann zeta function

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Bessel function of the second kind

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$$\big(\Delta - s(s-1)\big)E(s;\tau) = 0 \qquad E(s;\tau) \sim \tau_2^s \qquad g_{\mathrm{s}} = \tau_2^{-1} \to 0$$

[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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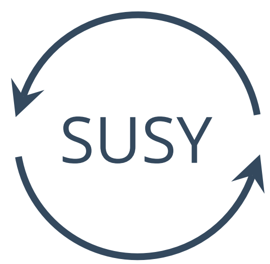

$$(\Delta - \tfrac{3}{4})\mathcal{E}_0(\tau) = 0$$

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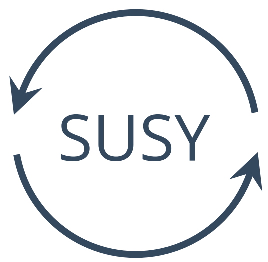

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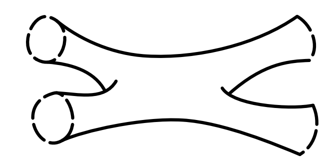

(Einstein frame)

$$\mathcal{E}_0(\tau) \sim 2\zeta(3)\tau_2^{3/2}$$

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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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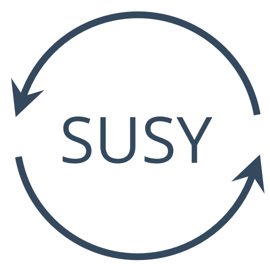

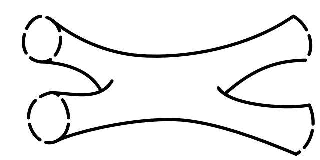

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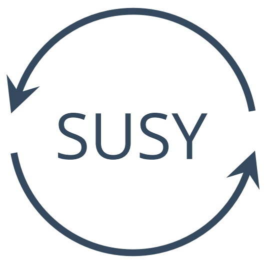
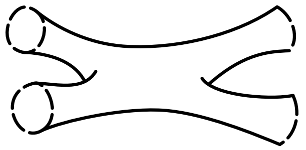
[Green-Gutperle, Pioline, Green-Russo-Vanhove]

Eisenstein series





$$\mathcal{E}_0(\tau) = 2\zeta(3)E(3/2;\tau)$$

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$\mathcal{E}_6(\tau)$ as a sum over images $\sum_{B(\mathbb{Z})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_{s} $\qquad\qquad$ $\tau = \chi + i g_{\mathrm{s}}^{-1}$

[Green-Gutperle]

Extracting physical information

Expand Bessel function in $g_{\rm s}$ $\tau = \chi + i g_{\rm s}^{-1}$

$$\mathcal{E}_0(\tau) = 2\zeta(3) g_{\rm s}^{-3/2} + 4\zeta(2) g_{\rm s}^{1/2} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\,\sigma_{-2}(m)\, e^{-2\pi|m|g_{\rm s}^{-1} + 2\pi i m\chi}\Big[1 + \mathcal{O}(g_{\rm s})\Big]$$

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$$\mathcal{E}_0(\tau) = \underbrace{2\zeta(3) g_{\rm s}^{-3/2} + 4\zeta(2) g_{\rm s}^{1/2}}_{\text{Perturbative (zero-mode)}} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\,\sigma_{-2}(m) e^{-2\pi |m| g_{\rm s}^{-1} + 2\pi i m \chi}\left[1 + \mathcal{O}(g_{\rm s})\right]$$

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Perturbative
(zero-mode)

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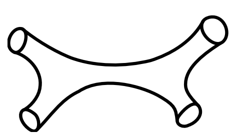

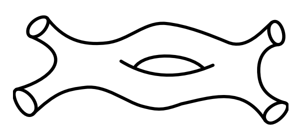

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Perturbative
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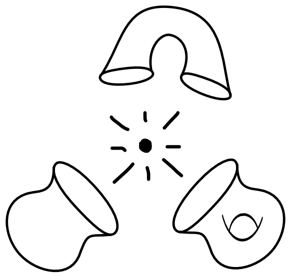
[Green-Gutperle]

Extracting physical information

Expand Bessel function in $g_{\rm s}$ $\tau = \chi + i g_{\rm s}^{-1}$



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Extracting physical information

Expand Bessel function in g_{s}

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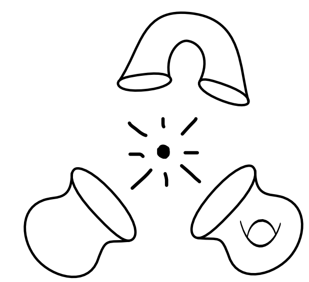

Instanton action

$$\mathcal{E}_0(\tau) = 2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\,\sigma_{-2}(m)\, e^{-2\pi |m| g_{\mathrm{s}}^{-1} + 2\pi i m \chi}\left[1 + \mathcal{O}(g_{\mathrm{s}})\right]$$

Perturbative
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Non-perturbative
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[Green-Gutperle]

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Perturbative
(zero-mode)

Non-perturbative
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$$\sigma_s(m) = \sum_{d\,|\,m} d^s$$

Sums over the number of ways the charge *m* can be factorised into two integers

[Green-Gutperle]

Extracting physical information

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Sums over the number of ways the charge *m* can be factorised into two integers

wrapping number of a T-dual D-particle and charge

[Green-Gutperle]

Lower dimensions

Lower dimensions

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z})\times\mathbb{Z}_2$
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$	$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5)\times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

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8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$	$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5)\times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
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3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

$$E(\chi;g)=\sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})}\chi(\gamma g)$$

Parabolic subgroups

Fourier expand in different directions $\longleftrightarrow$ Unipotent subgroup U

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$\uparrow$

Choice of parabolic subgroup P

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Choice of parabolic subgroup P

Σ choice of simple roots $\langle\Sigma\rangle$ generated root system

Parabolic subgroups

Fourier expand
in different directions $\longleftrightarrow$ Unipotent subgroup U

$\uparrow$

Choice of parabolic subgroup P

Σ choice of simple roots $\langle \Sigma \rangle$ generated root system

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) g \quad \forall h \in \mathfrak{h}\}$$

Cartan subalgebra

Parabolic subgroups

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$$\boxed{\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}}$$

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$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_\alpha$$

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$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) g \quad \forall h \in \mathfrak{h}\}$$

Cartan subalgebra

$$\boxed{\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}}$$

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle\Sigma\rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_\alpha \qquad \Delta(\mathfrak{u}) = \Delta_+ \setminus (\Delta_+ \cap \langle\Sigma\rangle)$$

Positive roots

Parabolic subgroups

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$\uparrow$

Choice of parabolic subgroup P

Σ choice of simple roots $\langle \Sigma \rangle$ generated root system

$G = SL(4)$  $\Sigma = \{\alpha_1\}$

Parabolic subgroups

Fourier expand
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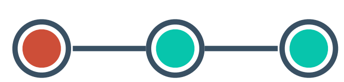
$\uparrow$

Choice of parabolic subgroup P

Σ choice of simple roots $\langle\Sigma\rangle$ generated root system

$G = SL(4)$  $\Sigma = \{\alpha_1\}$

$P = LU$

Parabolic subgroups

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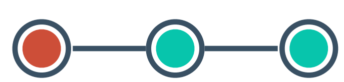
$\uparrow$

Choice of parabolic subgroup P

Σ choice of simple roots

$\boxed{\langle\Sigma\rangle \text{ generated root system}}$

$G = SL(4)$  $\Sigma = \{\alpha_1\}$

$$P = LU \qquad L = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\} \qquad U = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

Parabolic subgroups

Fourier expand
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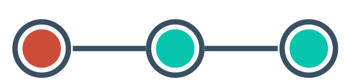

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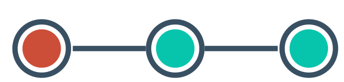

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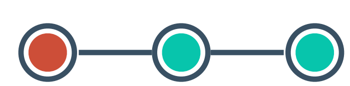
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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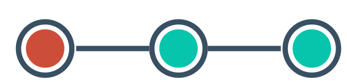
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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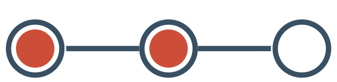

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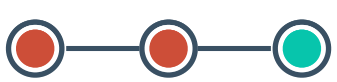

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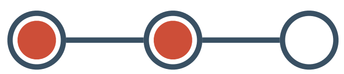
Parabolic subgroups

Parabolic subgroups

Maximal parabolic

Parabolic subgroups

Minimal parabolic
Borel



Maximal parabolic

Parabolic subgroups

Minimal parabolic
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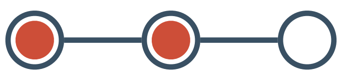
$$B = NA$$

$$N = \left\{ \begin{pmatrix} \boxed{1} & * & * & * \\ & \boxed{1} & * & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$



Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Fourier expansion

Fourier expansion

$\psi(u_1 u_2) = \psi(u_1)\psi(u_2)$

Let $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \underbrace{U(1)}_{\cong S^1}$ be a multiplicative character

Parametrised by $m_\alpha \in \mathbb{Z}$ called charges

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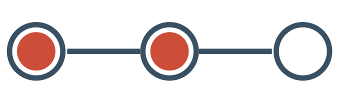

$$\psi_U\left(\begin{pmatrix} 1 & & & y_1 \\ & 1 & & y_2 \\ & & 1 & y_3 \\ & & & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 y_1 + m_2 y_2 + m_3 y_3)}$$

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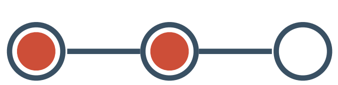

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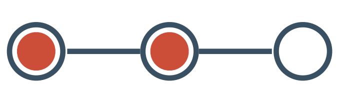

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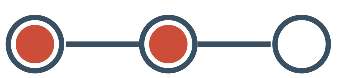
Parametrised by $m_\alpha \in \mathbb{Z}$ called charges



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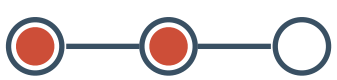

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$$F_U(\chi, \psi; g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi, ug)\overline{\psi(u)}\, du$$

Fourier expansion

Fourier expansion

$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi \neq 1} F_U(\chi, \psi; g)$$

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Fourier expansion

$$E(\chi;g) = F_U(\chi,1;g) + \sum_{\psi^{(1)}\neq 1} F_{U^{(1)}}(\chi,\psi^{(1)};g) + \sum_{\psi^{(2)}\neq 1} F_{U^{(2)}}(\chi,\psi^{(2)};g) + \ldots$$

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$$U^{(1)} = U \qquad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

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Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

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F_U

W_N

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

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Characters and coefficients with all $m_\alpha \neq 0$ are called generic otherwise they are called degenerate

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

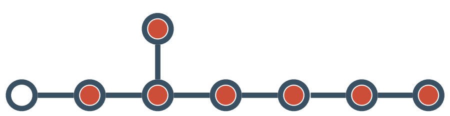
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

- String perturbation limit $g_{\mathrm{s}} \to 0$

  D-instantons | NS5-instantons



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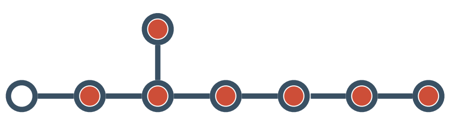
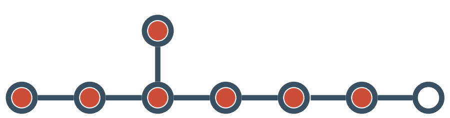


- **Decompactification limit**
  Higher dimensional black holes | BPS states

  Large radius for compactified circle



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  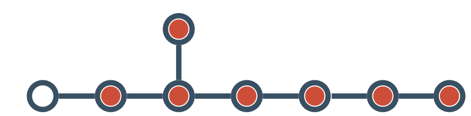
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  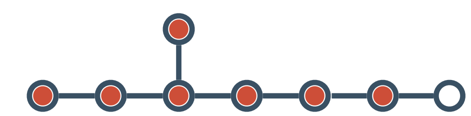
- M-theory limit
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  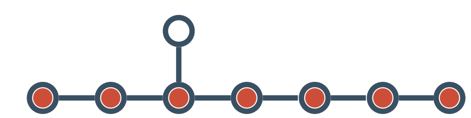

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  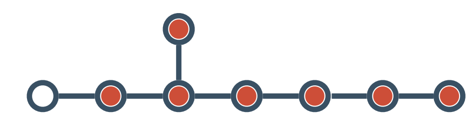
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  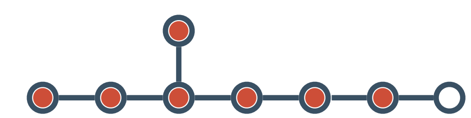
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  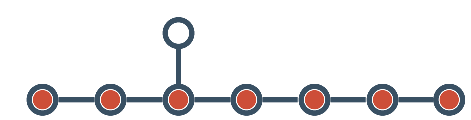

[Green-Miller-Vanhove]

Maximal parabolic subgroups

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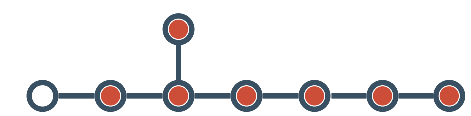

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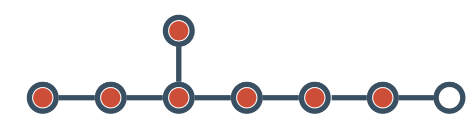

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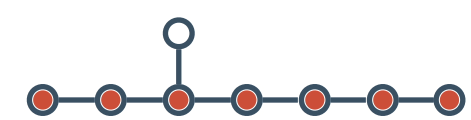

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

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  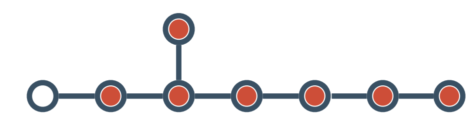

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  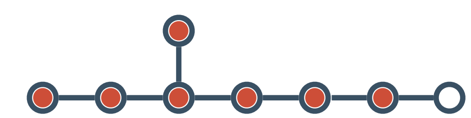

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  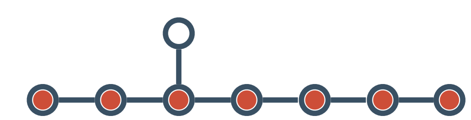

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

Recent result in [Bossard-Pioline]

Fourier expansion

Goal: find expressions for Fourier coefficients in terms of (known) Whittaker coefficients

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Would allow us to compute non-perturbative effects that capture information about instantons and black holes

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

— Robert P. Langlands*

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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Eisenstein series

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Adelic Eisenstein series

$\uparrow$ Lift

Eisenstein series

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$$\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] &
\end{tikzcd}$$

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\end{tikzcd}$$

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The adeles

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Product of Cauchy completions of $\mathbb{Q}$

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p$$

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Euclidean norm

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Automorphic form $: G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$

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Euclidean norm

p-adic numbers

p-adic norm

[FGKP15 §4.2.2]

Automorphic form $: \underline{G(\mathbb{Z})\backslash G(\mathbb{R})/K} \to \mathbb{C}$ $\quad \underline{G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}}$

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Euclidean norm

p-adic numbers
p-adic norm

[FGKP15 §4.2.2]

Automorphic form $: G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$ $\quad G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}$

$\mathbb{Q}$ is discrete in $\mathbb{A}$ taking the role of $\mathbb{Z}$ in $\mathbb{R}$

Much easier to work with since it is a field!

The adeles

Product of Cauchy completions of $\mathbb{Q}$

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p$$

$\mathbb{R} = \mathbb{Q}_\infty$

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Charges / Fourier modes

$m_\alpha \in \mathbb{Z}$

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Computing adelic Fourier coefficients

[FGKP15 §9-10]

W_N Whittaker coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

W_N Whittaker coefficients

Factorisation

Computing adelic Fourier coefficients

[FGKP15 §9-10]

W_N Whittaker coefficients

Factorisation $$\int (\text{complicated}) \longrightarrow \overset{\mathbb{R}}{\int} (\text{easier}) \cdot \prod_{p \text{ prime}} \overset{\mathbb{Q}_p}{\int} (\text{easier})$$

Computing adelic Fourier coefficients

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Computing adelic Fourier coefficients

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Casselman-Shalika formula

Computing adelic Fourier coefficients

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[GKP14] + ...

F_U Fourier coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

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Casselman-Shalika formula

[GKP14] + ...

F_U Fourier coefficients

~~Factorisation~~

Write in terms of Whittaker coefficients

Simplify drastically for certain Eisenstein series, or χ

Example of simplifications

$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

[FGKP15 §10.6]

Example of simplifications

$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \qquad \psi_{m_1,m_2}\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i (m_1 x_1 + m_2 x_2)}$$

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

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Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\\text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

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Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_{\#}(\ldots)\, K_{\#}(\ldots) \longrightarrow 0$$

[FGKP15 §10.6]

Example of simplifications

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_\#(\ldots)\, K_\#(\ldots) \longrightarrow 0$$

$$W_N(\chi, \psi_{m_1,0}; g) \propto K_\#(\ldots) + K_\#(\ldots) + K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

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To explain this, we need to study
small automorphic representations

[FGKP15 §10.6]

Automorphic representations

$G(\mathbb{A}) \circlearrowleft$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

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Automorphic representations

$G(\mathbb{A}) \circlearrowleft$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

(Fourier modes)

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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$$\psi \notin \mathrm{WF}(\pi) \implies F_U(\chi,\psi;g) = 0 \quad \text{for} \quad E(\chi;g) \in \pi$$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The wavefront set is described by nilpotent orbits

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The wavefront set is described by nilpotent orbits

$\mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\}$ $\qquad X \in \mathfrak{g}$ nilpotent

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Characters $\psi \longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

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So called admissible orbits

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Closure with respect to partial ordering

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Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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$$(p_1, p_2, \ldots)$$

Nilpotent orbits

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decreasing order

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Nilpotent orbits

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For $SL(n)$, orbits can be identified with partitions of n

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$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$ partial ordering

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \dots) \leq (q_1, q_2, \dots)$$

partial ordering

$$\iff$$

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad \forall k$$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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Illustrated by a Hasse diagram

Nilpotent orbits

[Collingwood-McGovern]

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partial ordering

$$\Longleftrightarrow$$

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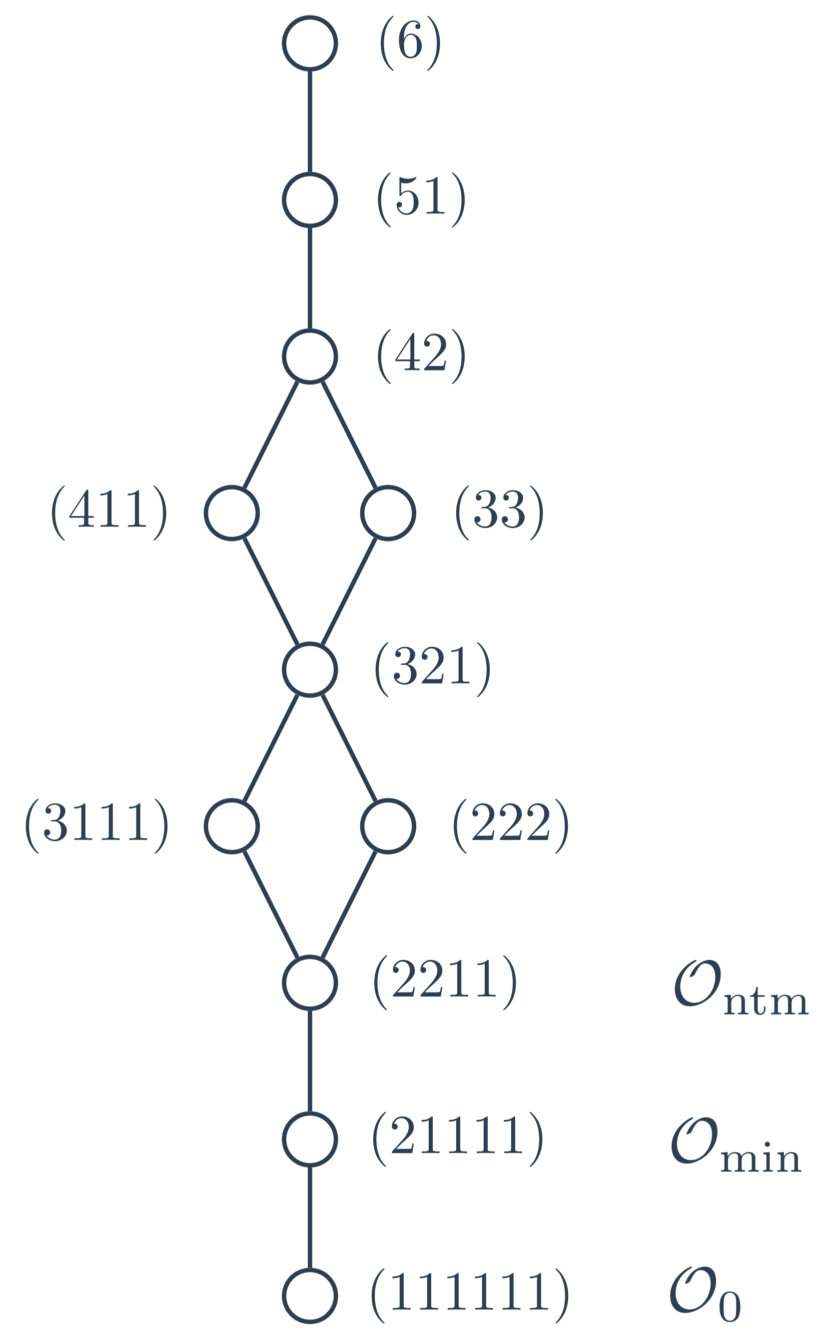
Illustrated by a Hasse diagram

$SL(6)$



Nilpotent orbits

[Collingwood-McGovern]

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$$\Longleftrightarrow$$

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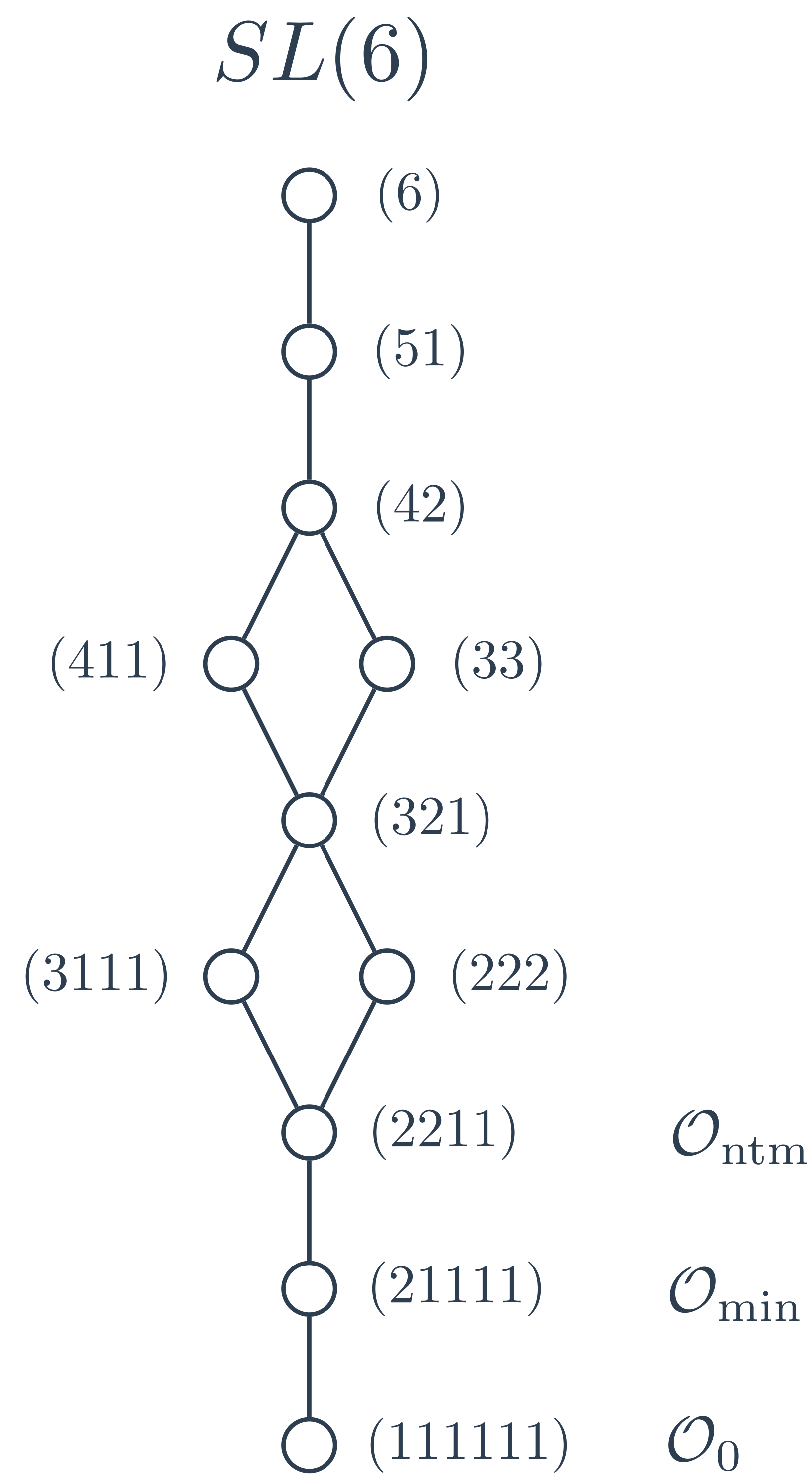
Illustrated by a Hasse diagram

Closure: $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$



Automorphic representations

Small representations

Automorphic representations

Small representations

$$\mathrm{WF}(\pi_{\mathrm{min}}) = \overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\quad$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ $\quad$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

Small representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

$\int K\,K \longrightarrow 0$

$\sum K \longrightarrow K$

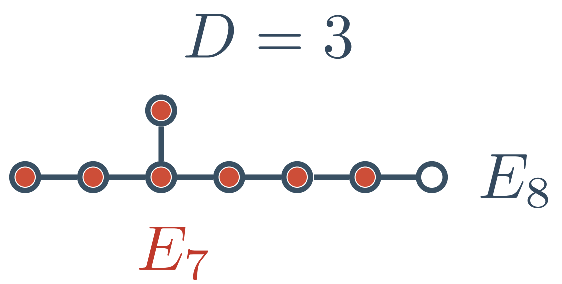
Automorphic representations

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Automorphic representations

- Decompactification limit
  Higher dimensional black holes | BPS states

Large radius for
compactified circle



[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Automorphic representations

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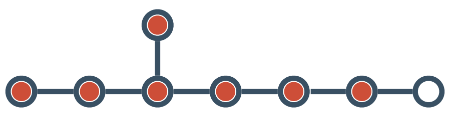
Large radius for compactified circle

$D = 3$



$\pi_{\rm min}$ $\pi_{\rm ntm}$ π_{3A_1} π_{A_2}

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Automorphic representations

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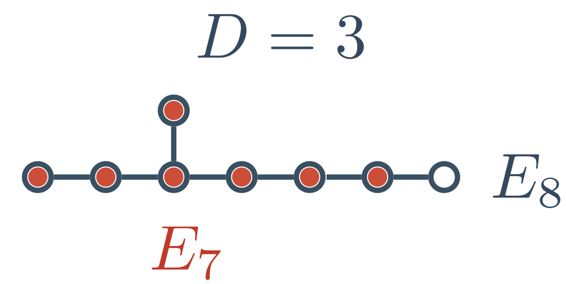

$\mathcal{E}_0^{(D)}$ $\mathcal{E}_4^{(D)}$

π_{min} π_{ntm} π_{3A_1} π_{A_2}

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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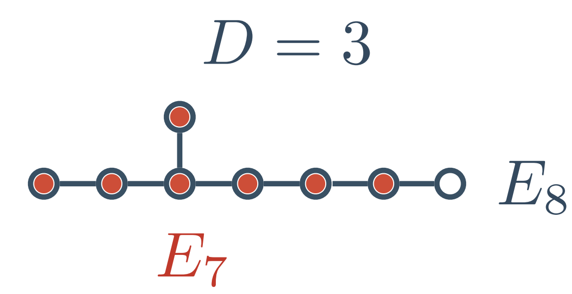

	$\mathcal{E}_0^{(D)}$	$\mathcal{E}_4^{(D)}$			
	π_{min}	π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \text{WF}\}$	28	45	55	56	[Miller-Sahi]

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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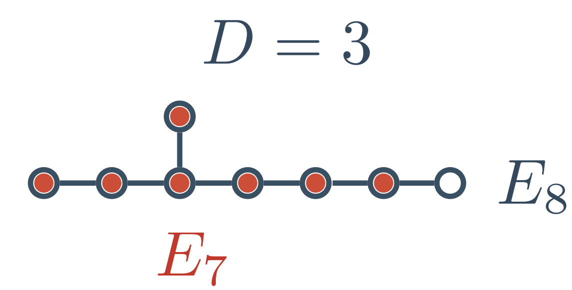

	$\mathcal{E}_0^{(D)}$	$\mathcal{E}_4^{(D)}$			
	π_{min}	π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \mathrm{WF}\}$	28	45	55	56	[Miller-Sahi]
$D=4$ BPS-orbits $E_7 \circlearrowleft \{\psi_U\}$	½ BPS	¼ BPS	⅛ BPS	⅛ BPS$^+$	

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients
using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\text{min}}$ is completely determined by maximally degenerate Whittaker coefficients

$$W_N \quad \text{with only one} \quad m_\alpha \neq 0$$

[Miller-Sahi]

Main results $SL(3), SL(4)$

[GKP14]

Main results $SL(3), SL(4)$

Theorem

For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\text{min}}$ is completely determined by maximally degenerate Whittaker coefficients.

[GKP14]

Main results $SL(3), SL(4)$

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More generally, for $\varphi \in \pi$

[GKP14]

Main results $SL(3), SL(4)$

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For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients.

More generally, for $\varphi \in \pi$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\min}$ maximally degenerate Whittaker coefficients

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\mathrm{min}}$ maximally degenerate Whittaker coefficients single root

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\min}$ single root

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\mathrm{min}}$ single root

$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

[GKP14]

Main results $SL(3), SL(4)$

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Other groups

$$SL(n)$$

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[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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Maximal parabolic
Fourier coefficient

Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple Lie groups

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Maximally degenerate

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

Would allow us to compute non-perturbative effects that capture information about instantons and black holes

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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- Other compactifications leading to automorphic forms on other groups. (more conjectural)

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- Simplification of Fourier coefficients with $\chi_{\min}$ for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

  [Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

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  How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

- $\mathcal{E}_6 D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

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Thank you!

Henrik Gustafsson

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