CHALMERS
UNIVERSITY OF TECHNOLOGY

Eisenstein series attached to small automorphic representations

Henrik Gustafsson

Lie Group/Quantum Mathematics Seminar

Rutgers 2016

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
`arXiv:1412.5625 [math.NT]` [GKP14]
Journal of Number Theory 166 (Sep, 2016) 344–399

Eisenstein series and automorphic representations
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Upcoming work with
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$SL(n)$ E_6, E_7, E_8

Outline

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- How can we compute them?
- What happens for small automorphic representations?
- What's next?

Motivation

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- Hecke eigenvalues
- Point counts of elliptic curves
- Langlands program
  L-functions | The Langlands–Shahidi method

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  Scattering amplitudes | Black hole microstate counting
- Statistical mechanics
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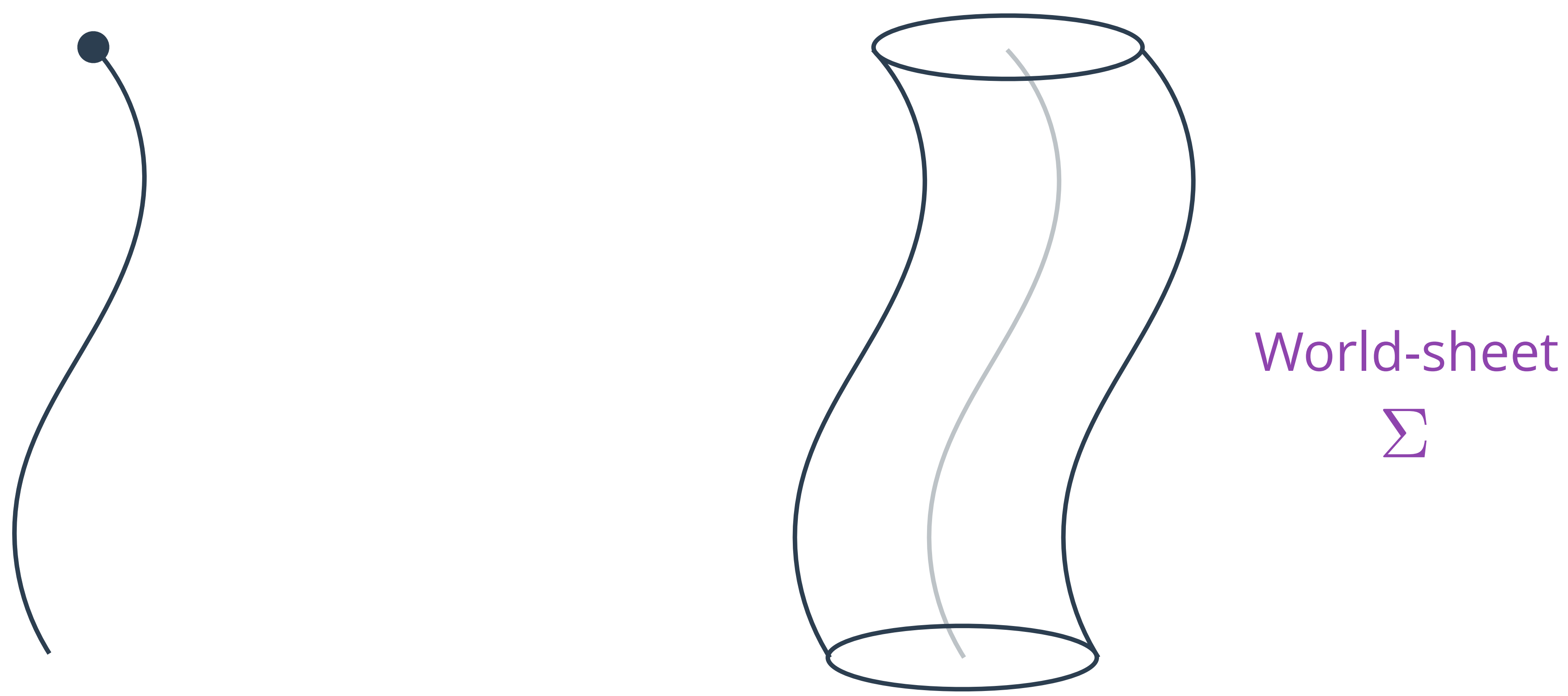
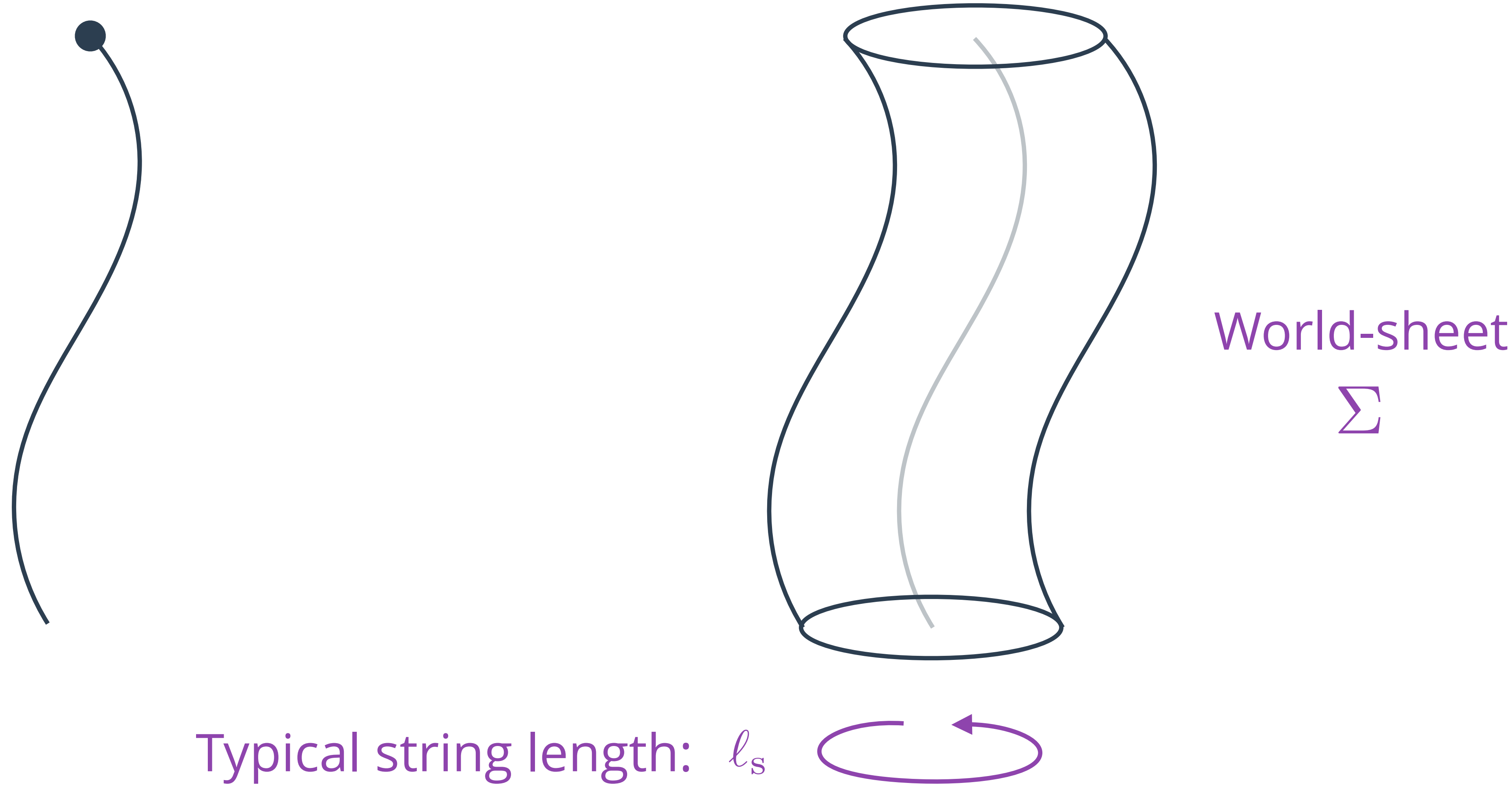
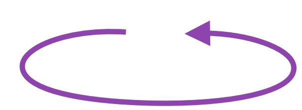


String theory



String theory

World-sheet
Σ

Typical string length: ℓ_{s} 

$\alpha' = \ell_{\mathrm{s}}^2$

String theory

Space-time is described by a Riemannian manifold M

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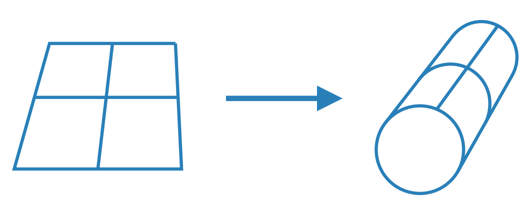
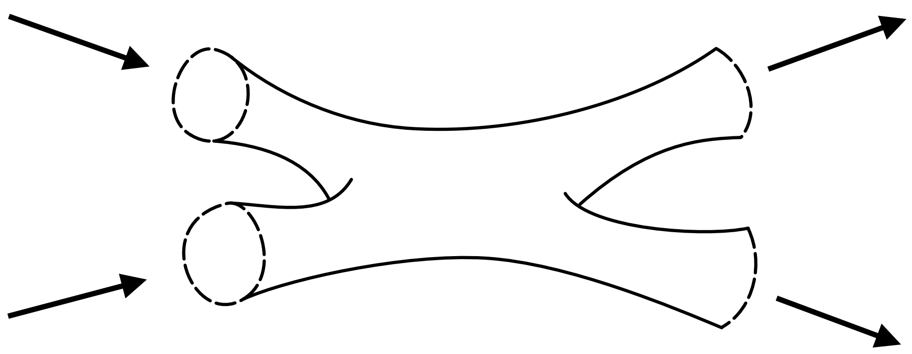
Toroidal compactifications



D dimensions

Interactions



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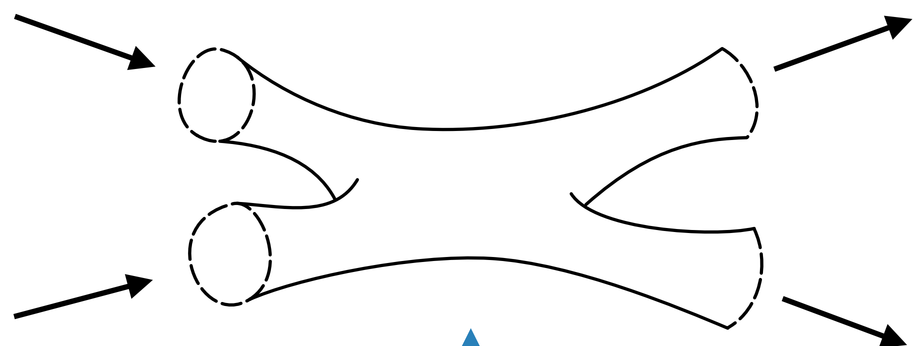
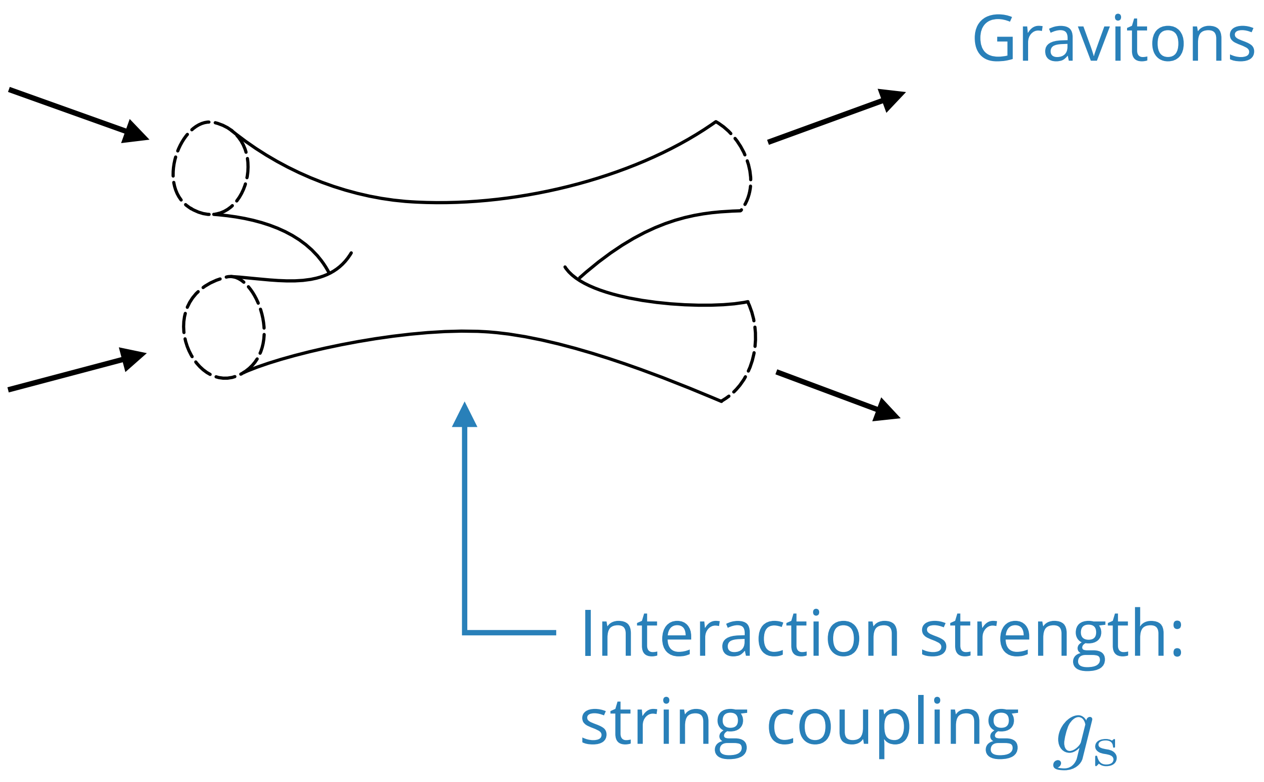


Interaction strength:
string coupling g_{s}

Interactions



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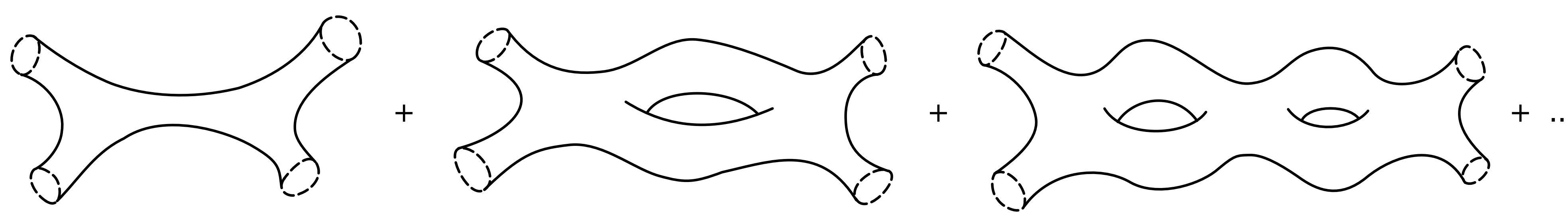
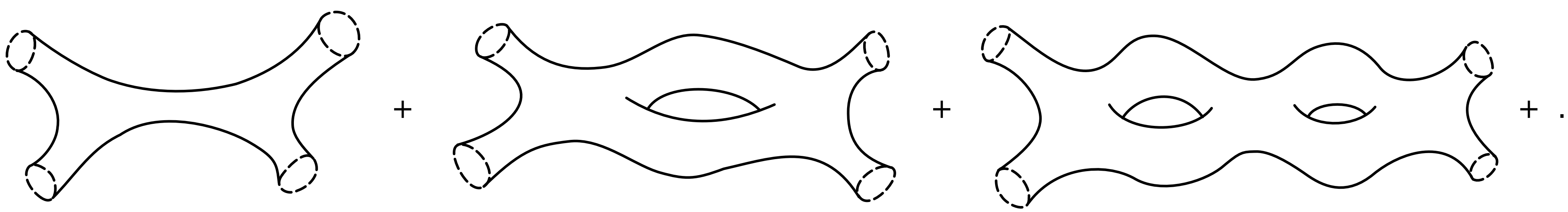


Interactions

Weighted by: $g_s^{-\chi_E}$ $\qquad -\chi_E = 2(\text{genus} - 1) + \text{boundaries}$



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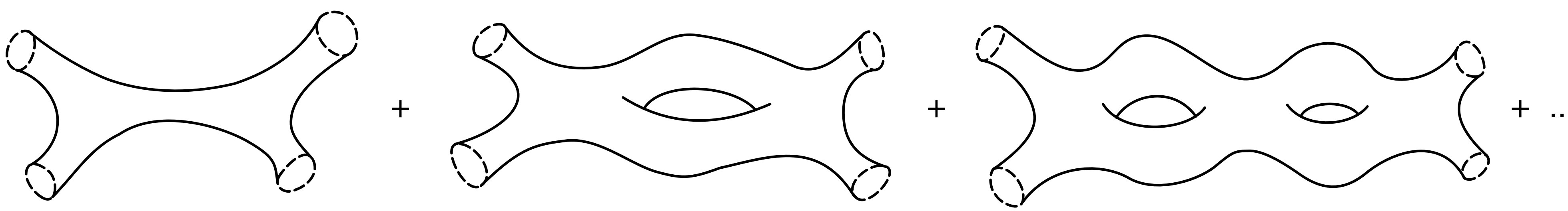

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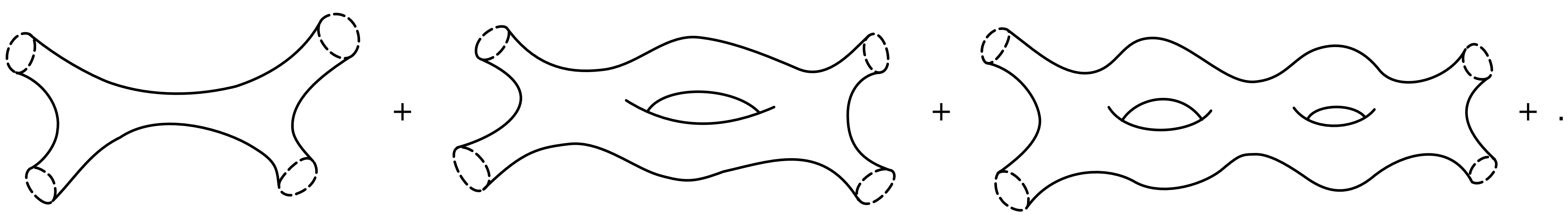

g_{s}^{-2}

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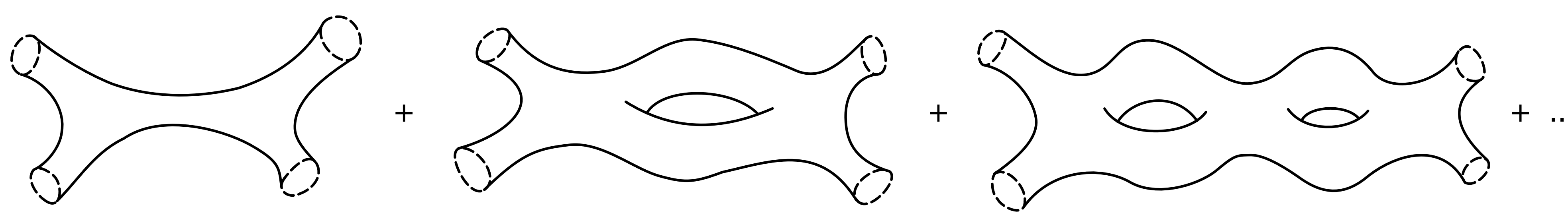

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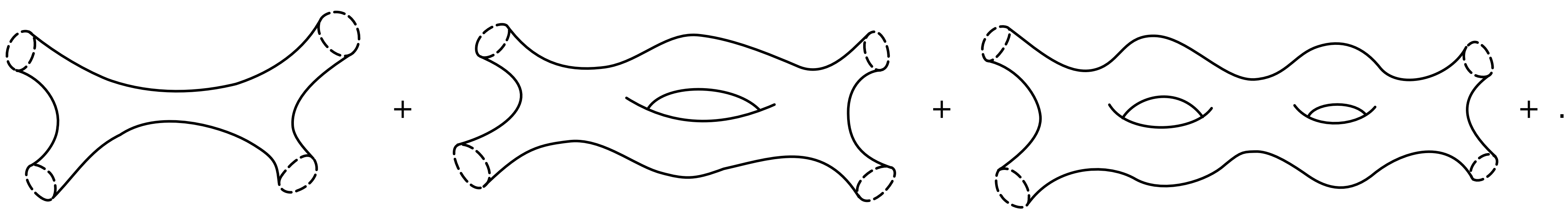
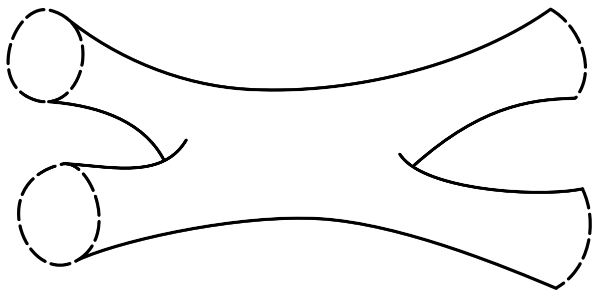


g_s^{-2} $\qquad\qquad 1 \qquad\qquad$ g_s^2

Interactions

Gravitons in D dimensions



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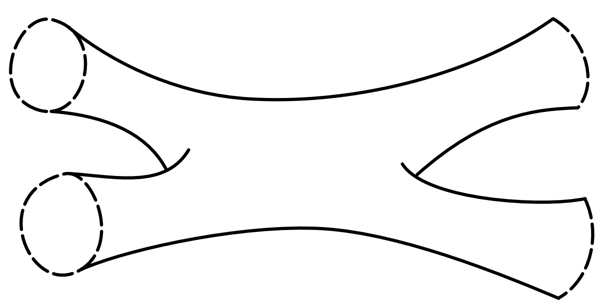
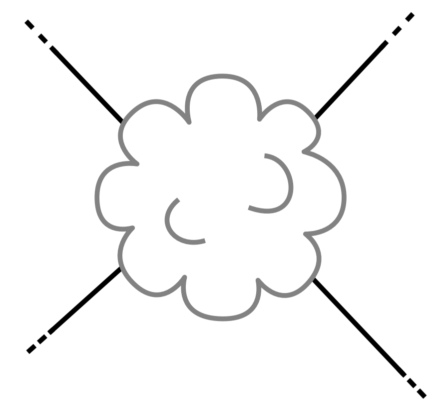
Gravitons in *D* dimensions



Effective field theory



$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \ldots$$

Expansion parameter

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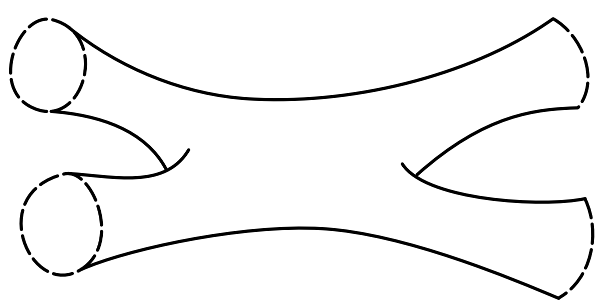

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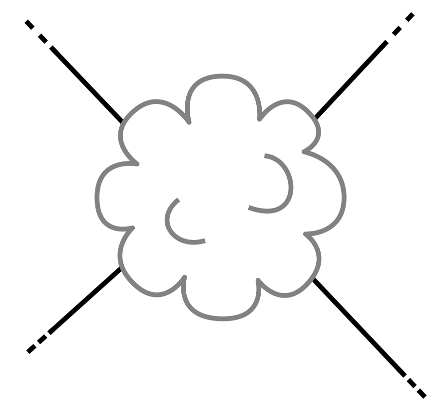

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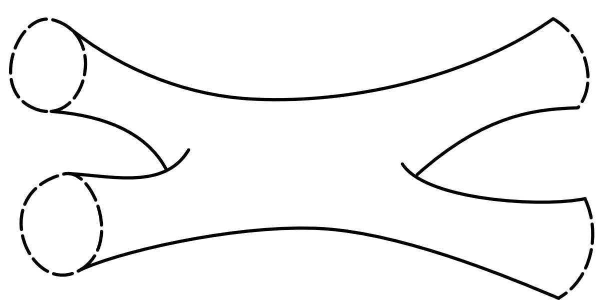

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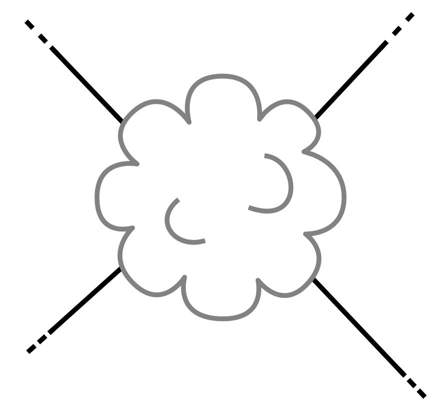

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[Cremmer-Julia]

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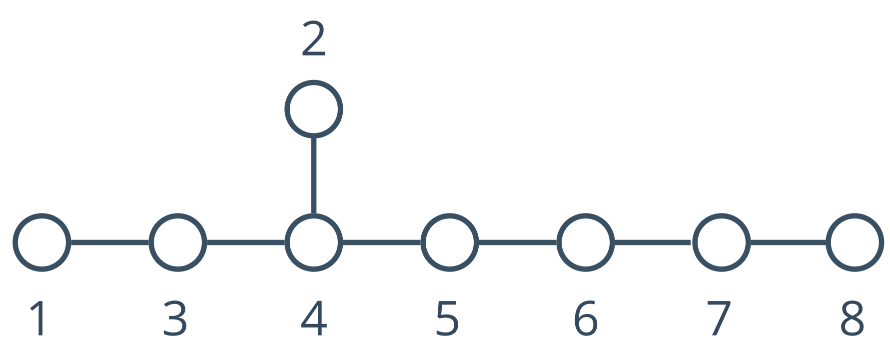

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$$\mathcal{E}_n(\tau) = \mathcal{E}_n^{(10)}(g)$$

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$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

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Quantization of charges

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All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

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$$\mathcal{E}_0^{(D)}(g),\quad \mathcal{E}_4^{(D)}(g),\quad \mathcal{E}_6^{(D)}(g)\qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K\to\mathbb{C}$$

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(D) Growth: for any norm $\|\cdot\|$ on $G(\mathbb{R})$ there exists a positive integer n and constant C such that $|\varphi(g)| \leq C\|g\|^n$

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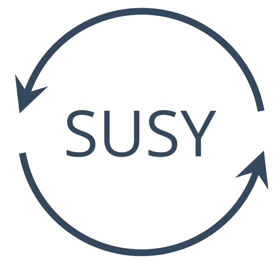
bosons  fermions

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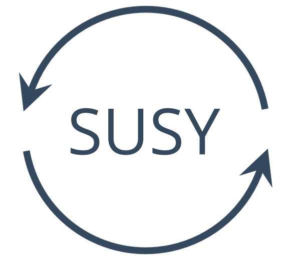
bosons  fermions

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Similarly for lower dimensions

Eisenstein series

$E(s;\tau) =$ $s \in \mathbb{C}$

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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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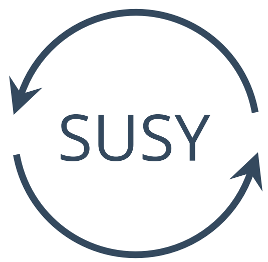

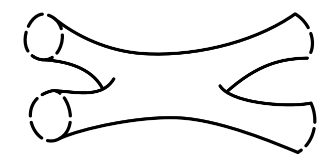

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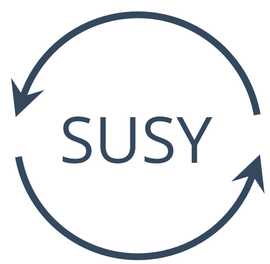

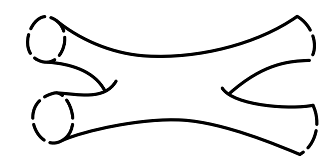

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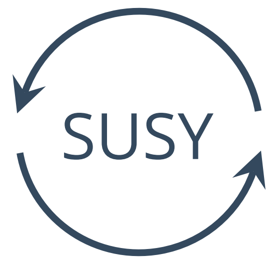

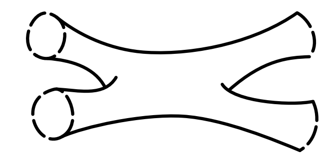

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$\mathcal{E}_6(\tau)$ as a sum over images $\sum\limits_{B(\mathbb{Q})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_{s} $\tau = \chi + i g_{\mathrm{s}}^{-1}$

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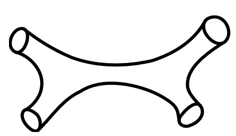 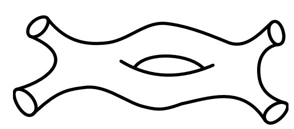

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Non-perturbative
(remaining modes)

$$e^{-\frac{1}{g_\mathrm{s}}}$$

[Green-Gutperle]

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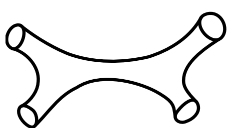 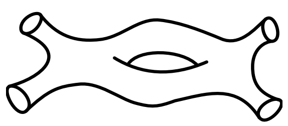

Instanton action

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Sums over the number of ways the charge *m* can be factorised into two integers

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wrapping number of a T-dual D-particle and charge

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arithmetic information
p-adic part

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[Green-Gutperle]

Lower dimensions

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D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

Lower dimensions

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

$$E(\chi; g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi(\gamma g)$$

Parabolic subgroups

Fourier expand
in different directions $\longleftrightarrow$ Unipotent subgroup U

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in different directions $\longleftrightarrow$ Unipotent subgroup U

$\uparrow$

Choice of parabolic subgroup P

Parabolic subgroups

$$\text{Fourier expand in different directions} \longleftrightarrow \text{Unipotent subgroup } U$$

$$\uparrow$$

Choice of parabolic subgroup P

Σ choice of simple roots $\langle\Sigma\rangle$ generated root system

Parabolic subgroups

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Choice of parabolic subgroup P

Σ choice of simple roots $\qquad$ $\langle\Sigma\rangle$ generated root system

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h) g \quad \forall h \in \mathfrak{h}\}$$

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Cartan subalgebra

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Cartan subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{p})} \mathfrak{g}_\alpha$$

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$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}_\alpha \qquad \Delta(\mathfrak{p}) = \Delta_+ \cup \langle\Sigma\rangle$$

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Positive roots

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Levi decomposition

Parabolic subgroups

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$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$$

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$$\Delta(\mathfrak{u}) = \Delta_+ \backslash (\Delta_+ \cap \langle \Sigma \rangle)$$

Parabolic subgroups

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Corresponding group P

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$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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Corresponding group P

$G = SL(4)$

$\Sigma = \{\alpha_1\}$

$$L = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\} \qquad U = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

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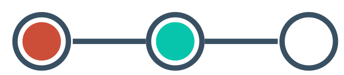
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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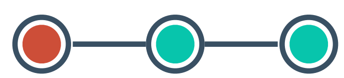
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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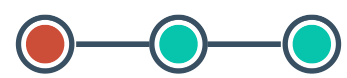

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$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

Parabolic subgroups

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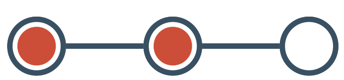

Maximal parabolic

Parabolic subgroups

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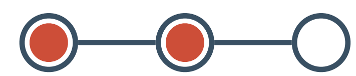
Corresponding group P

Minimal parabolic
Borel



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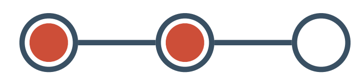
Corresponding group P

Minimal parabolic
Borel

$$B = NA$$

$$N = \left\{ \begin{pmatrix} \boxed{1} & * & * & * \\ & \boxed{1} & * & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$



Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Fourier expansion

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Let $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to U(1)$ be a multiplicative character

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$$u = \prod_{\alpha\in\Delta(\mathfrak{u})} \exp(u_\alpha E_\alpha) \mapsto \exp(2\pi i \sum_{\alpha\in\Delta^{(1)}(\mathfrak{u})} m_\alpha u_\alpha) \qquad m_\alpha \in \mathbb{Z} \text{ charges}$$

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Multiplicative: $\Delta^{(1)}(\mathfrak{u}) = \Delta(\mathfrak{u})\backslash\Delta([\mathfrak{u},\mathfrak{u}])$

$$F_U(\chi,\psi;g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi,ug)\overline{\psi(u)}\,du$$

Fourier expansion

Fourier expansion

$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi \neq 1} F_U(\chi, \psi; g)$$

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$$F_U(\chi, \psi; ug) = \psi(u) F_U(\chi, \psi; g) \qquad \psi(u_1 u_2) = \psi(u_1)\psi(u_2)$$

Fourier expansion

$$E(\chi;g) = F_U(\chi,1;g) + \sum_{\psi^{(1)}\neq 1} F_{U^{(1)}}(\chi,\psi^{(1)};g) + \sum_{\psi^{(2)}\neq 1} F_{U^{(2)}}(\chi,\psi^{(2)};g) + \dots$$

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Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi^{(1)} \neq 1} F_{U^{(1)}}(\chi, \psi^{(1)}; g) + \sum_{\psi^{(2)} \neq 1} F_{U^{(2)}}(\chi, \psi^{(2)}; g) + \dots$$

$$F_U(\chi, \psi; ug) = \psi(u) F_U(\chi, \psi; g) \qquad \psi(u_1 u_2) = \psi(u_1)\psi(u_2)$$

$$U^{(1)} = U \qquad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

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F_U W_N

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F_U W_N

$$W_N(\chi, \psi; g) = W_N(\chi, \psi; nak) = \psi(n) W_N(\chi, \psi; a)$$

Iwasawa decomposition

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

F_U W_N

$$W_N(\chi, \psi; g) = W_N(\chi, \psi; nak) = \psi(n) W_N(\chi, \psi; a)$$

Iwasawa decomposition

Characters and coefficients with all $m_\alpha \neq 0$ are called generic otherwise they are called degenerate

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

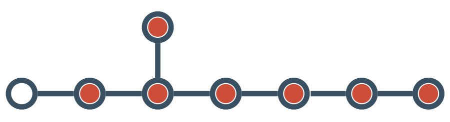
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

- String perturbation limit
  D-instantons | NS5-instantons

$g_{\rm s} \to 0$



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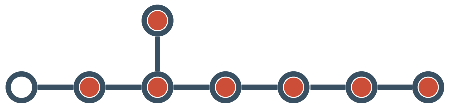
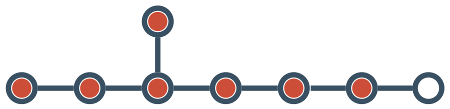


- Decompactification limit
  Higher dimensional black holes | BPS states

  Large radius for compactified circle



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  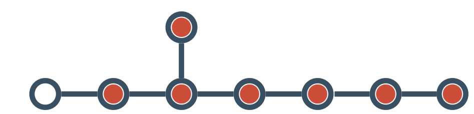

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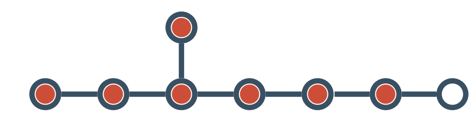
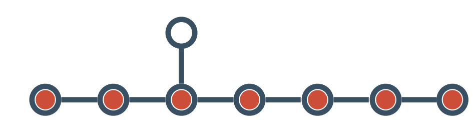
  

- **M-theory limit**
  M2, M5-instantons
  Large M-theory torus
  

[Green-Miller-Vanhove]

Fourier expansion

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  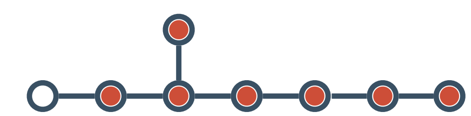

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  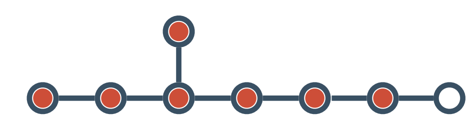

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  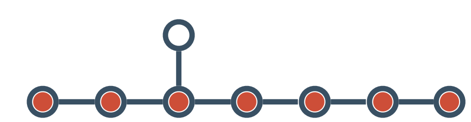

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Fourier expansion

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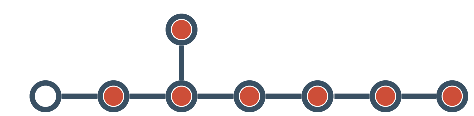

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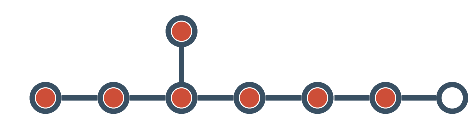

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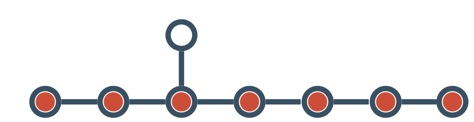

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

Fourier expansion

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients

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Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

— Robert P. Langlands*

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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Eisenstein series

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$$\begin{array}{c}\text{Adelic Eisenstein series}\\ \Big\uparrow \text{Lift}\\ \text{Eisenstein series}\end{array}$$

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$$
\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"']
\end{tikzcd}
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\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \text{Fourier coefficient}
\end{tikzcd}
$$

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The adeles

For a prime p

$\mathbb{Q}$

The adeles

For a prime p

$$\mathbb{Q} \xrightarrow{\text{Euclidean norm}} \mathbb{R}$$

The adeles

For a prime p

$$
\begin{tikzcd}[row sep=small]
 & \mathbb{R} \\
\mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\
 & \mathbb{Q}_p \quad p\text{-adic numbers}
\end{tikzcd}
$$

The adeles

For a prime p

$$
\begin{tikzcd}
 & \mathbb{R} \\
\mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\
 & \mathbb{Q}_p
\end{tikzcd}
$$

$\mathbb{Q}_p$ p-adic numbers

$\mathbb{R} = \mathbb{Q}_\infty$

The adeles

For a prime p

$$\mathbb{Q} \begin{array}{l} \xrightarrow{\text{Euclidean norm}} \mathbb{R} \\ \xrightarrow[\text{p-adic norm}]{} \mathbb{Q}_p \end{array}$$

p-adic numbers

$$\mathbb{R} = \mathbb{Q}_\infty$$

$$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$$

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p-adic numbers

$\mathbb{R} = \mathbb{Q}_\infty$

$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$

The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

The adeles

For a prime p

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

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$$\mathbb{Q} \hookrightarrow \mathbb{A}$$
$$q \mapsto (q; q, q, \ldots)$$

$\mathbb{Q}$ is discrete in $\mathbb{A}$ taking the role of $\mathbb{Z}$ in $\mathbb{R}$

Much easier to work with since it is a field!

Adelic framework

$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

Adelic framework

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Lift to the adeles [FGKP15 §4.2.2]

$$G(\mathbb{A}) = G(\mathbb{R}) \times {\prod_{p \text{ prime}}}' G(\mathbb{Q}_p) \qquad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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Adelic framework

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Eisenstein series $\longrightarrow$ Adelic Eisenstein series

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$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad\qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

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Eisenstein series $\longrightarrow$ Adelic Eisenstein series

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Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

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Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

$$\int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} E(\chi; ug) \overline{\psi_{\mathbb{R}}(u)} \, du \qquad \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\chi; ug) \overline{\psi_{\mathbb{A}}(u)} \, du$$

$$m_\alpha \in \mathbb{Z} \qquad m_\alpha \in \mathbb{Q}$$

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands’ constant term formula

Computing adelic Fourier coefficients

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Generic coefficient: Factorises over the primes. Casselman-Shalika formula

Computing adelic Fourier coefficients

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Degenerate coefficient: Reduction to generic coefficient on smaller group

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Generic coefficient: Factorises over the primes. Casselman-Shalika formula

Degenerate coefficient: Reduction to generic coefficient on smaller group

[GKP14]

Fourier coefficients

In terms of Whittaker coefficients
Simplify drastically for certain χ

Example of simplifications

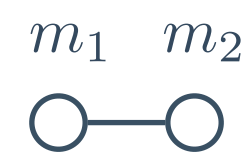
$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

[FGKP15 §10.6]

Example of simplifications

$G = SL(3)$ $\qquad$ $E(\chi; g)$ $\qquad$ $\chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ $\qquad$ $\psi_N\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i (m_1 x_1 + m_2 x_2)}$



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m_1 m_2

$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_{\#}(\ldots)\, K_{\#}(\ldots)$$

[FGKP15 §10.6]

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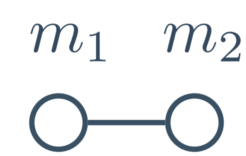

p-adic part

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\\text{factor}\end{smallmatrix}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

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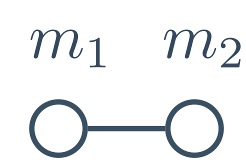

p-adic part

$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \left(\substack{\text{arithmetic} \\ \text{factor}}\right) \int K_{\#}(\ldots)\, K_{\#}(\ldots)$$

Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\ \text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots) \longrightarrow 0$$

[FGKP15 §10.6]

Example of simplifications

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[FGKP15 §10.6]

Example of simplifications

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[FGKP15 §10.6]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowleft$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

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Automorphic representations

$G(\mathbb{A}) \circlearrowleft$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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$$\psi \notin \mathrm{WF}(\pi) \implies F_U(\chi, \psi; g) = 0 \quad \text{for} \quad E(\chi; g) \in \pi$$

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Small automorphic representations have
few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters ψ $\longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Characters $\psi \longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

Nilpotent orbit $\quad \mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\} \quad X \in \mathfrak{g}$ nilpotent

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$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

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So called special orbits

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Closure with respect to partial ordering

$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

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Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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$$(p_1, p_2, \ldots)$$

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decreasing order

$$(p_1, p_2, \ldots)$$

Nilpotent orbits

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For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$

partial ordering

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \le (q_1, q_2, \ldots)$$

partial ordering

$$\iff$$

$$\sum_{i=1}^{k} p_i \le \sum_{i=1}^{k} q_i \quad \forall k$$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$ partial ordering

$$\Longleftrightarrow$$

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Illustrated by a Hasse diagram

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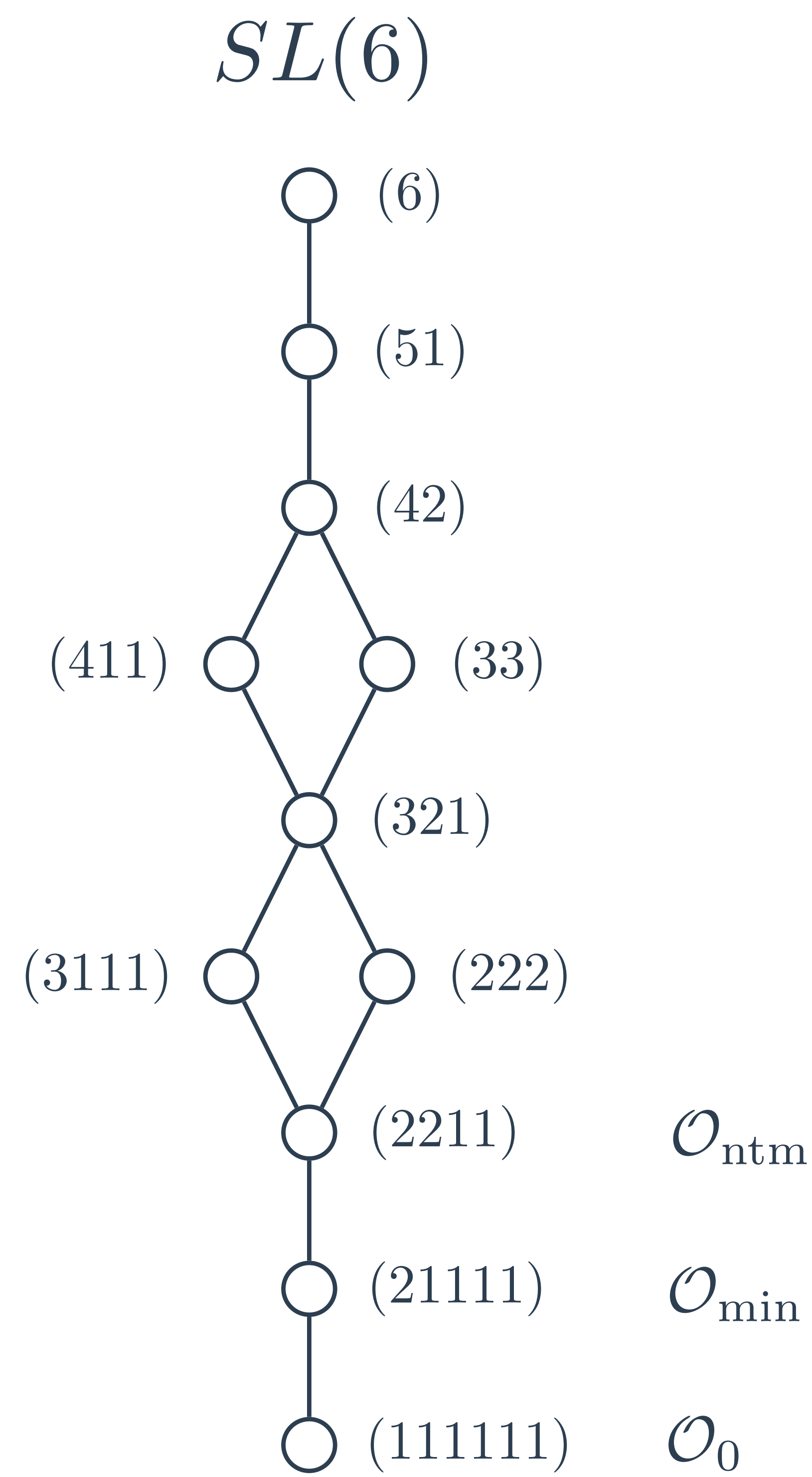


Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \dots) \leq (q_1, q_2, \dots) \qquad \text{partial ordering}$$

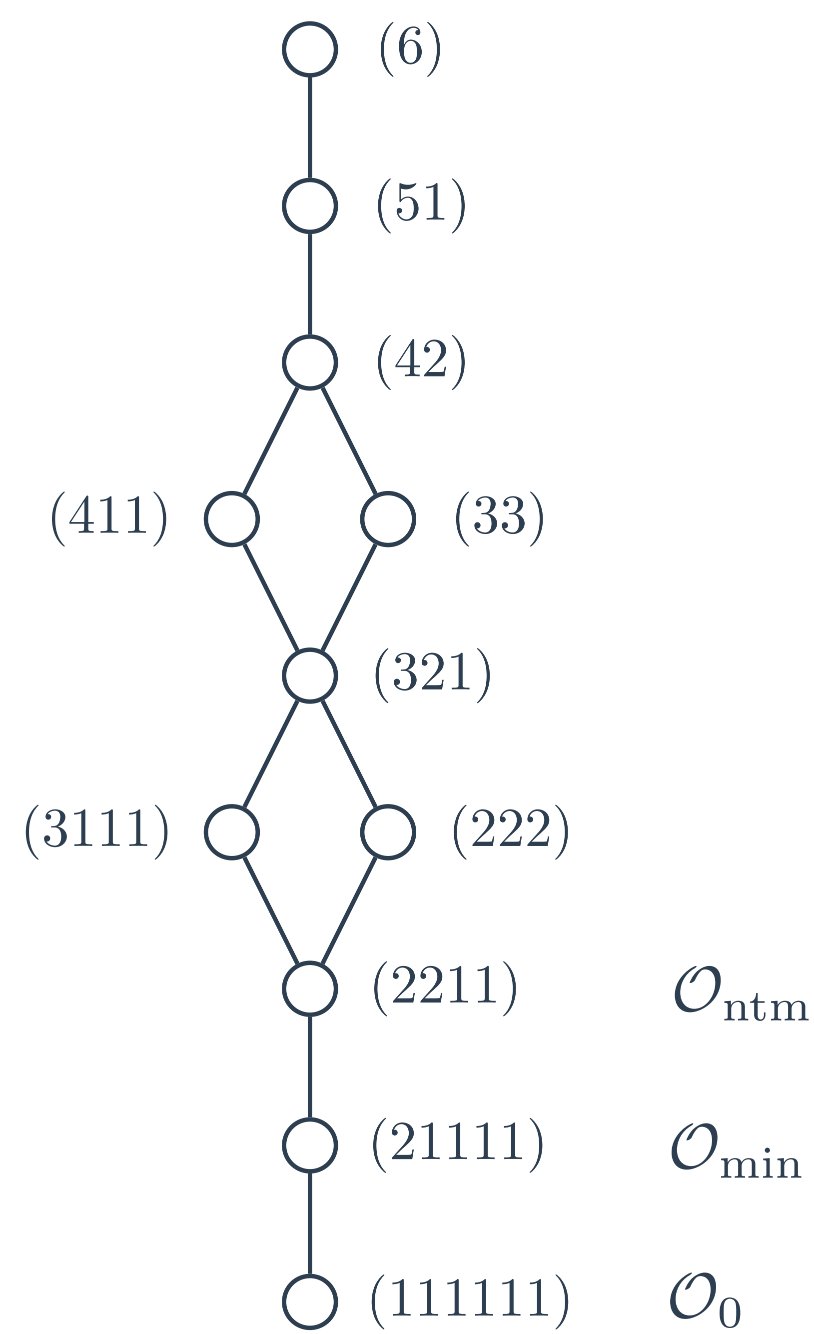
$$\Longleftrightarrow$$

$$\textstyle\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad \forall k$$

Illustrated by a Hasse diagram

Closure: $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$

$SL(6)$



Automorphic representations

Small representations

Automorphic representations

Small representations

$$\mathrm{WF}(\pi_{\mathrm{min}}) = \overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\quad$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ $\quad$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

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χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

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Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

Small representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

$$\int K\,K \longrightarrow 0$$

$$\sum K \longrightarrow K$$

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients
using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients

W_N with only one $m_\alpha \neq 0$

[Miller-Sahi]

Main results $SL(3), SL(4)$

[GKP14]

Main results $SL(3), SL(4)$

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For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients.

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[GKP14]

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More generally, for $\varphi \in \pi$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}} \qquad \text{where } \varphi_{\mathcal{O}} \text{ vanishes unless } \mathcal{O} \subseteq \mathrm{WF}(\pi)$$

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

[GKP14]

Main results

$SL(3), SL(4)$

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Corollary

$\varphi \in \pi_{\mathrm{min}}$ maximally degenerate Whittaker coefficients

[GKP14]

Main results $SL(3), SL(4)$

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Corollary

$\varphi \in \pi_{\min}$ maximally degenerate Whittaker coefficients single root

[GKP14]

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Corollary

$\varphi \in \pi_{\mathrm{min}}$ single root

$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

[GKP14]

Main results

$SL(3), SL(4)$

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Corollary

$\varphi \in \pi_{\mathrm{min}}$ single root

$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

strongly orthogonal

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

   $\pi_{\min}$

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

$\pi_{\min}$

Theorem

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

[GKP14]

Main results

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Maximal parabolic
Fourier coefficient

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

   $\pi_{\min}$

Known Whittaker coefficient

Theorem

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg) \qquad \text{with } l \in L(\mathbb{Q}) \text{ depending on } \psi$$

Maximal parabolic
Fourier coefficient

[GKP14]

Main results

$SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

   $\pi_{\min}$

Known Whittaker coefficient

Theorem

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic Fourier coefficient

Maximally degenerate

[GKP14]

Example

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$$\int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi_{\min}; ug)\overline{\psi_U(u)}\, du = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi_{\min}; nlg)\overline{\psi'_N(n)}\, dn$$

Example

Proof

$$\int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi_{\min}; ug)\overline{\psi_U(u)}\, du$$

Example

Proof (suppressing χ_{min})

$$\int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(ug)\overline{\psi_U(u)}\,du$$

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Proof (suppressing $\chi_{\min}$)

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Other groups

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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$$SL(n)$$

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg) \qquad \text{with } l \in L(\mathbb{Q}) \text{ depending on } \psi$$

Maximal parabolic
Fourier coefficient

Maximally degenerate

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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Maximal parabolic
Fourier coefficient

Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple, simply laced Lie groups

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Maximal parabolic
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[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Local spherical vectors

Checks for E_6, E_7, E_8

Local spherical vectors

Checks for E_6, E_7, E_8

$$\pi_{\min,p} \subset \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\min,p} \hookrightarrow \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p}$$

multiplicity one

[Gan-Savin]

Local spherical vectors

Checks for E_6, E_7, E_8

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$$\mathrm{Ind}_U^G \psi = \{ f : G \to \mathbb{C} \mid f(ug) = \psi(u) f(g), u \in U \}$$

Local spherical vectors

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$f^\circ_{\psi_{U,p}} \in \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p}$ computed in several cases $p \leq \infty$

[Dvorsky-Sahi, Kazhdan-Polishchuk, Kazhdan-Pioline, Savin-Woodbury]

Local spherical vectors

Local spherical vectors

$$\mathrm{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}\psi_U \ni F_U(\chi_{\min}, \psi_U; g) \stackrel{?}{=} W_N(\chi_{\min}, \psi'; lg)$$

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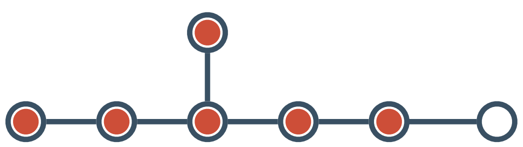
For example: E_7 with U from 

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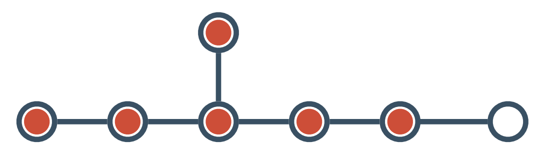
For example: E_7 with U from 

$$f^{\circ}_{\psi_{U,p}} = \frac{1-p^3\,|m|_p^{-3}}{1-p^3}$$

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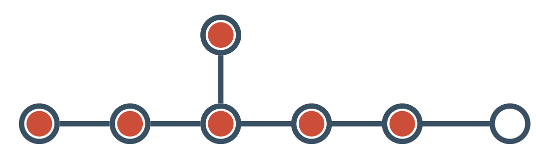
For example: E_7 with U from 

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[Savin-Woodbury]

$$f^{\circ}_{\psi_{U,\infty}} = m^{-3/2}K_{3/2}(m)$$

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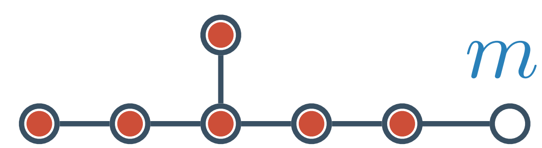
For example: E_7 with U from 

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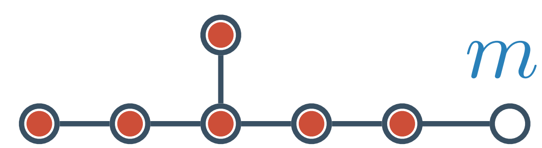
For example: E_7 with U from  $m \longleftarrow \psi'$

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$$W_{\psi'}(\chi_{\min}, 1) = \frac{2}{\xi(4)} \left(\prod_{p<\infty} \frac{1 - p^3 \left|m\right|_p^{-3}}{1 - p^3} \right) \left(\left|m\right|^{-3/2} K_{3/2}(m) \right)$$

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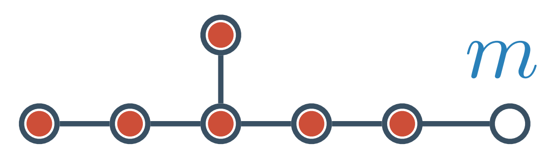
For example: E_7 with U from  $m \longleftarrow \psi'$

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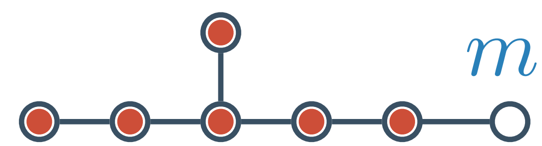
For example: E_7 with U from  $m \longleftarrow \psi'$

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Complete agreement for E_6, E_7, E_8 in both abelian and Heisenberg realisations.

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The Fourier coefficients capture information about BPS-states and black holes.

Whittaker pairs

Tools for proving the conjecture

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Whittaker pairs

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$(S, f) \in \mathfrak{g} \times \mathfrak{g}$ Whittaker pair [Gomez-Gourevitch-Sahi]

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Methods for relating different Whittaker pairs

$$(S, \psi) \longrightarrow (S', \psi')$$

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Eigenvalues of $\mathrm{ad}(S)$ in $\mathbb{Q}$

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Eigenvalues of $\mathrm{ad}(S)$ in $\mathbb{Q}$

$f \in \mathfrak{g}^S_{-2}$, that is, $[S, f] = -2f$

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Let $\mathfrak{u}_S = \mathfrak{g}^S_{>1} \oplus \mathfrak{g}^S_1 \cap \mathfrak{g}_f$ ⟵ Centralizer

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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If (h, f) is part of an SL_2-triple (Jacobson-Morozov triple)

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Methods for relating different Whittaker pairs

$$(h, \psi) \xrightarrow{h + tZ} (S = h + Z, \psi)$$

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Outlook

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Simplification of Fourier coefficients with $\chi_{\min}$ for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

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How to define “small automorphic representations” for Kac-Moody groups? What is the mechanism behind the vanishing properties?

$\mathcal{E}_6 D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

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Thank you!

Henrik Gustafsson

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