Whittaker functions, lattice models and (non)symmetric polynomials

Lecture notes for working seminar on symmetric functions at Rutgers, fall 2019

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Aims and objectives

The aim of these lecture is to:

— Understand the connections between (non)symmetric polynomials, Whittaker functions and lattice models.

— Gain familiarity with different techniques to compute and manipulate these objects.

After reading these notes the reader should be able to define, understand and relate the following concepts:

— Roots, weights and characters.

— Schur polynomials and their connection to representation theory.

— Spherical and Iwahori Whittaker Whittaker functions on $\mathrm{GL}_r(F)$ for a non-archimedean field F, in particular $F = \mathbb{Q}_p$ (the p-adic numbers).

— Compute standard p-adic integrals and in particular the spherical Whittaker function for $\mathrm{GL}_2(\mathbb{Q}_p)$.

— Two-dimensional lattice models whose partition functions describe spherical and Iwahori Whittaker functions.

— The Yang-Baxter equations for these lattice models and how it can be used to obtain functional relations for Whittaker functions.

Lecture 1

In this lecture we will first recall some facts about algebraic groups, weights and characters. We will then define the Schur polynomials and connect them to characters of GL_r. In the last part we will show that local Whittaker functions for $\mathrm{GL}_r(F)$ where F is a non-archimedean field also can be expressed in terms of Whittaker functions. Recommended reference literature: Bump, *Lie groups*; Hall *Lie groups, Lie algebras, and representations: An elementary introduction*; and Milne *Algebraic groups*. I can also recommend the notes http://sporadic.stanford.edu/bump/whittaker/whittaker.html by Bump.

1.1 Weights and characters

Let G be a *linear algebraic group* over $\mathbb{Q}$, that is, a subgroup of GL_r defined by polynomial equations in the matrix elements of $g \in \mathrm{GL}_r$ and $\det(g)^{-1}$ with rational coefficients. For a field F, the group $G(F)$ is then the solutions to these equations in F. In this lecture series we will take for example $F = \mathbb{C}$, or a local non-archimedean field like the p-adic numbers $\mathbb{Q}_p$.

Let T be a *maximal torus* of G. A *torus* is a subgroup T of G such that $T(\mathbb{C}) \cong (\mathbb{C}^\times)^r$ for some $r \geq 1$ and it is *maximal* if it is not contained in any other torus.

The *weight lattice* $\Lambda = X^*(T)$ is defined as the group of rational characters on T, that is the homomorphisms $T \to \boldsymbol{G}_m$ where $\boldsymbol{G}_m$ takes a field F to the multiplicative group of invertible elements $F^\times$. Since $T(\mathbb{C}) \cong (\mathbb{C}^\times)^r$ and $\boldsymbol{G}_m(\mathbb{C}) \cong \mathbb{C}^\times$, we may identify $\Lambda \cong \mathbb{Z}^r$ using maps

$$t = \mathrm{diag}(t_1, \ldots, t_r) \mapsto t^\lambda = \textstyle\prod_{i=1}^r t_i^{\lambda_i} \qquad \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r\,. \tag{1.1}$$

The group $X_*(T)$ of *co-characters* is similarly defined as the homomorphisms $\boldsymbol{G}_m \to T$.

Let (π, V) be an irreducible representation of G, then the restriction of π to T is not irreducible unless V is one-dimensional. The restriction decomposes into one-dimensional characters which are the weights in Λ associated to this particular representation. The module V decomposes into eigenspaces called *weight spaces* $V(\mu) = \{v \in V : \pi(t)v = t^\mu v\}$. In particular, if π is the adjoint representation: $\pi(g)h = ghg^{-1}$, then the associated weights are called the set of *roots* Φ of G.

Example 1.1

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G = \mathrm{GL}_2$ and $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$. Then,

$$tgt^{-1} = \begin{pmatrix} a & bt_1/t_2 \\ ct_2/t_1 & d \end{pmatrix} = t^{(1,-1)} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + t^{(-1,1)} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + t^{(0,0)} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \tag{1.2}$$

The three different terms correspond to α, $-\alpha$ and 0.

In general for GL_r the roots can be expressed in $\Lambda \cong \mathbb{Z}^r$ as $\varepsilon_i - \varepsilon_j$ where ε_i is the standard basis vector for $\mathbb{Z}^r$, and the *simple roots* can be expressed as $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. That is, $\alpha_1 = (1, -1, 0, \ldots, 0)$, $\alpha_2 = (0, 1, -1, 0, \ldots, 0)$ etc. Furthermore, Λ admits a Weyl invariant inner product which we will denote by $(\cdot, \cdot)$ for which $\{\varepsilon_i\}$ is an orthonormal basis. We will denote the *Weyl group* by W.

It is often useful to embed the weight lattice of SL_r into the weight lattice of GL_r as the hyperplane satisfying $\sum \lambda_i = 0$ in the coordinates of the above basis keeping the GL_r-description of the roots. Note that the maximal torus of $\mathrm{SL}_r(\mathbb{C})$ is isomorphic to $(\mathbb{C}^\times)^{r-1}$.

The *co-roots* $\alpha^\vee \in X_*(T)$ of G are defined by the dual pairing $\langle \alpha^\vee, \beta\rangle = 2(\alpha,\beta)/(\alpha,\alpha)$ for a root β, extended by linearity.

A weight λ is called *dominant* if $(\lambda, \alpha_i) \geq 0$ for all simple roots α_i. In the basis $\{\varepsilon_i\}_i$ we get that $\lambda_i \geq \lambda_{i+1}$ which means that $(\lambda_1, \dots, \lambda_n)$ is an integer *partition* if $\lambda_r \geq 0$.

Under some basic assumptions on the group G, there is a bijection $\lambda \longleftrightarrow (\pi_\lambda, V_\lambda)$ between dominant weights and isomorphism classes of *irreducible representations*. There is an ordering on the weights: $\mu \succeq \mu'$ if $\mu - \mu'$ is a positive linear combination of the simple roots, and λ is the unique *highest weight* for the representation (π_λ, V_λ).

The *character* χ_π of a representation (π, V) is $\chi_\pi(g) = \mathrm{tr}_V(\pi(g))$. We restrict to the maximal torus T, and recall that V decomposes into weight spaces $V(\mu)$. Let $m_\mu = \dim(V(\mu))$. Then, since $\pi(t)v = t^\mu v$ for $v \in V(\mu)$,

$$\chi_\pi(t) = \sum_{\mu\in\Lambda} \mathrm{tr}_{V(\mu)}(\pi(t)) = \sum_{\mu\in\Lambda} \mathrm{tr}_{V(\mu)}(\mathbb{1}_{V(\mu)}) t^\mu = \sum_{\mu\in\Lambda} m_\mu t^\mu \,. \tag{1.3}$$

For a highest weight representation (π_λ, V_λ), the character $\chi_\lambda := \chi_{\pi_\lambda}$ can be computed by the *Weyl character formula*:

$$\chi_\lambda(t) = \frac{\sum_{w\in W}(-1)^{\ell(w)} t^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)} t^{w(\rho)}}\,, \tag{1.4}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector. For $G = \mathrm{GL}_r$ we will shift this by a multiple of $(1,\dots,1)$ to $\rho = (r-1, r-2, \dots, 1, 0)$ to make all entries non-negative. This does not affect the Weyl character formula. We also note for $G = \mathrm{GL}_r$ that $\chi_{\lambda+(k,\dots,k)}(t) = \det(t)^k \chi_\lambda(t)$ which means that we may assume that λ is a partition for the computation of $\chi_\lambda(t)$.

1.2 Schur polynomials

There are several ways of defining Schur polynomials. One which will be convenient for us is the following determinant formula. Let λ be a partition of length r (which may be padded by zeroes to obtain this length). Then we define the *Schur polynomial* $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by

$$s_\lambda(z_1, \dots, z_r) := \frac{\det(z_i^{(\lambda+\rho)_j})}{\det(z_i^{\rho_j})}\,. \tag{1.5}$$

For brevity we will denote $\mathbf{z} = (z_1, \dots, z_r)$.

Both the numerator and the denominator are *alternating polynomials*, changing sign under permutations of the variables, because of the properties of the determinant. The denominator is called the *Vandermonde determinant* and can, by Gauss elimination, be shown to equal $\prod_{i<j}(z_i - z_j)$. Since the numerator is alternating it also has zeroes at $z_i = z_j$, $i < j$ and therefore contains the Vandermonde determinant as a factor. Thus $s_\lambda(z_1, \dots, z_r)$ is a *symmetric polynomial*.

By writing out the determinants we may compare (1.5) with the Weyl character formula (1.4) for $G = \mathrm{GL}_r$ which has the Weyl group $W \cong S_r$. We get that

$$s_\lambda(\mathbf{z}) = \frac{\sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \prod_i z_i^{(\lambda+\rho)_{\sigma_i}}}{\sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \prod_i z_i^{\rho_{\sigma_i}}} = \chi_\lambda(\mathbf{z})\,. \tag{1.6}$$

Noting that s_λ is polynomial and comparing with (1.3) we conclude that $m_\mu = 0$ unless μ is dominent, i.e. a partition.

Example 1.2

Consider $r = 2$ with $\lambda = (1, 0)$ and $S_2 = \{\mathbb{1}, \sigma\}$. We have that $s_\lambda(z_1, z_2) = \frac{\mathrm{sgn}(\mathbb{1}) z_1^2 + \mathrm{sgn}(\sigma) z_2^2}{\mathrm{sgn}(\mathbb{1}) z_1 + \mathrm{sgn}(\sigma) z_2} = z_1 + z_2$.

1.3 Whittaker functions and the Langlands dual group

To the group G we may associate the *Langlands dual group* $\hat{G}(\mathbb{C})$ as follows. Its maximal torus $\hat{T}$ is dual to T in the sense that $X_*(T)$ is identified with $X^*(\hat{T}) := \hat{\Lambda}$ and the co-roots of G are the roots of $\hat{G}$. The Langlands dual associated to GL_r is $\mathrm{GL}_r(\mathbb{C})$.

We will now define the local Whittaker functions. Let $G = \mathrm{GL}_r$ and N be the maximal unipotent subgroup of G consisting of lower triangular matrices with unit diagonal. Let F be a *local non-archimedean field* with ring of integers $\mathfrak{o}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ with generator $\varpi \in \mathfrak{p}$ and residue cardinality $q = |\mathfrak{o}/\mathfrak{p}|$. We fix a unitary character ψ on $N(F)$ such that ψ restricted to any single matrix element $n_{i,i+1}$ is a character on F trivial on $\mathfrak{o}$ but no larger fractional ideal.

For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ let $\varpi^\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_r})$. These form a full set of representatives for the coset space $T(F)/T(\mathfrak{o})$. We will consider an unramified character $\tau_\mathbf{z}$ on $T(F)$ trivial on $T(\mathfrak{o})$ and defined by $\tau_\mathbf{z}(\varpi^\lambda) = \mathbf{z}^\lambda = \prod_i z_i^{\lambda_i}$. For $F = \mathbb{Q}_p$ we have that $\tau_\mathbf{z}(t) = \prod_i z_i^{\mathrm{ord}_p(t_i)}$.

We trivially extend $\tau_\mathbf{z}$ to a character on the upper triangular matrices $B(F)$ and then as a function on $\mathrm{GL}_r(F)$. It is a *spherical* function since $\tau_\mathbf{z}(gk) = \tau_\mathbf{z}(g)$ for k in the maximal compact subgroup $\mathrm{GL}_r(\mathfrak{o})$ of $\mathrm{GL}_r(F)$. Let $f^\circ_\mathbf{z}(ntk) = \delta^{1/2}(t)\tau_\mathbf{z}(t)$ for $t \in T(F)$, $n \in N(F)$ and $k \in \mathrm{GL}_r(\mathfrak{o})$ where $\delta^{1/2}$ is the *modular quasicharacter* which may be evaluated as $\delta^{1/2}(t) = t^{\overline{\rho}}$ where $\overline{\rho} = \frac{1}{2}\sum_{\alpha>0} \alpha$ without a shift unlike ρ. We define the *spherical Whittaker function* as

$$W^\circ_\mathbf{z}(g) = \int\limits_{N(F)} f^\circ_\mathbf{z}(w_0 n g)\psi^{-1}(n)\, dn \tag{1.7}$$

which is determined by its values on $T(F)/T(\mathfrak{o})$.

The values $W^\circ_\mathbf{z}(\varpi^\lambda)$ can be computed using the *Casselman-Shalika formula* which can be reinterpreted as the Weyl character formula (1.4) for the Langlands dual group $\mathrm{GL}_r(\mathbb{C})$ which we showed in (1.6) was a Schur polynomial

$$W^\circ_\mathbf{z}(\varpi^\lambda) = \prod_{\alpha>0}(1 - q^{-1}\mathbf{z}^\alpha) \cdot \begin{cases} \delta^{1/2}(\varpi^\lambda) s_\lambda(\mathbf{z}) & \lambda \in \hat{\Lambda} \text{ is dominant} \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.8}$$

Lecture 2

Last time we defined Schur polynomials using the determinant formula and showed that they are indeed symmetric polynomials. We recalled some facts about representation theory, especially for GL_r, noting that characters of irreducible representations are Schur polynomials. We also defined spherical Whittaker functions for $\mathrm{GL}_r(F)$ where F is a local non-archimedean field and mentioned they are also related to Schur polynomials.

Today we will, as requested, compute the spherical Whittaker functions for $\mathrm{GL}_2(\mathbb{Q}_p)$ by directly computing the p-adic integrals. We will therefore start by recalling some useful facts for p-adic numbers and computing some integrals that will be used later. While going through the proof in the next section I advice to take a look at the summary of the steps at the end of the section.

2.1 p-adic numbers and integrals

The ring of integers for $F = \mathbb{Q}_p$ is $\mathfrak{o} = \mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathfrak{p} = p\mathbb{Z}_p$ for which we may take the generator $\varpi = p$ and we have that $q := |\mathfrak{o}/\mathfrak{p}| = p$. We will also use the facts

(i) $\mathbb{Z}_p^\times := \{x \in \mathbb{Z}_p : |x|_p = 1\} = \mathbb{Z}_p \backslash p\mathbb{Z}_p$.

(ii) $\mathbb{Z}_p = \bigsqcup_{k=0}^{\infty} p^k \mathbb{Z}_p^\times$.

(iii) $\mathbb{Q}_p \backslash \mathbb{Z}_p = \bigsqcup_{k=1}^{\infty} p^{-k} \mathbb{Z}_p^\times$.

These can all be deduced from the formal Laurent series presentation of $x \in \mathbb{Q}_p$:

$$x = x_k p^k + x_{k+1} p^{k+1} + \dots \text{ with } x_i \in \mathbb{Z}/p\mathbb{Z} \text{ such that } x_k \neq 0 \text{ and } |x|_p = p^{-k}. \tag{2.1}$$

Note that terms further to the right have smaller and smaller p-adic norm.

We have an additive measure dx on $\mathbb{Q}_p$ that is invariant under translations $d(x+a) = dx$ and scales as $d(ax) = |a|_p\, dx$ for $a \in \mathbb{Q}_p$. It is normalized such that $\int_{\mathbb{Z}_p} dx = 1$.

Let $\mathbf{e} : \mathbb{Q}_p \to \mathbb{C}^\times$ be a unitary additive character on $\mathbb{Q}_p$ trivial on $\mathbb{Z}_p$ but no larger fractional ideal $\frac{m}{n}\mathbb{Z}_p$. Being additive it satisfies $\mathbf{e}(x+y) = \mathbf{e}(x)\,\mathbf{e}(y)$. Note that $\mathbf{e}(\frac{1}{p})$ is a p-th root of unity since $1 = \mathbf{e}(1) = \mathbf{e}(\frac{1}{p} + \cdots + \frac{1}{p}) = \mathbf{e}(\frac{1}{p})^p$ where the sum is over p terms. We have that $\sum_{n=0}^{p-1} \mathbf{e}(\frac{n}{p}) = (1 - \mathbf{e}(\frac{1}{p})^p)/(1 - \mathbf{e}(\frac{1}{p})) = 0$. We will make use of the following integral.

Proposition 2.1. *Let $k, l \in \mathbb{Z}$. Then,*

$$\int_{p^{-k}\mathbb{Z}_p} \mathbf{e}(p^l x)\, dx = p^k \begin{cases} 1 & l-k \geq 0 \\ 0 & l-k < 0 \end{cases} \tag{2.2}$$

Proof. By the variable substitution $x = p^{-k}x'$ we can reduce to the case $k = 0$, that is, the integral $\int_{\mathbb{Z}_p} \mathbf{e}(p^l x)\, dx$. For $l \geq 0$ the argument $p^l x \in \mathbb{Z}_p$ on which $\mathbf{e}$ is trivial, which leaves us with $\int_{\mathbb{Z}_p} dx = 1$. For $l < 0$ we split the integration domain into a sum over p parts: $0 \leq n \leq p-1$ where the elements of part n has p^{-l-1}-coefficient equal to n. We get an overall factor that is the sum over all p-roots of unity and hence the integral is zero. $\square$

2.2 Detailed computation of Whittaker functions on $\mathrm{GL}_2(\mathbb{Q}_p)$

We start by recalling the spherical function $f^\circ_{\mathbf{z}}$ we integrate over to obtain the Whittaker function on $\mathrm{GL}_r(\mathbb{Q}_p)$. By definition $f^\circ_{\mathbf{z}}$ satisfies $f^\circ_{\mathbf{z}}(ntg) = f^\circ_{\mathbf{z}}(nt) f^\circ_{\mathbf{z}}(g)$ for $n \in N(F)$ and $t \in T(F)$ and $f^\circ_{\mathbf{z}}(gk) = f^\circ_{\mathbf{z}}(g)$ for $k \in G(\mathfrak{o})$. It is given by $f^\circ_{\mathbf{z}}(ntk) = \left|t^{\overline{\rho}}\right|_p \prod_i z_i^{\mathrm{ord}_p(t_i)}$.

Let $\mu_{\mathbf{z}} = (\mu_1, \dots, \mu_r) \in \mathbb{C}^r$ where $\mu_i = -\log z_i / \log p$, such that $z_i = p^{-\mu_i}$ and therefore

$$z_i^{\mathrm{ord}_p(t_i)} = p^{-\mathrm{ord}_p(t_i)\mu_i} = |t_i|_p^{\mu_i} \,. \tag{2.3}$$

For the remainder of this section we will consider the group $\mathrm{GL}_2(\mathbb{Q}_p)$. The Whittaker function defined in (1.7) can then be expressed as

$$W^\circ_{\mathbf{z}}(t) = \int\limits_{\mathbb{Q}_p} f_{\mathbf{z}}\Big(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} t\Big) \mathbf{e}(x)\, dx \tag{2.4}$$

which is determined by its values at $t = \mathrm{diag}(t_1, t_2) = \mathrm{diag}(p^{\lambda_1}, p^{\lambda_2})$ and where

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)\,. \tag{2.5}$$

Theorem 2.2. *Let λ be a partition of length two, that is, $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$. Then, the spherical Whittaker function on $\mathrm{GL}_2(\mathbb{Q}_p)$ evaluated at $t = \mathrm{diag}(p^\lambda_1, p^\lambda_2)$ equals*

$$W^\circ_{\mathbf{z}}(t) = p^{\frac{1}{2}(\lambda_2 - \lambda_1)} \Big(1 - p^{-1} \frac{z_1}{z_2}\Big) s_\lambda(z_1, z_2)\,. \tag{2.6}$$

This agrees with (1.8) where $\delta^{1/2}(t) = p^{\frac{1}{2}(\lambda_2 - \lambda_1)}$ and $\mathbf{z}^\alpha = z_1/z_2$ for the single simple root α. The different steps enumerated in the margin will be summarized after the proof.

(A) *Proof.* We have that

$$w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} t = w_0 t w_0 \begin{pmatrix} 1 & \\ x t_2/t_1 & 1 \end{pmatrix} w_0 \tag{2.7}$$

and $f^\circ_{\mathbf{z}}\big(w_0 t w_0 \big(\begin{smallmatrix} 1 & \\ xt_2/t_1 & 1 \end{smallmatrix}\big) w_0\big) = f^\circ_{\mathbf{z}}(w_0 t w_0) f^\circ_{\mathbf{z}}\big(\big(\begin{smallmatrix} 1 & \\ xt_2/t_1 & 1 \end{smallmatrix}\big)\big)$ where $f^\circ_{\mathbf{z}}(w_0 t w_0) = |t_2|_p^{\mu_1 + \frac{1}{2}} |t_1|_p^{\mu_2 - \frac{1}{2}}$.

Thus, $W^\circ_{\mathbf{z}}(t)$ equals

$$|t_2|_p^{\mu_1 + \frac{1}{2}} |t_1|_p^{\mu_2 - \frac{1}{2}} \int\limits_{\mathbb{Q}_p} f_{\mathbf{z}}\Big(\begin{pmatrix} 1 & \\ xt_2/t_1 & 1 \end{pmatrix}\Big) \mathbf{e}(x)\, dx = |t_2|_p^{\mu_1 - \frac{1}{2}} |t_1|_p^{\mu_2 + \frac{1}{2}} \int\limits_{\mathbb{Q}_p} f_{\mathbf{z}}\Big(\begin{pmatrix} 1 & \\ x' & 1 \end{pmatrix}\Big) \mathbf{e}(x' t_1/t_2)\, dx' \tag{2.8}$$

where we have made the substitution $x' = xt_2/t_1$.

If $x' \in \mathbb{Z}_p$ then the argument of $f^\circ_{\mathbf{z}}$ is in $\mathrm{GL}_2(\mathbb{Z}_p)$ on which $f^\circ_{\mathbf{z}}$ is trivial. Otherwise, $x'^{-1} \in \mathbb{Z}_p$
(B) and we make the following, so called, *p-adic Iwasawa decomposition* with factors in $N(F)$,

$T(F)$ and $\mathrm{GL}_2(\mathbb{Z}_p)$ respectively.

$$\begin{pmatrix} 1 & \\ x' & 1 \end{pmatrix} = \begin{pmatrix} 1 & x'^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} x'^{-1} & \\ & x' \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & x'^{-1} \end{pmatrix} \tag{2.9}$$

Since $\rho = (1,0)$ we have that $f^\circ_{\mathbf{z}}(\left(\begin{smallmatrix} x'^{-1} & \\ & x' \end{smallmatrix}\right)) = \left|x'^{-1}\right|_p^{\mu_1+\frac{1}{2}} \left|x'\right|_p^{\mu_2-\frac{1}{2}} = |x'|_p^{\mu_2-\mu_1-1}$ and thus

$$f^\circ_{\mathbf{z}}\left(\begin{pmatrix} 1 & \\ x' & 1 \end{pmatrix}\right) = \begin{cases} 1 & \text{if } x' \in \mathbb{Z}_p \\ |x'|_p^{\mu_2-\mu_1-1} & \text{otherwise} \end{cases} \tag{2.10}$$

We can thus split up the integral as follows

$$\int\limits_{\mathbb{Q}_p} f^\circ_{\mathbf{z}}\left(\begin{pmatrix} 1 & \\ x' & 1 \end{pmatrix}\right)\mathbf{e}(x't_1/t_2)\, dx' = \int\limits_{\mathbb{Z}_p} \mathbf{e}(x't_1/t_2)\, dx' + \int\limits_{\mathbb{Q}_p\backslash\mathbb{Z}_p} \left|x'\right|_p^{\mu_2-\mu_1-1} \mathbf{e}(x't_1/t_2)\, dx' \tag{2.11}$$

We have that $t_1/t_2 = p^{\lambda_1-\lambda_2}$ where $\lambda_1 - \lambda_2 \geq 0$. Using Proposition 2.1, the first integral in (2.8) becomes

$$\int\limits_{\mathbb{Z}_p} \mathbf{e}(x'p^{\lambda_1-\lambda_2})\, dx' = 1 \tag{2.12}$$

(C) Since $\mathbb{Q}_p\backslash\mathbb{Z}_p = \bigsqcup_{k=1}^\infty p^{-k}\mathbb{Z}_p^\times$ we get that the second integral in (2.8) becomes

$$\int\limits_{\mathbb{Q}_p\backslash\mathbb{Z}_p} |x|_p^{\mu_2-\mu_1-1}\, \mathbf{e}(xp^{\lambda_1-\lambda_2})\, dx = \sum_{k=1}^\infty p^{k(\mu_2-\mu_1-1)} \int\limits_{p^{-k}\mathbb{Z}_p^\times} \mathbf{e}(xp^{\lambda_1-\lambda_2})\, dx \tag{2.13}$$

Note that since $\mu_i = -\log z_i/\log p$ we have that $p^{k\mu_i} = p^{-k\log z_i/\log p} = e^{-k\log z_i} = z_i^{-k}$. Thus, we get the prefactor $p^{k(\mu_2-\mu_1-1)} = z_2^{-k} z_1^k p^{-k}$

$$\sum_{k=1}^\infty \left(\frac{z_1}{pz_2}\right)^k \int\limits_{p^{-k}\mathbb{Z}_p^\times} \mathbf{e}(xp^{\lambda_1-\lambda_2})\, dx \tag{2.14}$$

(D) We have that $\mathbb{Z}_p^\times = \mathbb{Z}_p\backslash p\mathbb{Z}_p$. Thus, with $l = \lambda_1 - \lambda_2$,

$$\int\limits_{p^{-k}\mathbb{Z}_p^\times} \mathbf{e}(p^l x)\, dx = \int\limits_{p^{-k}\mathbb{Z}_p} \mathbf{e}(p^l x)\, dx - \int\limits_{p^{-k+1}\mathbb{Z}_p} \mathbf{e}(p^l x)\, dx \tag{2.15}$$

Using Proposition 2.1 we get that (2.14) equals

$$\sum_{k=1}^\infty \left(\frac{z_1}{pz_2}\right)^k \left\{\begin{matrix} p^k(1-\frac{1}{p}) & \text{if } \lambda_1-\lambda_2-k \geq 0 \\ -p^{k-1} & \text{if } \lambda_1-\lambda_2-k = -1 \\ 0 & \text{if } \lambda_1-\lambda_2-k < -1 \end{matrix}\right\} = -\frac{1}{p}\left(\frac{z_1}{z_2}\right)^{\lambda_1-\lambda_2+1} + (1-\tfrac{1}{p}) \sum_{k=1}^{\lambda_1-\lambda_2} \left(\frac{z_1}{z_2}\right)^k \tag{2.16}$$

(E) In summary, by reorganizing the sums we get that,

$$\begin{aligned} W_{\mathbf{z}}^{\circ}(t) &= p^{\frac{1}{2}(\lambda_2-\lambda_1)} z_1^{\lambda_2} z_2^{\lambda_1} \left(1 - \frac{1}{p}\left(\frac{z_1}{z_2}\right)^{\lambda_1-\lambda_2+1} + (1-\tfrac{1}{p}) \sum_{k=1}^{\lambda_1-\lambda_2} \left(\frac{z_1}{z_2}\right)^k \right) \\ &= p^{\frac{1}{2}(\lambda_2-\lambda_1)} \left(\sum_{k=0}^{\lambda_1-\lambda_2} z_1^{\lambda_2+k} z_2^{\lambda_1-k} - \frac{1}{p}\left(\frac{z_1}{z_2}\right) \sum_{k=0}^{\lambda_1-\lambda_2} z_1^{\lambda_2+k} z_2^{\lambda_1-k} \right) \\ &= p^{\frac{1}{2}(\lambda_2-\lambda_1)} \left(1 - p^{-1}\frac{z_1}{z_2}\right) \sum_{k=0}^{\lambda_1-\lambda_2} z_1^{\lambda_2+k} z_2^{\lambda_1-k} \end{aligned} \tag{2.17}$$

which agrees with (2.6) where indeed $s_{(\lambda_1,\lambda_2)}(z_1,z_2) = \sum_{k=0}^{\lambda_1-\lambda_2} z_1^{\lambda_2+k} z_2^{\lambda_1-k}$. □

We now summarize the steps:

(A) We move the torus element to the front conjugated by w_0, and the remaining w_0 to the back so that we may use the transformation properties of $f_{\mathbf{z}}^{\circ}$.

(B) The remaining argument of $f_{\mathbf{z}}^{\circ}$ is lower triangular. If the argument is not in $\mathrm{GL}_2(\mathbb{Z}_p)$ on which $f_{\mathbf{z}}^{\circ}$ is trivial we have to factorize it further and pick up the remaining torus element, again using the transformation properties of $f_{\mathbf{z}}^{\circ}$. The integral is then split up into two pieces depending on whether the argument is already in $\mathrm{GL}_2(\mathbb{Z}_p)$ or if we need to make the factorization.

(C) The first piece was evaluated by Proposition 2.1, but for the second piece we need to split up the integration domain further into "shells of equal p-adic norm".

(D) Each such shell is the difference of two integrals that can be computed by Proposition 2.1. Because of the condition for this integral to be non-vanishing, we get a bound for which shells can contribute.

(E) Putting it all together and reorganizing the sums the different contributing shells become exactly the terms in the Schur polynomials.

Lecture 3

Last time we computed the spherical Whittaker functions on $GL_2(\mathbb{Q}_p)$ by directly computing the p-adic integrals.

Today we will return to the general case $GL_r(F)$ where F is a non-archimedean field and describe how the spherical Whittaker coefficients can be expressed as the partition function of a two dimensional solvable lattice model.

3.1 The spherical lattice model

We will now construct the lattice model whose partition function gives the spherical Whittaker vectors. The lattice model is also sometimes called the *Tokuyama model* or the *spherical model*.

Construct a grid as in Figure 1 where we label the rows from 1 to r in increasing order from top to bottom and the columns from 0 to some sufficiently large N increasing from right to left. Our results will not depend on N.

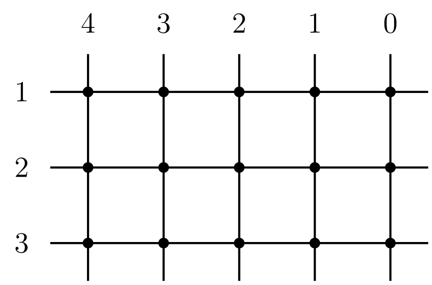


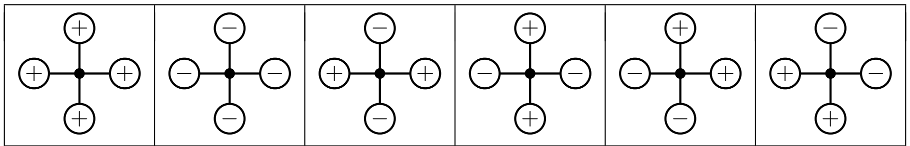
Figure 1: Lattice with numbering of rows and columns and vertex marked as dots.

Each crossing is called a vertex and each of the four lines bordering a vertex is called an edge. To each edge we assign one of two edge states: $\oplus$ or $\ominus$. But we may only make assignments according to the following rules. Let $\rho = (r-1, r-2, \ldots, 0)$.

1. The edges around a vertex must match one of the allowed configurations shown in Table 1.
2. The *boundary edges* must be assigned edge states according to the following rules:
   (a) The left and bottom boundary edges must be $\oplus$.
   (b) The right boundary edges must be $\ominus$.
   (c) For a given partition λ of length r we assign $\ominus$ to the columns with labels $\lambda_i + \rho_i$ on the top boundary. The remaining top boundary edges are chosen to be $\oplus$.

Note that, while the boundary is fixed by λ there are generally several ways to fill in the interior while still satisfying rule 1. A configuration of allowed edges for a given boundary is called a lattice state, or *state* for short. The set of states for a given λ is called an *ensemble* and will be denoted by $\mathfrak{S}_\lambda$. Note also that if three edge states around a vertex is given,

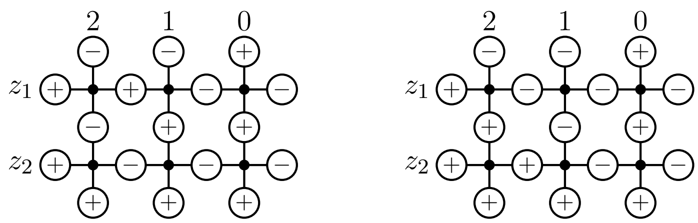
Table 1: Allowed vertex configurations and their Boltzmann weights.



1	z	$-v$	z	$(1-v)z$	1

there is at most one allowed vertex configuration for that vertex (or, equivalently, at most one allowed edge state for the remaining edge). Finally, note that the number of $\ominus$ at the north and west edges of a vertex equal the number of $\ominus$ at the south and east edges — we have a conservation of $\ominus$. A similar conservation law follows for whole rows of vertices as well.

We show an example of all possible states given $\lambda = (1,1)$ in Figure 2. Exercise: show that we cannot have a state where the vertical edges between the horizontal rows 1 and 2 are $\oplus\oplus\ominus$ from left to right.



$$\begin{bmatrix} -v & 1 & z_1 \\ 1 & z_2 & z_2 \end{bmatrix} \qquad \begin{bmatrix} 1 & z_1 & z_1 \\ 1 & 1 & z_2 \end{bmatrix}$$

Figure 2: The possible states for $\lambda = (1,1)$, i.e. $\lambda + \rho = (2,1)$ together with the Boltzmann weight for each vertex underneath in a similar grid pattern.

Each row of horizontal edges, labeled by the row number i is assigned a complex variable z_i and each vertex x in this row is given a weight $\beta(x) \in \mathbb{C}$ according to the second row in Table 1 with $z = z_i$ and v a parameter which we will eventually set as $v^{-1} = q = |\mathfrak{o}/\mathfrak{p}|$ (which was equal to p for $F = \mathbb{Q}_p$ in Lecture 2). These weights are called *Boltzmann weights*. A lattice state s is then given a total weight which is the product of its vertex weights

$$\beta(s) = \prod_{\text{vertex } x \in s} \beta(x) \tag{3.1}$$

which is a function in $z_1, \ldots, z_r$. The weight for each state in Figure 2 is shown beneath the state.

Given a partition λ, which specifies an ensemble $\mathfrak{S}_\lambda$ we can define the associated partition function Z_λ° as

$$Z_\lambda(z_1, \ldots, z_r) = \sum_{s \in \mathfrak{S}_\lambda} \beta(s) \tag{3.2}$$

We note that increasing N only adds extra factors of 1 and thus does not change the weights of the states nor the partition function.

For the example shown in Figure 2 we get that

$$Z^{\circ}_{(1,1)}(z_1,z_2) = -vz_1z_2^2 + z_1^2z_2 = z_1\Big(1 - v\frac{z_2}{z_1}\Big)s_{(1,1)}(z_1,z_2) \tag{3.3}$$

where $s_{(1,1)}(z_1,z_2) = z_1z_2$.

In general we have the following equality

$$Z^{\circ}_{\lambda}(\mathbf{z}) = \mathbf{z}^{\rho}\prod_{i<j}\Big(1 - q^{-1}\frac{z_j}{z_i}\Big)s_{\lambda}(\mathbf{z}) = \mathbf{z}^{\rho}\delta^{-1/2}(\varpi^{\lambda})W^{\circ}_{w_0\mathbf{z}}(\varpi^{\lambda}) \tag{3.4}$$

We will postpone the proof/motivation of this statement to Lecture 5.

Lecture 4

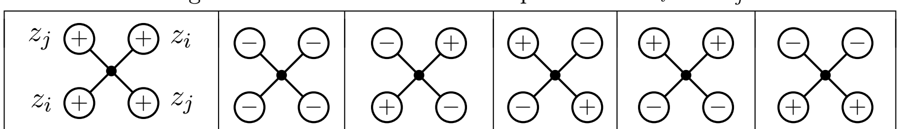
Last time we introduced a lattice model whose partition function we claimed computed the spherical Whittaker functions of $\mathrm{GL}_r(F)$ which are given by Schur polynomials. We will prove this statement in the next lecture.

Today we will introduce an important tool that will be used in that proof called the Yang-Baxter equation; a tool used in statistical mechanics to 'solve' lattice models. If there is time we will also discuss another important step of the proof which is to split up the spherical Whittaker function into smaller pieces called Iwahori Whittaker functions.

4.1 The Yang-Baxter equation

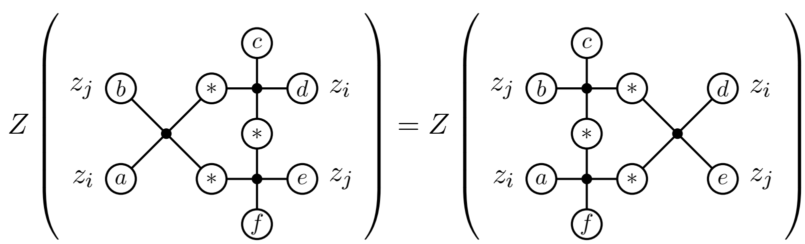
In the Yang-Baxter equation we introduce another type of vertices shown in Table 2. These are called *R-matrix configurations* and should be interpreted as mixing two horizontal rows of edges in the lattice model.

Table 2: R-matrix configurations. The location of the parameters z_i and z_j follow the first entry.



$z_j - vz_i$	$z_i - vz_j$	$v(z_i - z_j)$	$z_i - z_j$	$(1-v)z_i$	$(1-v)z_j$

The Yang-Baxter equation for this system can then be expressed as



(4.1)

where the interior states $\circledast$ are summed over. It is customary to drop the function Z and just write equality of the systems. The validity of this equation can easily be checked either by hand (tedious) or writing a simple computer program.

Note that the Yang-Baxter equation also holds as part of a bigger system, and that we may use it repeatedly to move the R-matrix further and further to the left.

As a warmup exercise for what we will do in the next lecture let us now apply the Yang-Baxter equation to the partition function $Z_\lambda(\mathbf{z})$ defined in the previous lecture. A schematical illustration of the procedure is shown in Figure 3.

We start by attaching an R-matrix between rows k and $k+1$ to the left boundary (which consists of $\oplus$). By this we mean that we impose $a = b = \oplus$ in (4.1) and assign the remaining edge states according to the rules of the previous lecture, with the following exception. We

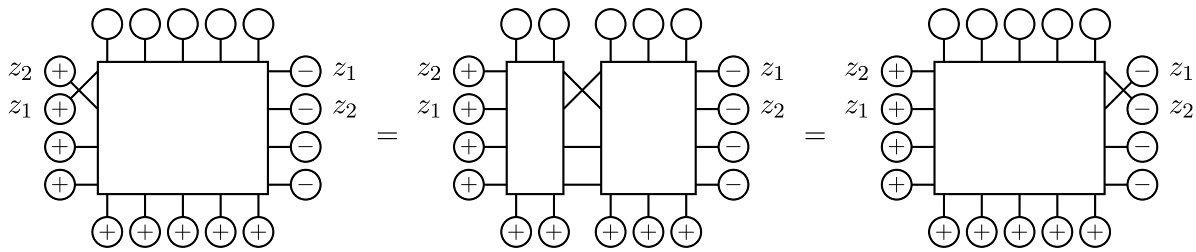


Figure 3: Repeated use of the Yang-Baxter equation on the partition function. The boxes correspond to areas of interior edge states that are summed over.

should now sum over the two edge states that used to belong to the boundary, but are now attached to the R-matrix; in a way, they have become internal edge states.

Looking at Table 2 we see that there is only one R-matrix element (the first entry) contributing in the left-hand side of (4.1), with weight $z_{k+1} - vz_k$. The old boundary for Z_λ thus remain unchanged.

With repeated use of (4.1) in the interior of $Z_\lambda(\mathbf{z})$ we end up with a right-hand side where the R-matrix is attached to the right boundary (which consists of $\ominus$). By the right-hand side of the Yang-Baxter equation we need to impose $d = e = \ominus$. This only gives one contribution (from the second entry in Table 2) of weight $z_k - vz_{k+1}$, and yet again the old boundary for $Z_\lambda(\mathbf{z})$ is unchanged. By this manipulation we note from (4.1) that z_k and z_{k+1} have switched places.

We get that

$$(z_{k+1} - vz_k)Z_\lambda(\mathbf{z}) = (z_k - vz_{k+1})Z_\lambda(w_k\mathbf{z}) \tag{4.2}$$

where w_k swaps z_k and z_{k+1}.

We have that

$$w_k\mathbf{z}^\rho = \frac{z_{k+1}}{z_k}\mathbf{z}^\rho \qquad w_k\Big[\prod_{i<j}\Big(1 - v\frac{z_j}{z_i}\Big)\Big] = \frac{\Big(1 - v\frac{z_k}{z_{k+1}}\Big)}{\Big(1 - v\frac{z_{k+1}}{z_k}\Big)}\prod_{i<j}\Big(1 - v\frac{z_j}{z_i}\Big) \tag{4.3}$$

Thus, inserting (3.4) into (4.2) we get that

$$\begin{aligned}\mathbf{z}^\rho\prod_{i<j}\Big(1 - v\frac{z_j}{z_i}\Big)s_\lambda(\mathbf{z}) = Z_\lambda(\mathbf{z}) &= \frac{z_k - vz_{k+1}}{z_{k+1} - vz_k}Z_\lambda(w_k\mathbf{z}) \\ &= \underbrace{\frac{z_k - vz_{k+1}}{z_{k+1} - vz_k}\frac{z_{k+1}}{z_k}\frac{\Big(1 - v\frac{z_k}{z_{k+1}}\Big)}{\Big(1 - v\frac{z_{k+1}}{z_k}\Big)}}_{=1}\mathbf{z}^\rho\prod_{i<j}\Big(1 - v\frac{z_j}{z_i}\Big)s_\lambda(w_k\mathbf{z})\end{aligned} \tag{4.4}$$

Thus, we have used the Yang-Baxter equation to show that $s_\lambda(w_k\mathbf{z}) = s_\lambda(\mathbf{z})$, that is, that s_λ is symmetric.

4.2 Iwahori Whittaker functions

For simplicity we will restrict to $F = \mathbb{Q}_p$, but the general case follows analogously.

We recall from Lectures 1 and 2 that the spherical Whittaker function was defined as

$$W_{\mathbf{z}}^{\circ}(g) = \int_{N(\mathbb{Q}_p)} f_{\mathbf{z}}^{\circ}(w_0 n g)\psi^{-1}(n)\, dn \qquad f_{\mathbf{z}}^{\circ}(ntk) = \left|t^{\bar{\rho}}\right|_p \prod_i z_i^{\mathrm{ord}_p(t_i)} \tag{4.5}$$

where t is a torus element, n upper triangular and $k \in K = \mathrm{GL}_r(\mathbb{Z}_p)$.

Instead of considering functions right-invariant under K, we will relax and only require invariance under a subgroup of K called the Iwahori subgroup. The Iwahori subgroup J is the elements of $\mathrm{GL}_r(\mathbb{Z}_p)$ which are upper triangular mod p.

One can show that (see for example arXiv:1002.2996)

$$\mathrm{GL}_r(\mathbb{Q}_p) = \bigsqcup_{w\in W} B(\mathbb{Q}_p) w J \qquad K = \mathrm{GL}_r(\mathbb{Z}_p) = \bigsqcup_{w\in W} B(\mathbb{Z}_p) w J \tag{4.6}$$

Using this decomposition we define the so called *Iwahori standard basis* elements for $g = bw'j$ with $b \in B(\mathbb{Q}_p)$ and $j \in J$ as

$$f_{\mathbf{z}}^{w}(bw'j) = \begin{cases} f_{\mathbf{z}}^{\circ}(b) & \text{if } w = w' \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Let $f(g) = \sum_{w\in W} f_{\mathbf{z}}^{w}(g)$. We can decompose $g = bk$ and $k = \hat{b}w'j$ where $\hat{b} \in B(\mathbb{Z}_p) \subset K$. Then $f(bk) = f(b\hat{b}w'j) = f_{\mathbf{z}}^{w'}(b\hat{b}) = f_{\mathbf{z}}^{\circ}(b\hat{b}) = f_{\mathbf{z}}^{\circ}(b)$ which shows that $\sum_{w\in W} f_{\mathbf{z}}^{w}(g)$ is a spherical function and indeed equal to $f_{\mathbf{z}}^{\circ}$.

We also define the corresponding Iwahori Whittaker function as

$$W_{\mathbf{z}}^{w}(g) = \int_{N(\mathbb{Q}_p)} f_{\mathbf{z}}^{w}(w_0 n g)\psi^{-1}(n)\, dn\,. \tag{4.8}$$

Now the spherical Whittaker function can now, similar to above, be expressed as

$$W_{\mathbf{z}}^{\circ}(g) = \sum_{w\in W} W_{\mathbf{z}}^{w}(g)\,. \tag{4.9}$$

Instead of being related to Schur polynomials like $W_{\mathbf{z}}^{\circ}$, the Iwahori Whittaker functions $W_{\mathbf{z}}^{w}$ are related to non-symmetric variants of Hall-Littlewood polynomials (a limit of non-symmetric Macdonald polynomials).

Lecture 5

In Lecture 3 we claimed that the partition function of the spherical lattice model is a spherical Whittaker function as detailed in (3.4). In Lecture 4 we described the Yang-Baxter equation for the spherical lattice model and described a generalization of spherical Whittaker functions which are called Iwahori Whittaker functions.

Today we will give (a sketch of) a proof of (3.4) following [BBBG19]. To match the notation therein we will slightly rewrite the right-hand side using $s_{-w_0\lambda}(\mathbf{z}^{-1}) = s_\lambda(w_0\mathbf{z}) = s_\lambda(\mathbf{z})$ which can be shown from the determinant formula. For simplicity we will still work with $F = \mathbb{Q}_p$. We thus want to show that

$$Z^\circ_\lambda(\mathbf{z}) = \mathbf{z}^\rho \delta^{-1/2}(p^{-w_0\lambda}) W^\circ_{\mathbf{z}^{-1}}(p^{-w_0\lambda})\,. \tag{5.1}$$

where $p^\lambda = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_r})$. Throughout this lecture we will let $v = p^{-1}$.

The strategy is to construct a new *Iwahori lattice model* and use its associated Yang-Baxter equations to show that this new partition function $Z^w_\lambda(\mathbf{z})$ equals $W^w_\lambda(\mathbf{z})$ (up to normalization). The Iwahori lattice model will be constructed in such a way that $\sum_w Z^w_\lambda(\mathbf{z}) = Z^\circ_\lambda(\mathbf{z})$, thus proving (5.1).

5.1 Some facts about Iwahori Whittaker functions

Before we introduce the Iwahori lattice model and prove (5.1) we will need the following facts which are proven in [BBBG19].

Let us denote

$$\phi^w_\lambda(\mathbf{z}) = \delta^{-1/2}(p^{-w_0\lambda}) W^w_{\mathbf{z}^{-1}}(p^{-w_0\lambda})\,. \tag{5.2}$$

Fact 1: $\phi^1_\lambda(\mathbf{z}) = \mathbf{z}^\lambda$

Let

$$\mathfrak{T}_k f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_k\mathbf{z})}{\mathbf{z}^{\alpha_k} - 1} - v\frac{f(\mathbf{z}) - \mathbf{z}^{-\alpha_k} f(s_k\mathbf{z})}{\mathbf{z}^{\alpha_k} - 1}\,. \tag{5.3}$$

Fact 2:

$$\phi^{s_k w}_\lambda(\mathbf{z}) = \begin{cases} \mathfrak{T}_k \phi^w_\lambda(\mathbf{z}) & \text{if } \ell(s_k w) > \ell(w), \\ \mathfrak{T}_k^{-1} \phi^w_\lambda(\mathbf{z}) & \text{if } \ell(s_k w) < \ell(w). \end{cases} \tag{5.4}$$

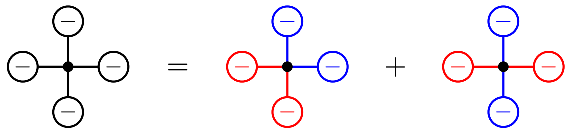
A special case of the this fact was first proven in [BBL15].

Fact 3: We recall from Lecture 4 that

$$W^\circ_{\mathbf{z}}(g) = \sum_{w\in W} W^w_{\mathbf{z}}(g) \tag{5.5}$$

5.2 Iwahori lattice model

The Iwahori lattice model is very similar to the spherical model we introduced in Lecture 3 but with a few important distinctions: Each $\ominus$ will be replaced by a color from an ordered palette of r colors (for GL_r). This will introduce some extra possible vertex configurations and states. Recall that the $\ominus$ in the spherical model traces out different paths in the lattice (going from north-west to south-east). We construct the boundaries and admissible states in such a way that each path is now assigned its unique color starting with a certain order of colors on the top boundary. For weights where two paths cross we ensure that the sum over possibilities (with fixed north and west input edges) matches the weight for the spherical model:

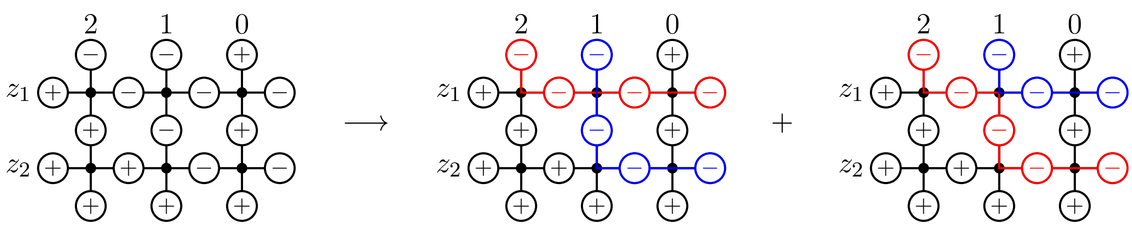


(5.6)

The actual weights can be found in [BBBG19], but we will not need to display all of them here.

For the Iwahori model we will specify an order of the colors on the right boundary (read from top to bottom) as a permutation w of the order of colors on the top boundary (read from left to right). We then define $Z_\lambda^w(\mathbf{z})$ to be the partition function with boundary given by λ for the top column positions and w for the order of the colors on the right boundary.

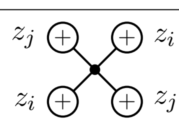
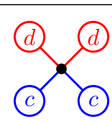
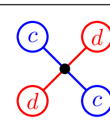
The state on the right of Figure 2 splits up into two cases:



$w = 1$ $\qquad$ $w = s_1$

There is a new Yang-Baxter equation for this model which we will use to relate different partition functions. Similar to arguments in Lecture 4, we will only need the R-matrix vertices shown in Table 3. The rest can be found in [BBBG19, Fig 9]

Table 3: R-matrix configurations for the Iwahori model. The colors c and d satisfy $c > d$ if the c comes before b in the order of colors on the top boundary read from left to right.

			...
$z_j - vz_i$	$\begin{cases}(1-v)z_i & \text{if } c < d\\(1-v)z_j & \text{if } c > d\end{cases}$	$\begin{cases}z_i - z_j & \text{if } c < d\\v(z_i - z_j) & \text{if } c > d\end{cases}$	...

Theorem 5.1. *The partition function for the Iwahori lattice model computes Iwahori Whittaker functions:*

$$Z^w_\lambda(\mathbf{z}) = \mathbf{z}^\rho \phi^w_\lambda(\mathbf{z}) \tag{5.7}$$

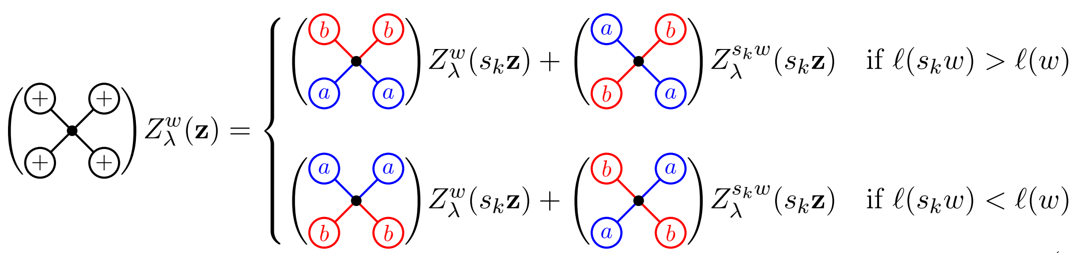
Proof. We will make an iteration over the length of w. The base case is when $w = 1$. In this case the partition function consists of only one term: the first path must go one step down and directly to the right, the second path two steps down and directly to the right etc. just as in the $w = 1$ case in the figure above.

With the full Boltzmann weights from [BBBG19] the partition function is easily computed as

$$Z^1_\lambda(\mathbf{z}) = \mathbf{z}^{\lambda+\rho} = \mathbf{z}^\rho \phi^1_\lambda(\mathbf{z}) \tag{5.8}$$

where the last equality follows from Fact 1.

When (repeatedly) applying the Yang-Baxter equation on rows k and $k+1$ of a general system with partition function $Z^w_\lambda(\mathbf{z})$, the condition $c < d$ is equivalent to $\ell(s_k w) > \ell(w)$. In the same way as in Lecture 4 we get the following functional relation where it is assumed that $a < b$



(5.9)

Applying s_k to this whole expression (acting on the $\mathbf{z}$) and solving for $Z^{s_k w}_\lambda(\mathbf{z})$ one finds that

$$Z^{s_k w}_\lambda(\mathbf{z}) = \begin{cases} T_k Z^w_\lambda(\mathbf{z}) & \text{if } \ell(s_k w) > \ell(w), \\ T_k^{-1} Z^w_\lambda(\mathbf{z}) & \text{if } \ell(s_k w) < \ell(w). \end{cases} \tag{5.10}$$

where $T_k = z^\rho \mathfrak{T}_k z^{-\rho}$.

With a reduced word $w = s_{i_1} \cdots s_{i_m}$ we therefore get that

$$Z^w_\lambda(\mathbf{z}) = T_{i_1} \cdots T_{i_m} Z^1_\lambda(\mathbf{z}) = T_{i_1} \cdots T_{i_m} \mathbf{z}^\rho \phi^1_\lambda(\mathbf{z}) = \mathbf{z}^\rho \mathfrak{T}_{i_1} \cdots \mathfrak{T}_{i_m} \phi^1_\lambda(\mathbf{z}) = \mathbf{z}^\rho \phi^w_\lambda(\mathbf{z}) \tag{5.11}$$

where we in the last step have used Fact 2. □

Corollary 5.2. *It follows from Fact 3 and Theorem 5.1 that*

$$\mathbf{z}^\rho \delta^{-1/2}(p^{-w_0\lambda}) W^\circ_{\mathbf{z}^{-1}}(p^{-w_0\lambda}) = \sum_{w\in W} Z^w_\lambda(\mathbf{z}) \tag{5.12}$$

To prove the relation between the spherical model and the spherical Whittaker function (5.1), it remains to show that $Z^\circ_\lambda(\mathbf{z}) = \sum_{w\in W} Z^w_\lambda(\mathbf{z})$. From how we created the Iwahori model with the procedure pictured in (5.6) this seems plausible, but I have deliberately skipped over some details like the fact that the Yang-Baxter equation requires the existence of extra vertex configurations with vertical edges having more than one color. We call these non-strict states and they do not have a counterpart in the spherical model. The fact that the contributions from these non-strict states cancel out when taking the sum over Weyl words was shown in [BBBG19, §6], but the proof actually used (5.1) to do this.

We believe there should also be a combinatorial proof the cancellation of these states without relying on (5.1), which we would need give an independent proof thereof. The original proof of (5.1) does, of course, not rely on this cancellation and is proved in a completely different manner in [BBF11].

References

[BBBG19] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson, "Colored five-vertex models and Demazure atoms," `arXiv:1902.01795 [math.CO]`.

[BBF11] B. Brubaker, D. Bump, and S. Friedberg, "Schur Polynomials and The Yang-Baxter Equation," *Communications in Mathematical Physics* **308** no. 2, (Oct, 2011) 281, `arXiv:0912.0911 [math.CO]`.

[BBL15] B. Brubaker, D. Bump, and A. Licata, "Whittaker functions and Demazure operators," *J. Number Theory* **146** (2015) 41–68, `arXiv:1111.4230 [math.RT]`.