CHALMERS
UNIVERSITY OF TECHNOLOGY

Eisenstein series attached to small automorphic representations

Henrik Gustafsson

Automorphic forms, mock modular forms and string theory

Simons Center for Geometry and Physics 2016

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
arXiv:1412.5625 [math.NT] [GKP14]
Journal of Number Theory 166 (Sep, 2016) 344–399

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$SL(n)$ E_6, E_7, E_8

Outline

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- Why do string theorists study Eisenstein series?

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- What happens for small automorphic representations?
- What's next?

Motivation

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- Hecke eigenvalues
- Point counts of elliptic curves
- Langlands program
  L-functions | The Langlands–Shahidi method

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  Scattering amplitudes | Black hole microstate counting
- Statistical mechanics
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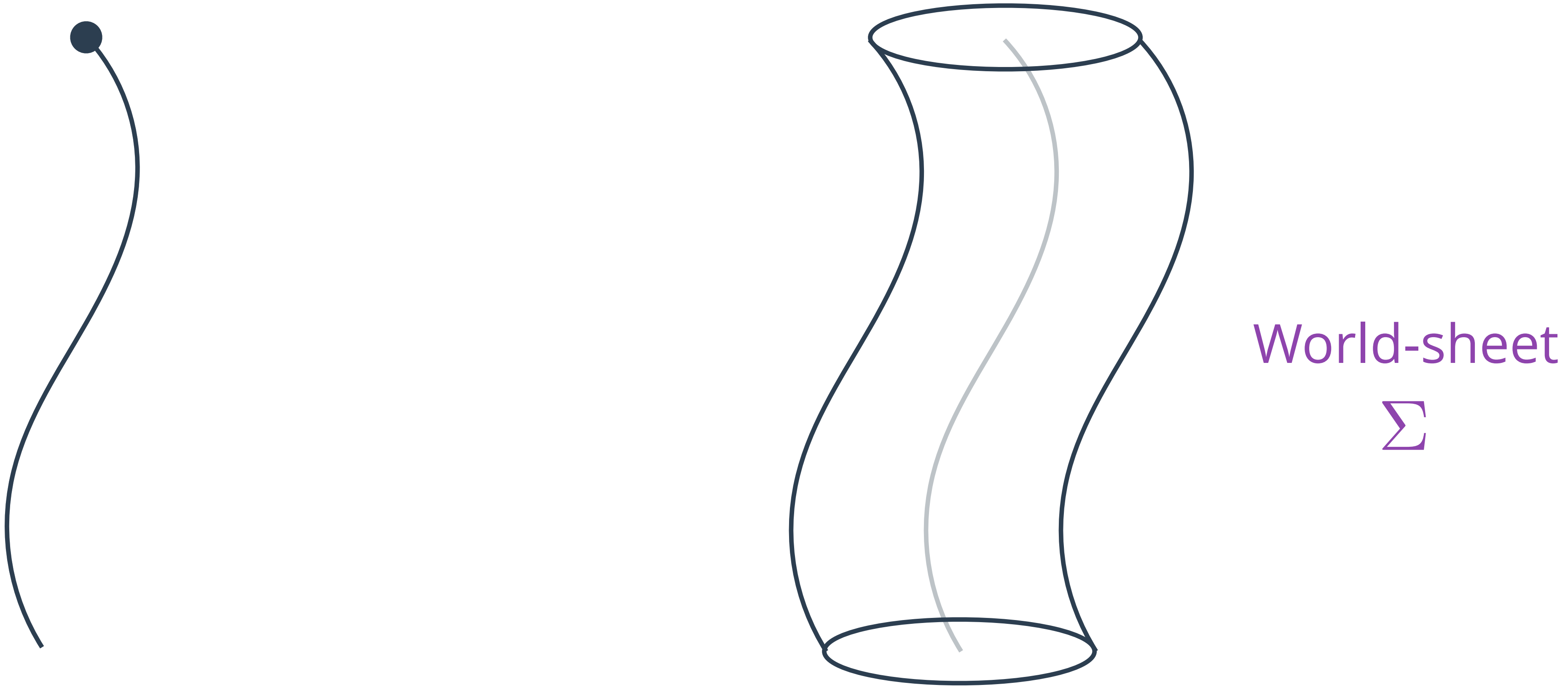

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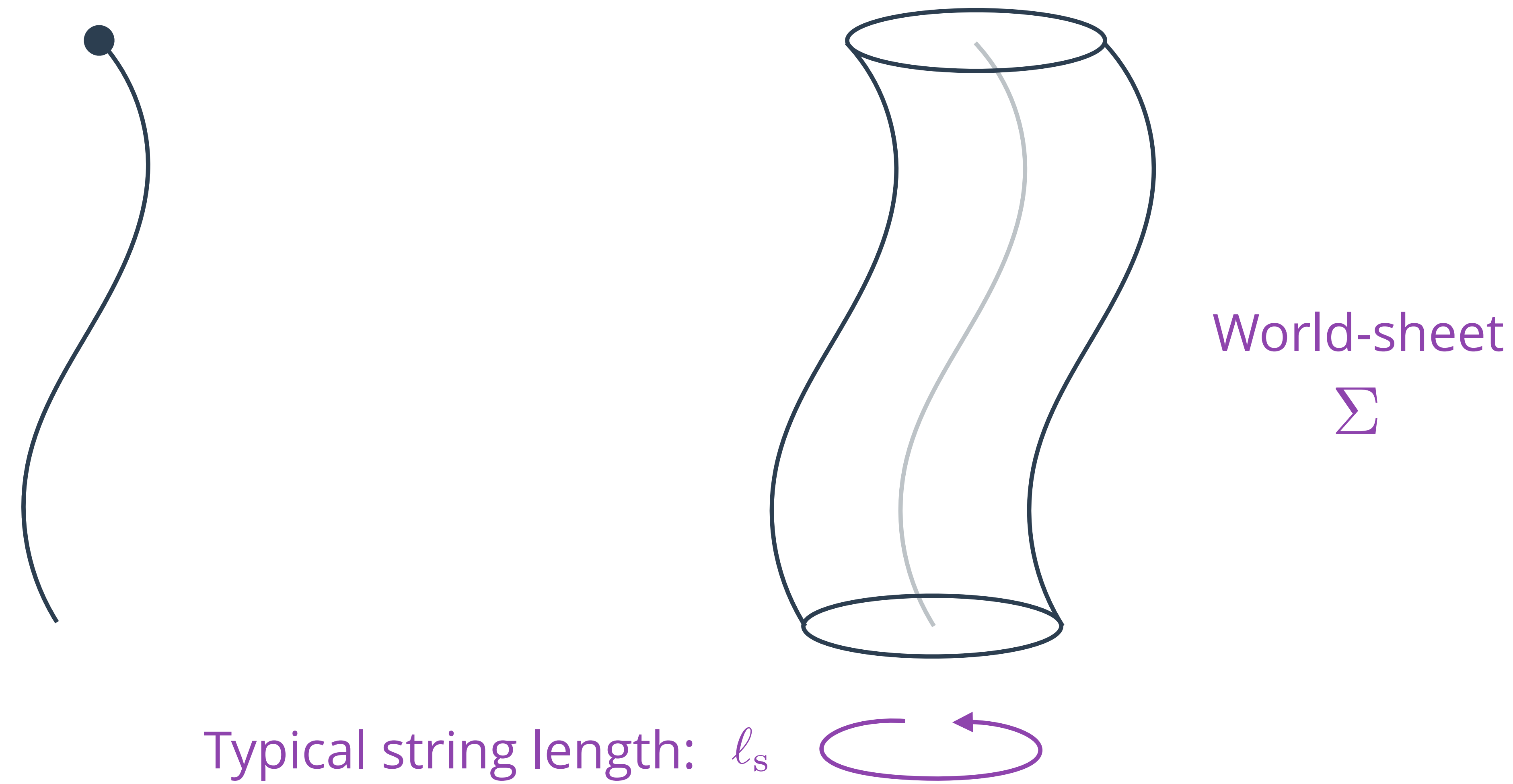

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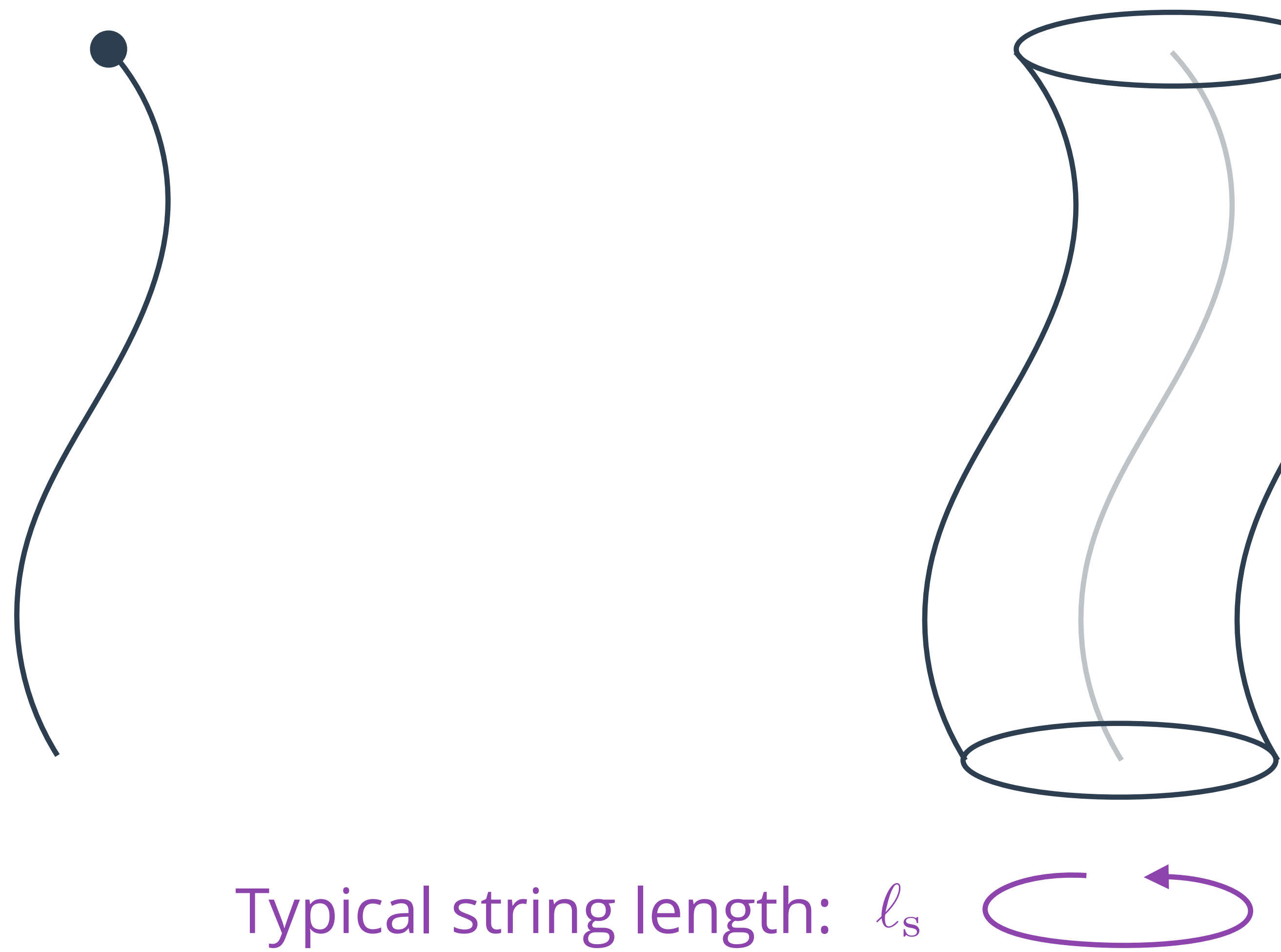

World-sheet
Σ

Typical string length: ℓ_{s}

$\alpha' = \ell_{\mathrm{s}}^2$

String theory

Space-time is described by a Riemannian manifold M

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$$X \quad : \quad \underbrace{\Sigma}_{\text{world-sheet}} \quad \to \quad \underbrace{M}_{\text{space-time}}$$

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We will focus on Type IIB

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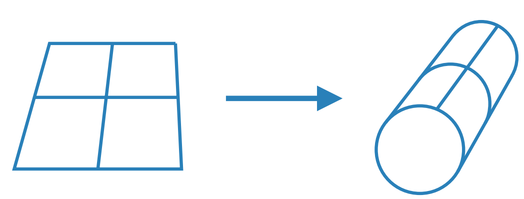
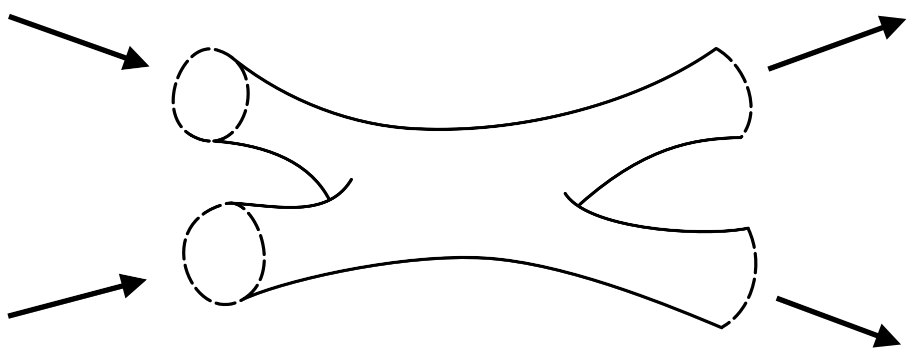
Toroidal compactifications



D dimensions

Interactions



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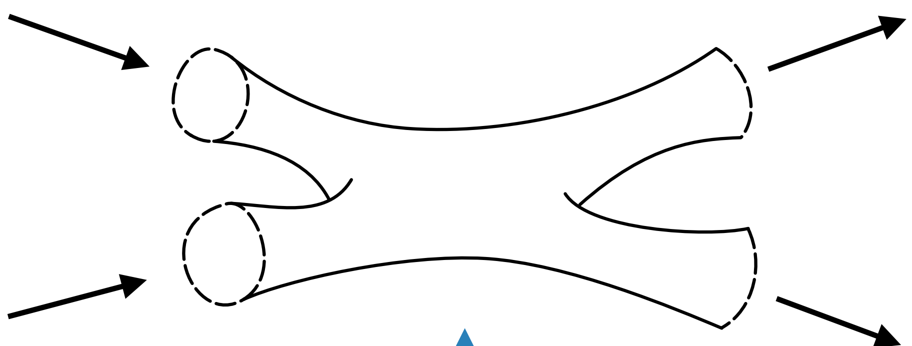
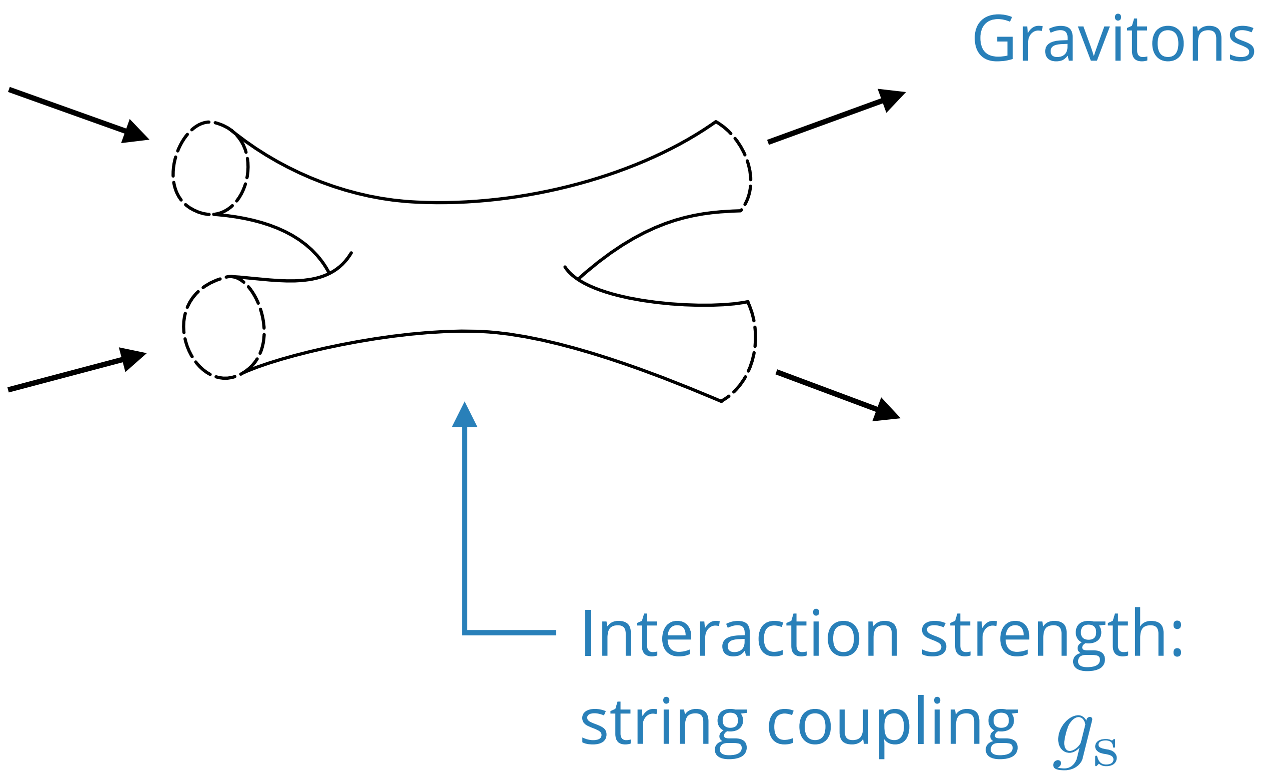


Interaction strength:
string coupling g_{s}

Interactions



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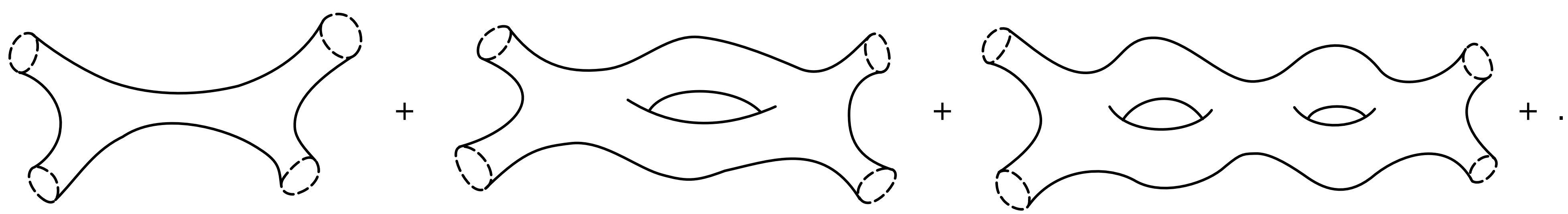
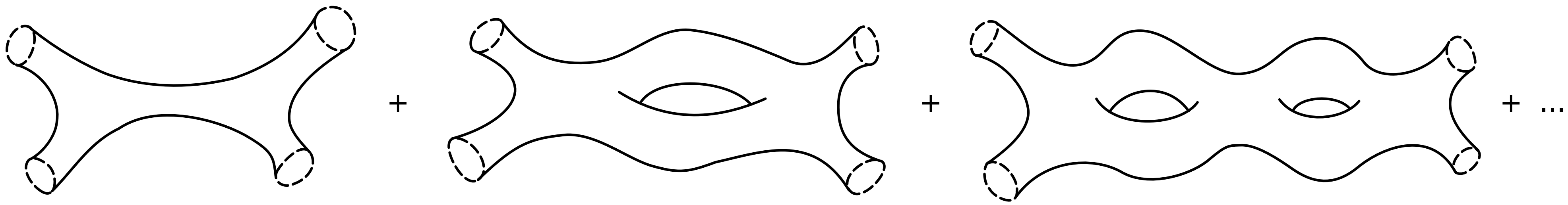


Interactions

Weighted by: $g_{\mathrm{s}}^{-\chi_E}$ $\qquad -\chi_E = 2(\text{genus} - 1) + \text{boundaries}$



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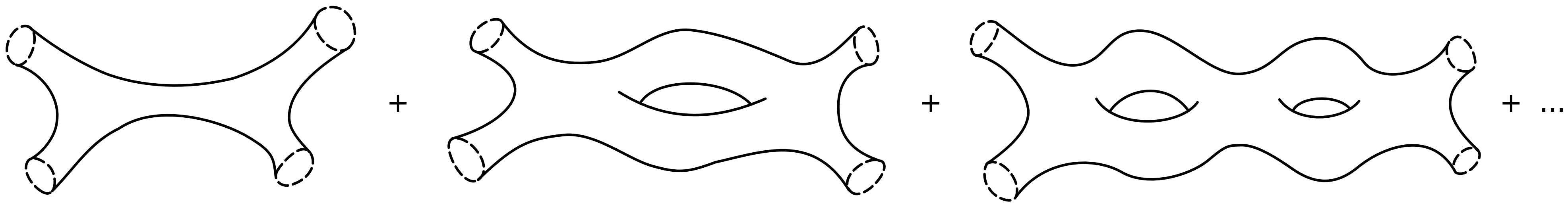

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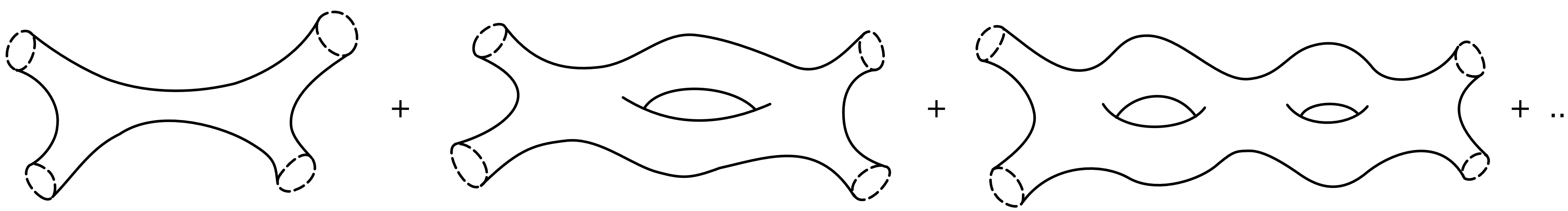

g_s^{-2}

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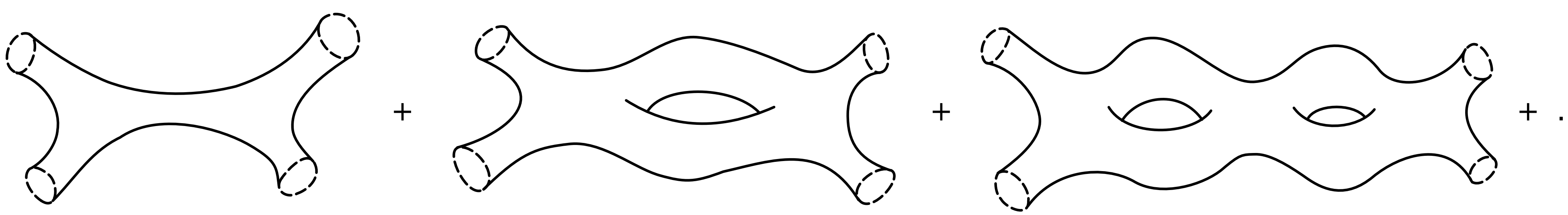

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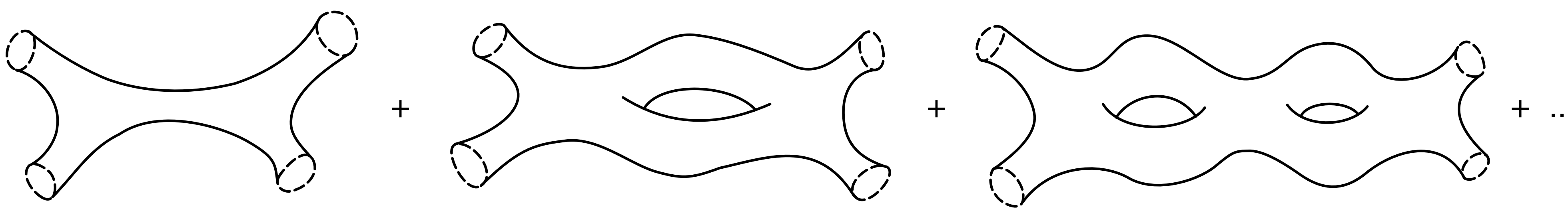

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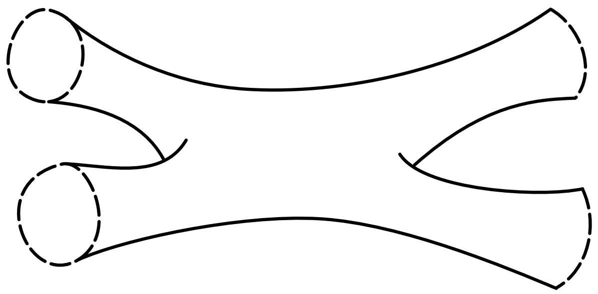
Interactions

Gravitons in D dimensions



Interactions

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Gravitons in *D* dimensions

$$R + (\alpha')^3 \mathcal{E}^{(D)}_{(0,0)}(g) R^4 + (\alpha')^5 \mathcal{E}^{(D)}_{(1,0)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}^{(D)}_{(0,1)}(g) D^6 R^4 + \ldots$$

Expansion parameter

Interactions

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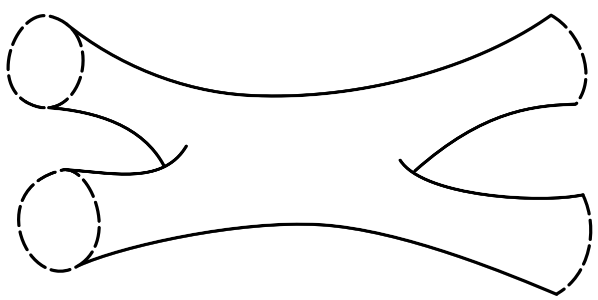
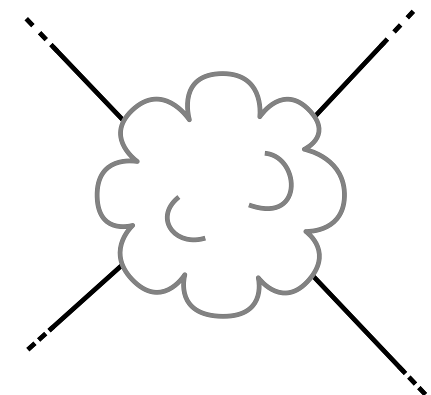


Effective field theory



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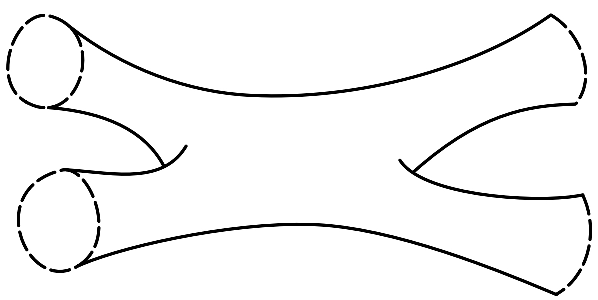

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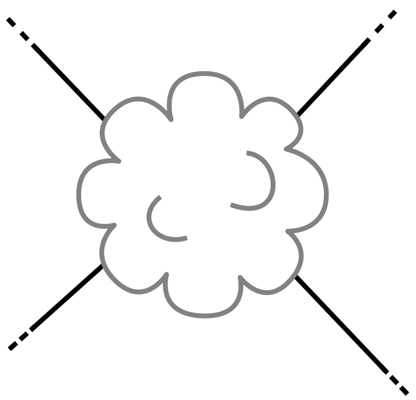

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Contractions of derivatives and 4 Riemann tensors (known)

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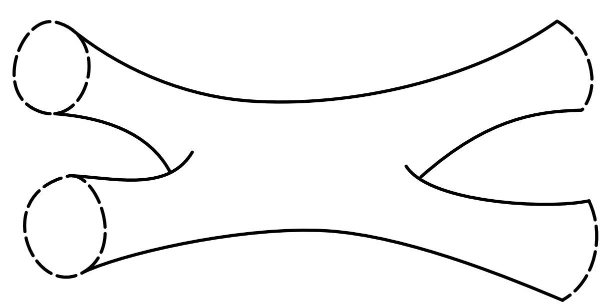

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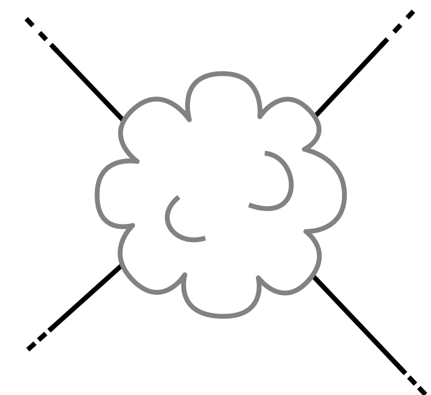

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[Cremmer-Julia]

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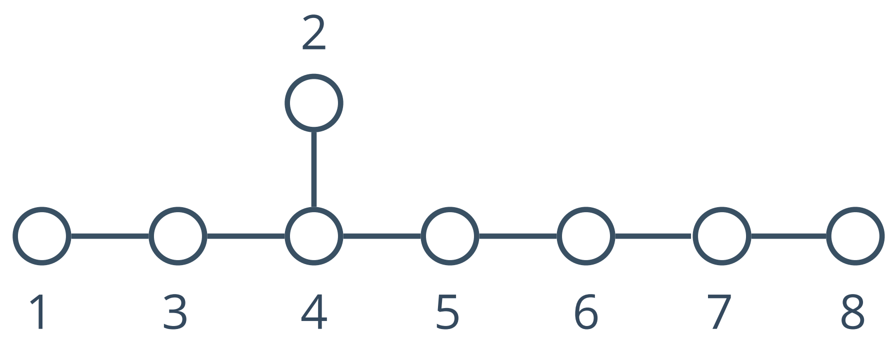

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$$\mathcal{E}_{(p,q)}(\tau) = \mathcal{E}^{(10)}_{(p,q)}(g)$$

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$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

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Quantization of charges

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Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

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All observables are invariant under $G(\mathbb{Z})$

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(D) Growth: for any norm $\|\cdot\|$ on $G(\mathbb{R})$ there exists a positive integer n and constant C such that $|\varphi(g)| \leq C\|g\|^n$

$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$

Automorphic forms

An *automorphic form* is a smooth function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ satisfying the following conditions

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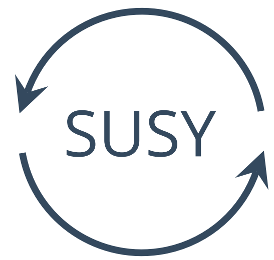
bosons SUSY fermions

10 dimensions: $$\Delta = 4\tau_2^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \overline{\tau}}$$ Laplacian

Supersymmetry constraints

bosons 

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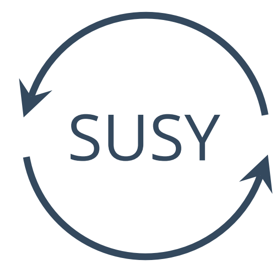
bosons 

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Similarly for lower dimensions

Eisenstein series

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$\tau = \tau_1 + i\tau_2$ $\quad$ $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ $\quad$ $\gamma(\tau) = \dfrac{a\tau + b}{c\tau + d}$

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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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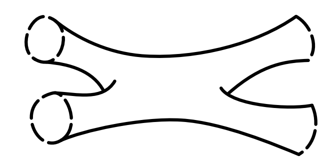

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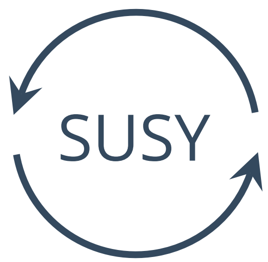

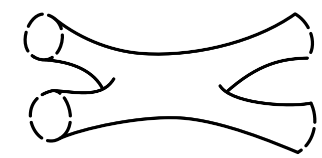

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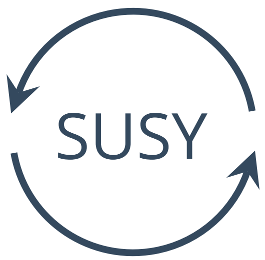

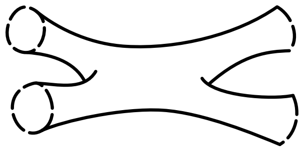

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$\mathcal{E}_{(0,1)}(\tau)$ as a sum over images $\sum_{B(\mathbb{Q})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_{s} $\tau = \chi + i g_{\mathrm{s}}^{-1}$

[Green-Gutperle]

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Interaction strength

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Non-perturbative
(remaining modes)

$e^{-\frac{1}{g_{\mathrm{s}}}}$

[Green-Gutperle]

Extracting physical information

Expand Bessel function in $g_{\rm s}$ $\qquad \tau = \chi + i g_{\rm s}^{-1}$

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Interaction strength

Instanton action

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Perturbative
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$$\sigma_s(m) = \sum_{d\,|\,m} d^s$$

Sums over the number of ways the charge *m* can be factorised into two integers

[Green-Gutperle]

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wrapping number of a T-dual D-particle and charge

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arithmetic information
p-adic part

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[Green-Gutperle]

Lower dimensions

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D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

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6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

$$E(\chi; g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi(\gamma g)$$

Parabolic subgroups

Fourier expand
in different directions $\longleftrightarrow$ Unipotent subgroup U

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$\uparrow$

Choice of parabolic subgroup P

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$$L = \left\{ \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix} \right\} \qquad U = \left\{ \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

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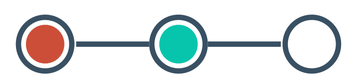

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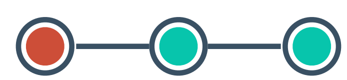

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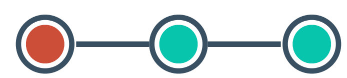

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Minimal parabolic
Borel

$$B = NA$$

$$N = \left\{ \begin{pmatrix} \boxed{1} & * & * & * \\ & \boxed{1} & * & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

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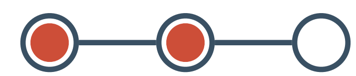

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Fourier expansion

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$$F_U(\chi,\psi;g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi,ug)\overline{\psi(u)}\,du$$

Fourier expansion

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$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

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Fourier expansion

$$E(\chi;g) = F_U(\chi,1;g) + \sum_{\psi^{(1)}\neq 1} F_{U^{(1)}}(\chi,\psi^{(1)};g) + \sum_{\psi^{(2)}\neq 1} F_{U^{(2)}}(\chi,\psi^{(2)};g) + \dots$$

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$$U^{(1)} = U \qquad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

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Iwasawa decomposition

Characters and coefficients with all $m_\alpha \neq 0$ are called generic otherwise they are called degenerate

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

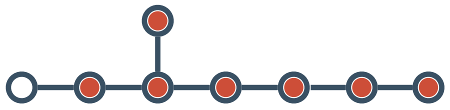
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

- String perturbation limit $g_{\mathrm{s}} \to 0$

  D-instantons | NS5-instantons



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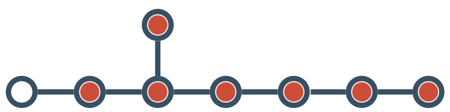
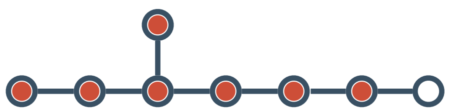


- **Decompactification limit**
  Higher dimensional black holes | BPS states
  Large radius for compactified circle



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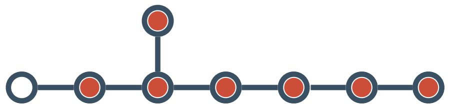

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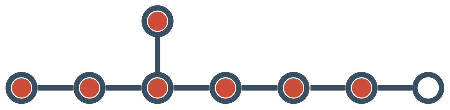
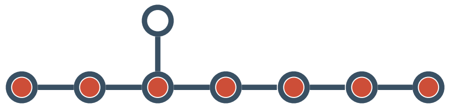


- **M-theory limit**
  M2, M5-instantons
  Large M-theory torus



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  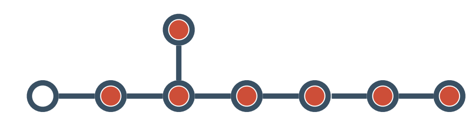

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  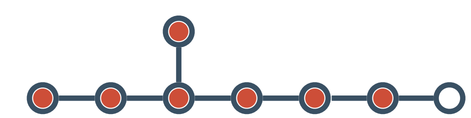

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  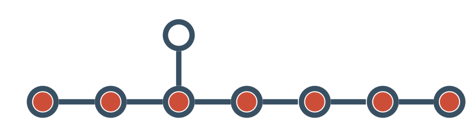

[Green-Miller-Vanhove]

Maximal parabolic subgroups

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  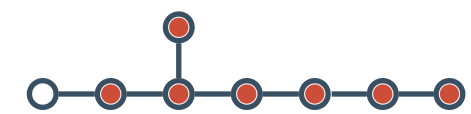

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  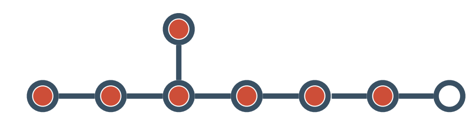

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  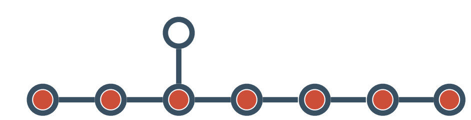

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

Fourier expansion

Goal: find expressions for Fourier coefficients
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Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

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*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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\begin{array}{c}
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$$
\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \text{Fourier coefficient}
\end{tikzcd}
$$

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$$\mathcal{E}^{(D)}_{(0,0)}(g),\ \mathcal{E}^{(D)}_{(1,0)}(g),\ \mathcal{E}^{(D)}_{(0,1)}(g)\ : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

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Lift to the adeles [FGKP15 §4.2.2]

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} \qquad G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p) \qquad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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Adelic framework

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$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

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Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

Adelic framework

Eisenstein series $\longrightarrow$ Adelic Eisenstein series

$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi; ug)\overline{\psi_{\mathbb{R}}(u)}\,du \qquad \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi; ug)\overline{\psi_{\mathbb{A}}(u)}\,du$$

$$m_\alpha \in \mathbb{Z} \qquad m_\alpha \in \mathbb{Q}$$

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Computing adelic Fourier coefficients

[FGKP15 §9-10]

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Generic coefficient: Factorises over the primes. Casselman-Shalika formula

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

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Degenerate coefficient: Reduction to generic coefficient on smaller group

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

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[GKP14]

Fourier coefficients

In terms of Whittaker coefficients
Simplify drastically for certain χ

Example of simplifications

$G = SL(3)$ $\qquad$ $E(\chi; g)$ $\qquad$ $\chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

[FGKP15 §10.6]

Example of simplifications

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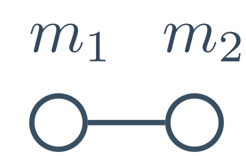
$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ $\qquad$ $\psi_N\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 x_1 + m_2 x_2)}$



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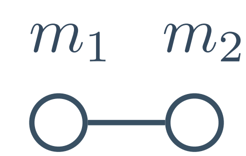

$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

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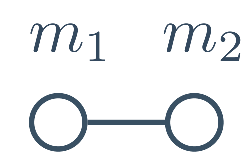

p-adic part

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots)$$

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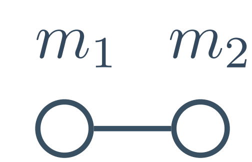

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Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

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Example of simplifications

Certain (s_1, s_2)

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[FGKP15 §10.6]

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[FGKP15 §10.6]

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$$W_N(\chi, \psi_{m_1,0}; g) \propto K_\#(\ldots) + K_\#(\ldots) + K_\#(\ldots)$$

[FGKP15 §10.6]

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[FGKP15 §10.6]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

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Automorphic representation π = an irreducible component of the above space under this action

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Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters ψ $\longleftrightarrow$ Nilpotent elements in $\mathfrak{g}$

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Nilpotent orbit $\quad \mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\} \quad X \in \mathfrak{g}$ nilpotent

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[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Closure with respect to partial ordering

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So called special orbits

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[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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decreasing order

$$(p_1, p_2, \dots)$$

Nilpotent orbits

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For $SL(n)$, orbits can be identified with partitions of n

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$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$ partial ordering

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[Collingwood-McGovern]

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$$\Longleftrightarrow$$

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Illustrated by a Hasse diagram

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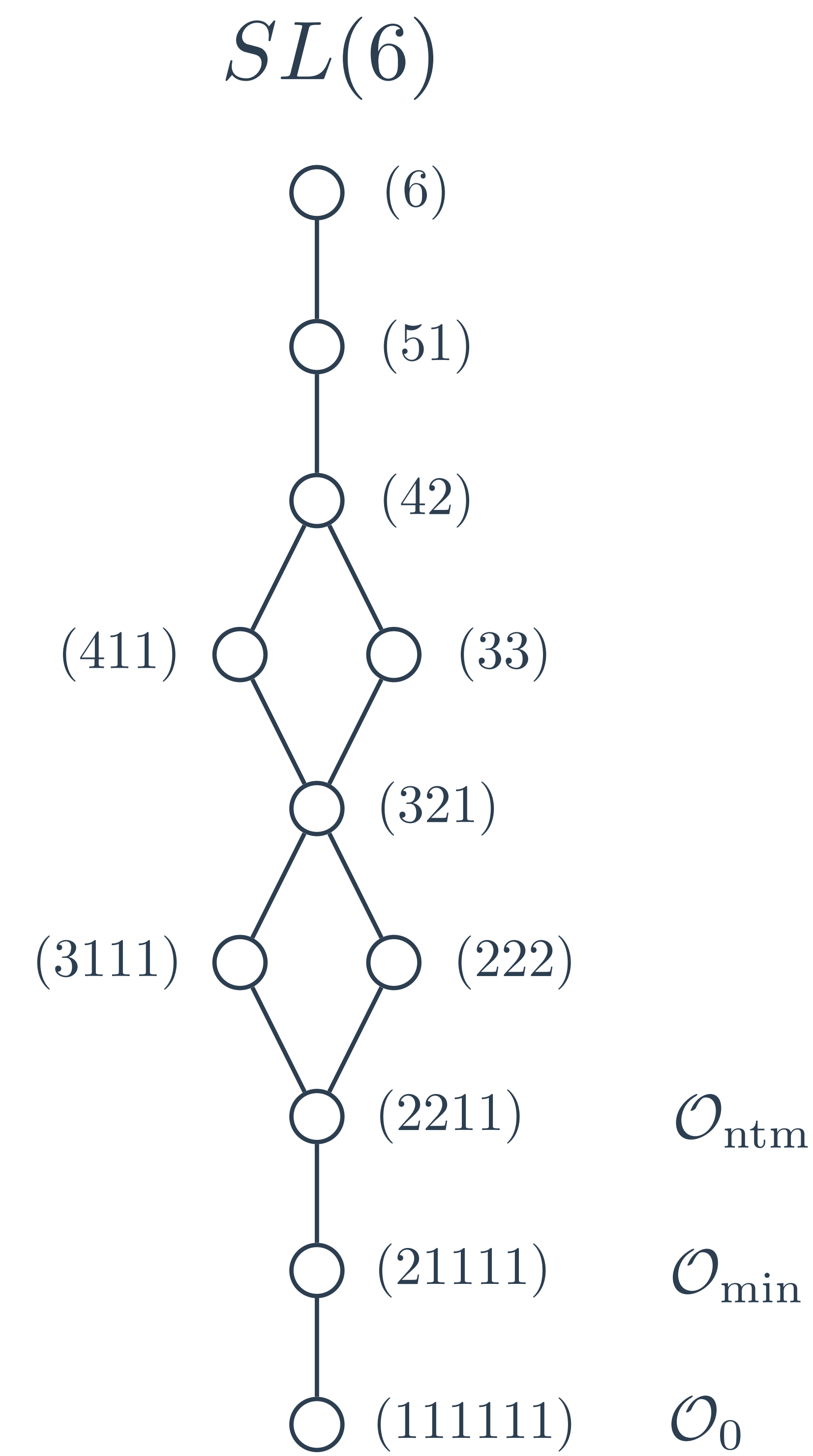

Nilpotent orbits

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$$\Longleftrightarrow$$

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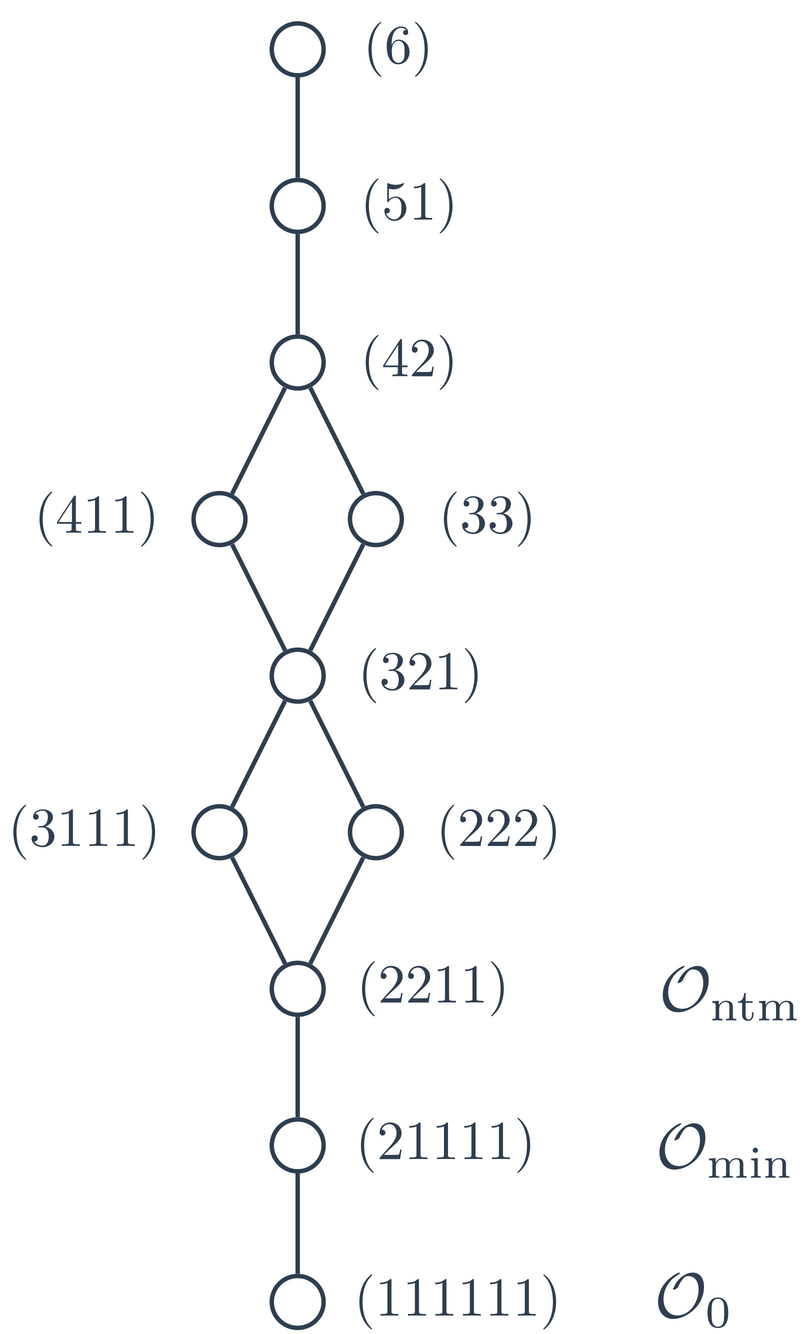
Illustrated by a Hasse diagram

Closure: $\overline{\mathcal{O}} = \displaystyle\bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$

$SL(6)$



Automorphic representations

Small representations

Automorphic representations

Small representations

$$\mathrm{WF}(\pi_{\mathrm{min}}) = \overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

Automorphic representations

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$\mathcal{E}^{(D)}_{(0,0)} \in \pi_{\mathrm{min}}$ $\mathcal{E}^{(D)}_{(1,0)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

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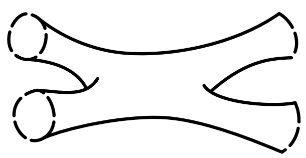
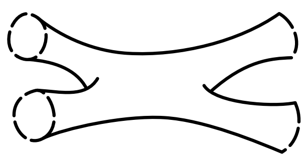
$\int K\,K \longrightarrow 0$

$\sum K \longrightarrow K$



$$R + (\alpha')^3 \mathcal{E}^{(D)}_{(0,0)}(g) R^4 + (\alpha')^5 \mathcal{E}^{(D)}_{(1,0)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}^{(D)}_{(0,1)}(g) D^6 R^4 + \ldots$$



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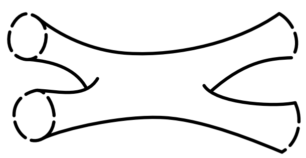
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$



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$$\mathcal{E}_{(0,0)}(\tau) = 2\zeta(3) E(3/2;\tau) \qquad \mathcal{E}_{(1,0)}(\tau) = \zeta(5) E(5/2;\tau)$$

$$F_U(\chi,\psi;g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi;ug)\overline{\psi(u)}\,du$$

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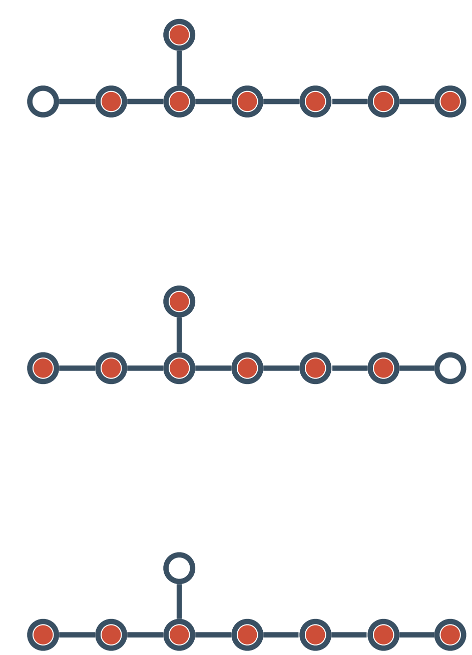

Maximal parabolic subgroups

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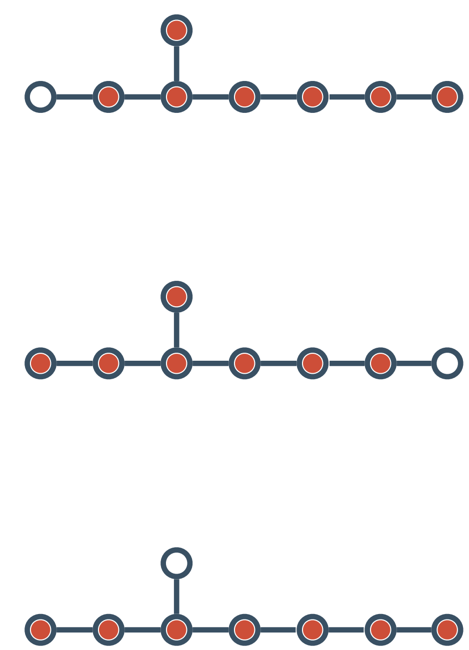
$$\mathbb{Z}\backslash\mathbb{R} \longrightarrow \mathbb{Q}\backslash\mathbb{A}$$



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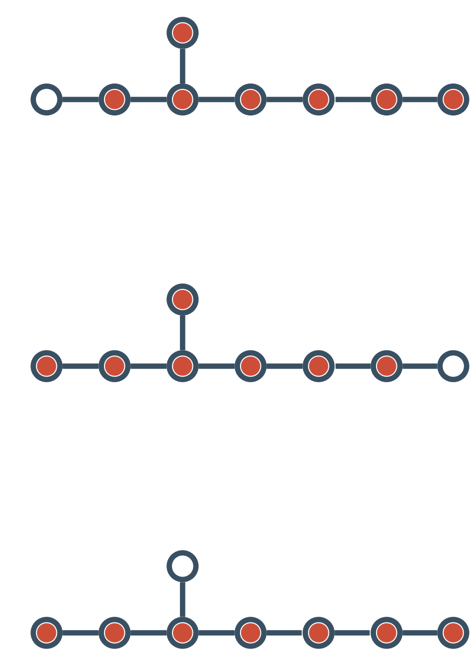
Known Whittaker coefficients W_N



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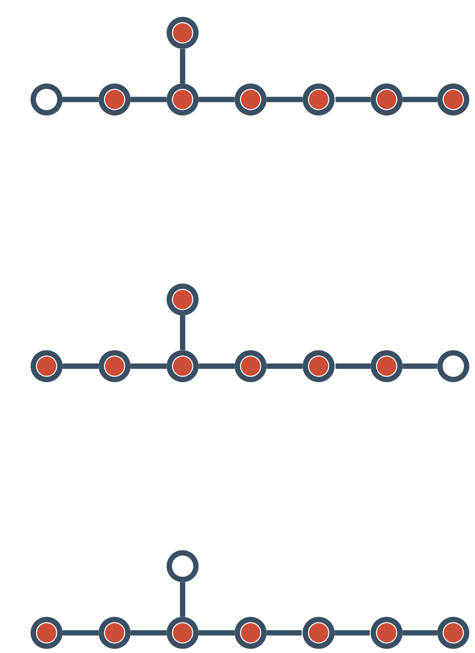
Automorphic representation π



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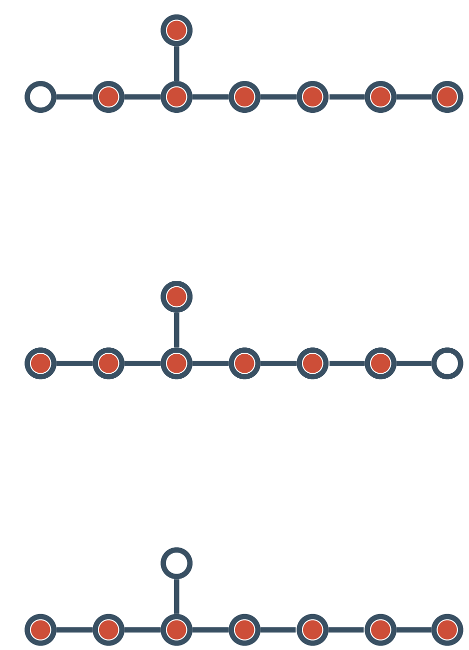
Known Whittaker coefficients W_N

Automorphic representation π

Vanishing properties $\mathbf{WF}(\pi)$



Maximal parabolic subgroups

Goal: find expressions for Fourier coefficients in terms of (known) Whittaker coefficients using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients

$$W_N \quad \text{with only one} \quad m_\alpha \neq 0$$

[Miller-Sahi]

Main results $SL(3), SL(4)$

[GKP14]

Main results $SL(3), SL(4)$

Theorem

For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients.

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More generally, for $\varphi \in \pi$

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For $G = SL(3), SL(4)$, an automorphic form $\varphi \in \pi_{\text{min}}$ is completely determined by maximally degenerate Whittaker coefficients.

More generally, for $\varphi \in \pi$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}} \qquad \text{where } \varphi_{\mathcal{O}} \text{ vanishes unless } \mathcal{O} \subseteq \mathrm{WF}(\pi)$$

[GKP14]

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$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

[GKP14]

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strongly orthogonal

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups

  

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Fourier coefficients on maximal parabolic subgroups

  

in the minimal representation

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$$\mathcal{E}^{(D)}_{(0,0)} \in \pi_{\min}$$

Main results

$SL(3), SL(4)$



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[GKP14]

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Theorem

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

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Other groups

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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Maximal parabolic Fourier coefficient

Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple, simply laced Lie groups

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic
Fourier coefficient

Maximally degenerate

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Local spherical vectors

Checks for E_6, E_7, E_8

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$$\pi_{\min,p} \subset \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\min,p} \hookrightarrow \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p}$$

multiplicity one

[Gan-Savin]

Local spherical vectors

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$f^\circ_{\psi_{U,p}} \in \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p}$ computed in several cases $p \leq \infty$

[Dvorsky-Sahi, Kazhdan-Polishchuk, Kazhdan-Pioline, Savin-Woodbury]

Local spherical vectors

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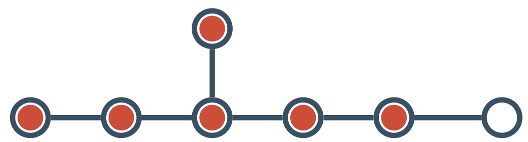
Factorises

For example: E_7 with U from 

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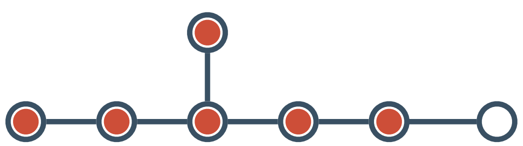
For example: E_7 with U from 

$$f^{\circ}_{\psi_{U,p}} = \frac{1 - p^3 \, |m|_p^{-3}}{1 - p^3}$$

[Savin-Woodbury]

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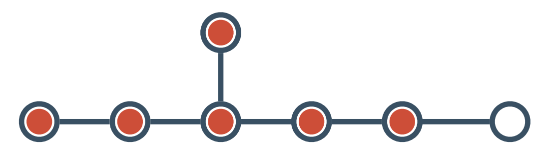
For example: E_7 with U from 

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[Savin-Woodbury] [Dvorsky-Sahi]

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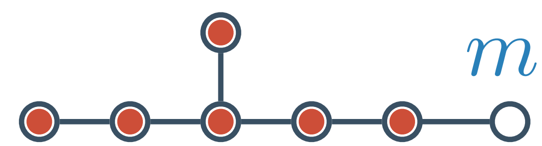
For example: E_7 with U from  $m \longleftarrow \psi'$

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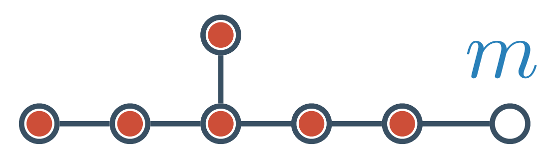
For example: E_7 with U from  $m \longleftarrow \psi'$

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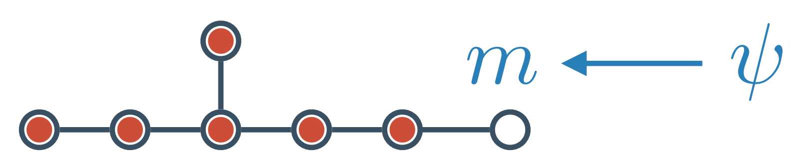

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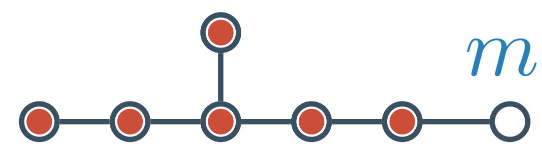
For example: E_7 with U from  $m \longleftarrow \psi'$

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Complete agreement for E_6, E_7, E_8 in both abelian and Heisenberg realisations

Outlook

Tools for proving the conjecture

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Outlook

Tools for proving the conjecture

$(S, \psi) \in \mathfrak{g} \times \mathfrak{g}^*$ Whittaker pair [Gomez-Gourevitch-Sahi]

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Describes the integration domain and character for a Fourier coefficient

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Semi-simple

Describes the integration domain and character for a Fourier coefficient

Methods for relating different Whittaker pairs

$$(S, \psi) \longrightarrow (S', \psi')$$

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Outlook

Outlook

Simplification of Fourier coefficients with χ_{min} for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

[Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

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How to define “small automorphic representations” for Kac-Moody groups? What is the mechanism behind the vanishing properties?

$\mathcal{E}_{(0,1)} D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

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Thank you!

Henrik Gustafsson

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