A vertex model for Iwahori Whittaker functions

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Solvable Lattice Models Seminar – Stanford University

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Rutgers University | University of Gothenburg and Chalmers | Swedish Research Council

Papers

Joint work with Ben Brubaker, Valentin Buciumas and Daniel Bump

Colored five-vertex models and Demazure atoms

Journal of Combinatorial Theory, Series A 178 (Feb, 2021)

arXiv:1902.01795

Colored vertex models and Iwahori Whittaker functions

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Metaplectic Iwahori Whittaker functions and supersymmetric lattice models

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Iwahori Whittaker functions of unramified principal series on $\mathrm{GL}_r(F)$

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non-archimedean local field

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Solvable vertex model represented by colored lines and associated with the quantum group $U_q(\widehat{\mathfrak{gl}}(r|1))$

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$\mathbf{z} = (z_1, \ldots, z_r) \in (\mathbb{C}^\times)^r$

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$w \in W = S_r$

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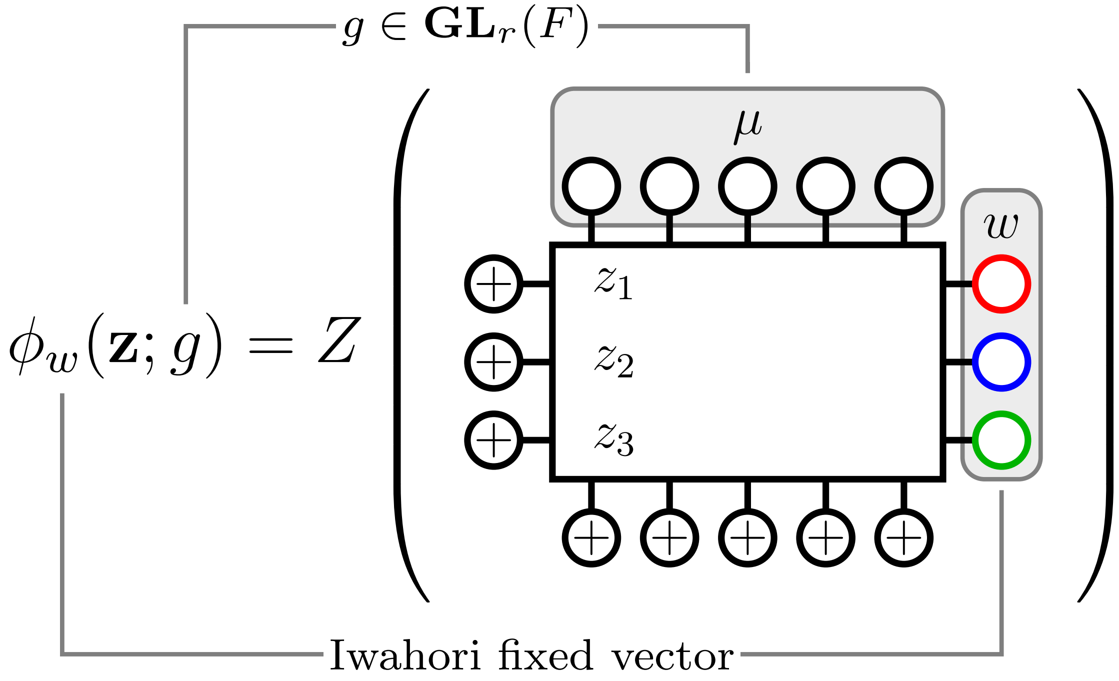

The lattice model

Features

Described by colored lines moving right- and downwards on a grid from the top boundary to the right boundary.

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$c_1 < \cdots < c_r$

Palette of r colors: B $<$ G $<$ R (ordered)

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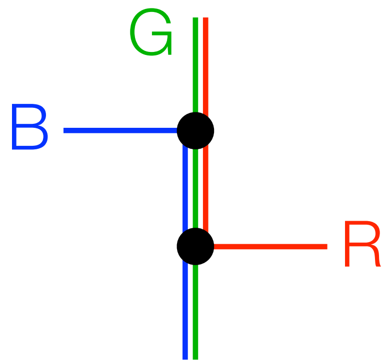

Vertical edges described by subsets of palette

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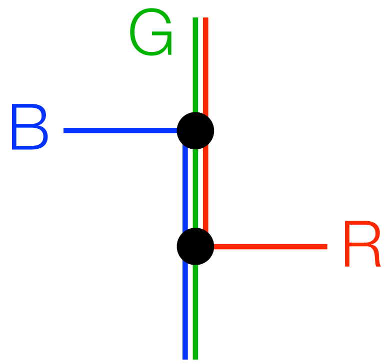

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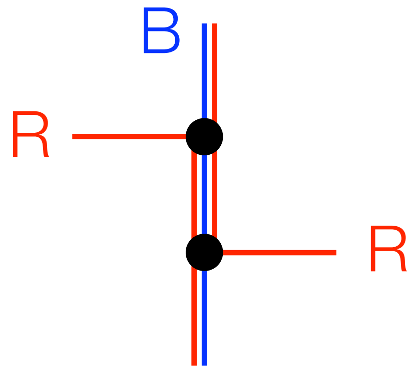

subsets with multiplicities

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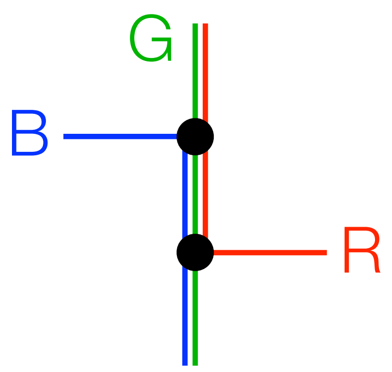
fermionic



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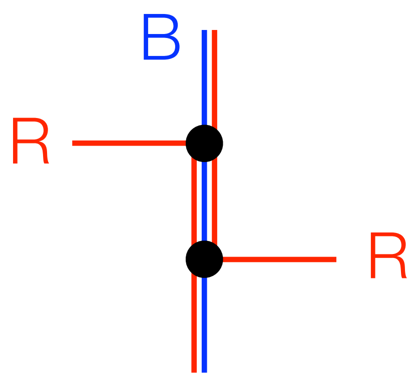

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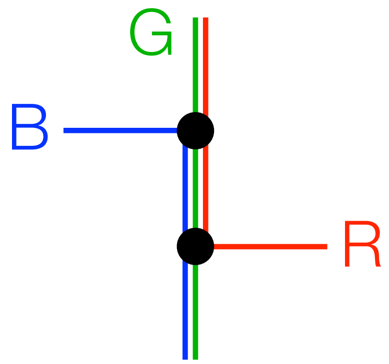

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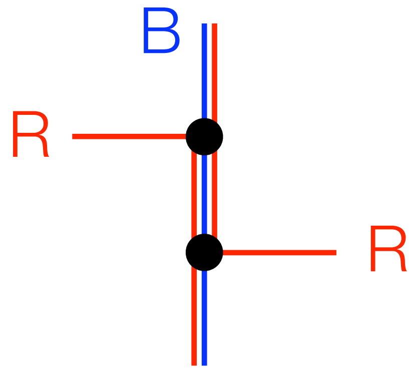
bosonic



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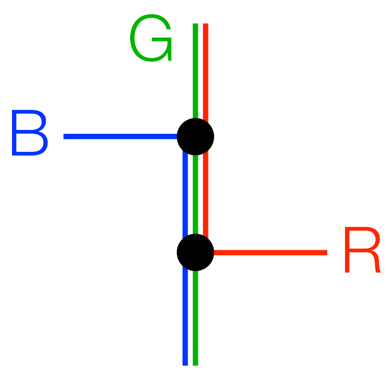
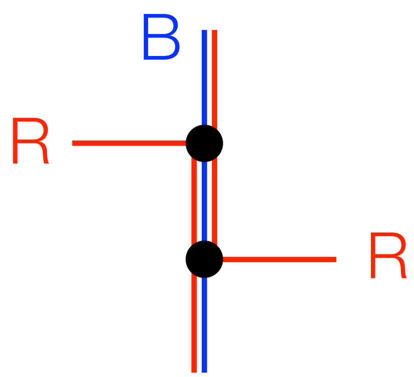


Vertical edges described by subsets of palette

See above papers and [Aggarwal–Borodin–Wheeler arXiv:2101.01605]

bosonic



subsets with multiplicities

See for example [Borodin–Wheeler arXiv:1808.01866]

Features

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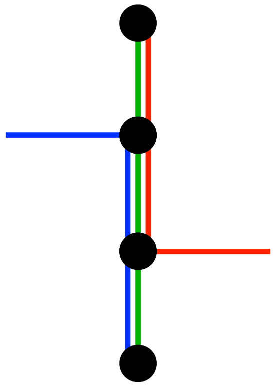

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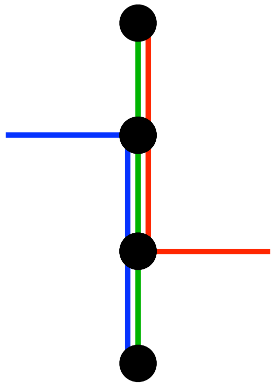

Vertical edges described by subsets of palette

Binary representation

Features

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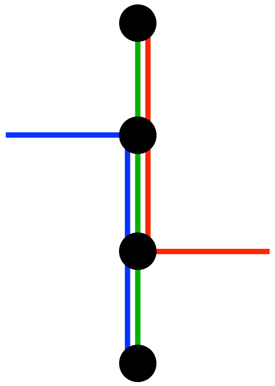


Vertical edges described by subsets of palette

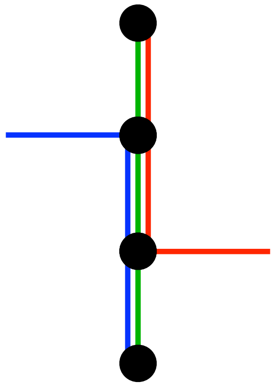
B G R

⊕ ⊖ ⊖

Binary representation

Features

fermionic



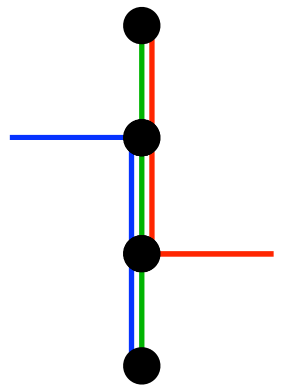
Vertical edges described by subsets of palette

B	G	R
⊕	⊖	⊖
⊖	⊖	⊖

Binary representation

Features

fermionic



Vertical edges described by subsets of palette

B	G	R
⊕	⊖	⊖
⊖	⊖	⊖
⊖	⊖	⊕

Binary representation

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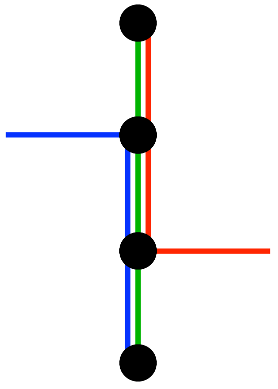
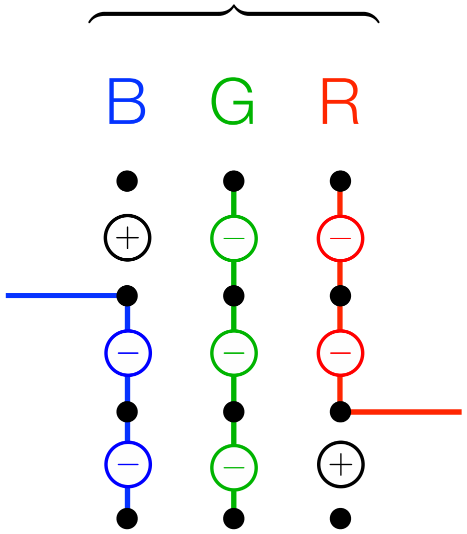


Vertical edges described
by subsets of palette

color block



Binary representation

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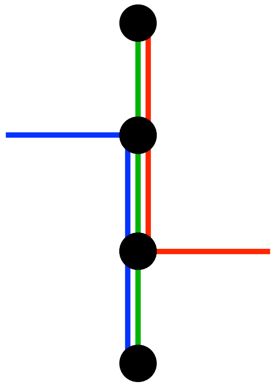

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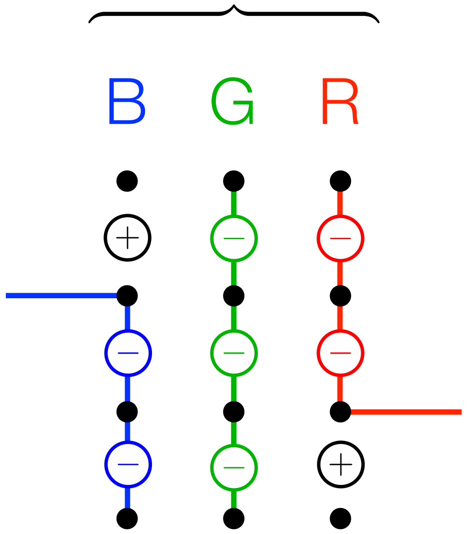
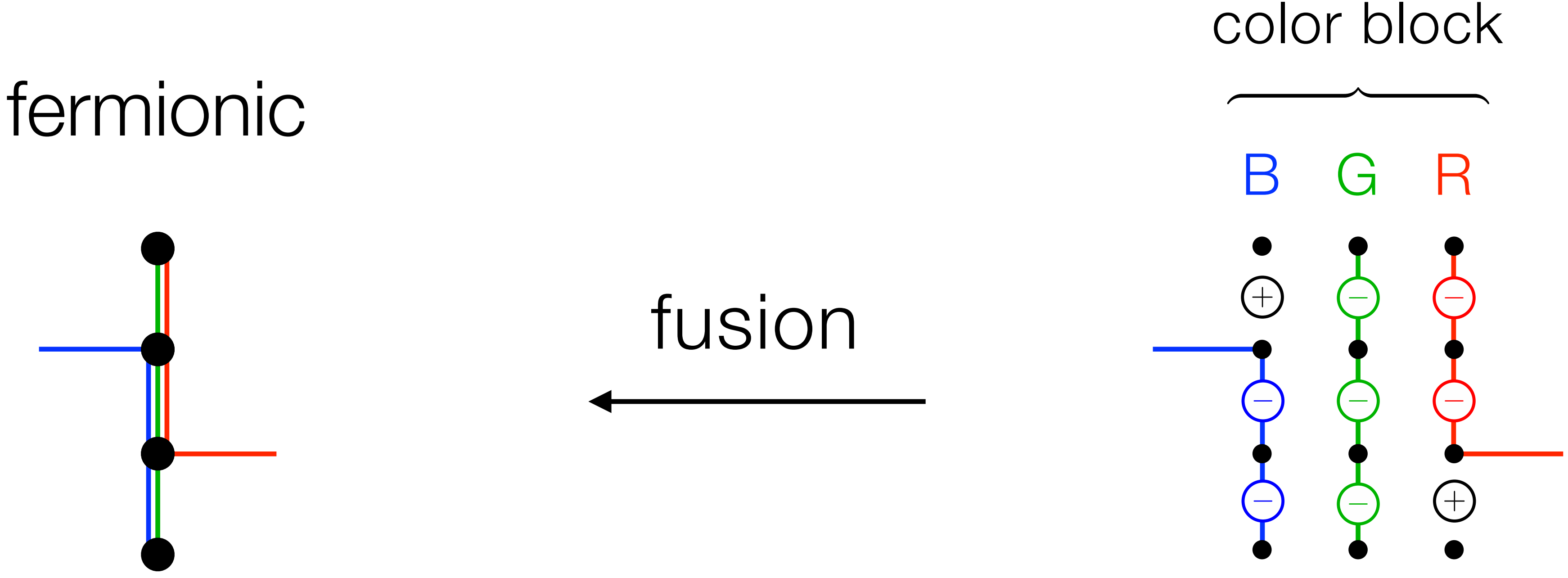


Binary representation

Each column can only carry a specific color

Features



Vertical edges described by subsets of palette

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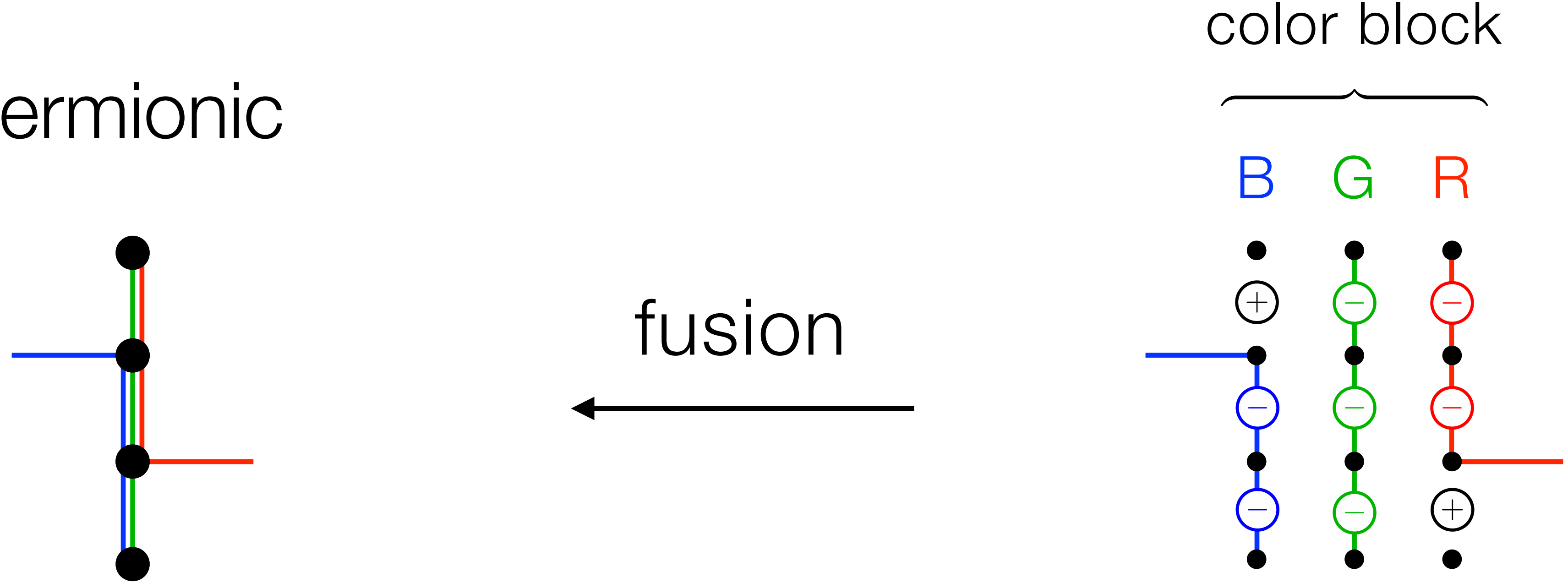


fermionic

fusion

color block

B G R

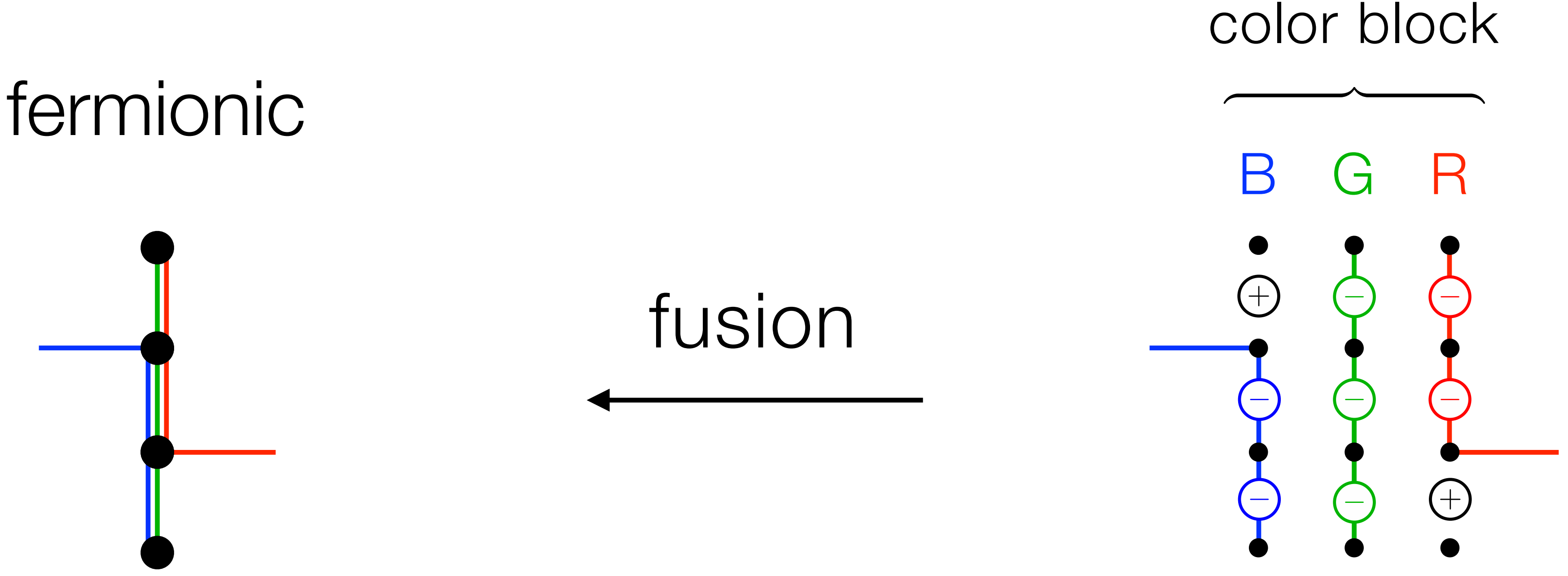
Vertical edges described by subsets of palette

Binary representation

Each column can only carry a specific color

Easier to describe

Features



Vertical edges described
by subsets of palette

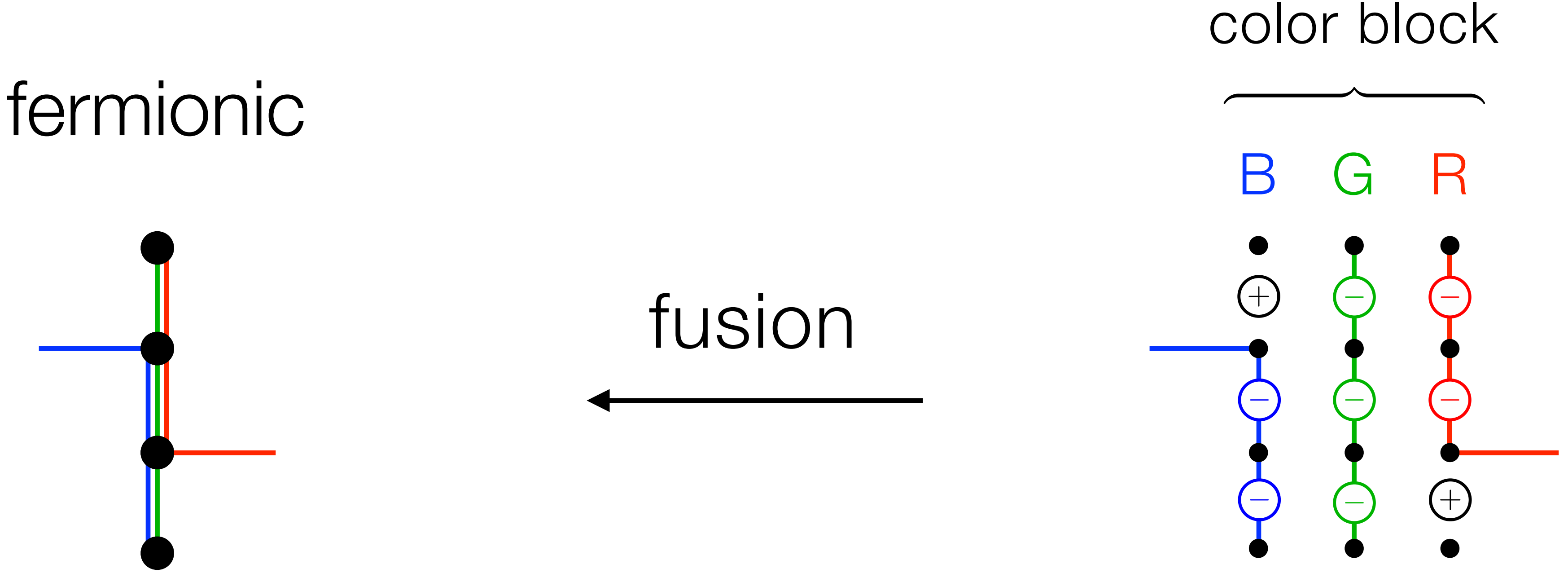
Binary representation

Each column can only
carry a specific color

Quantum group connection

Easier to describe

Features



Vertical edges described by subsets of palette

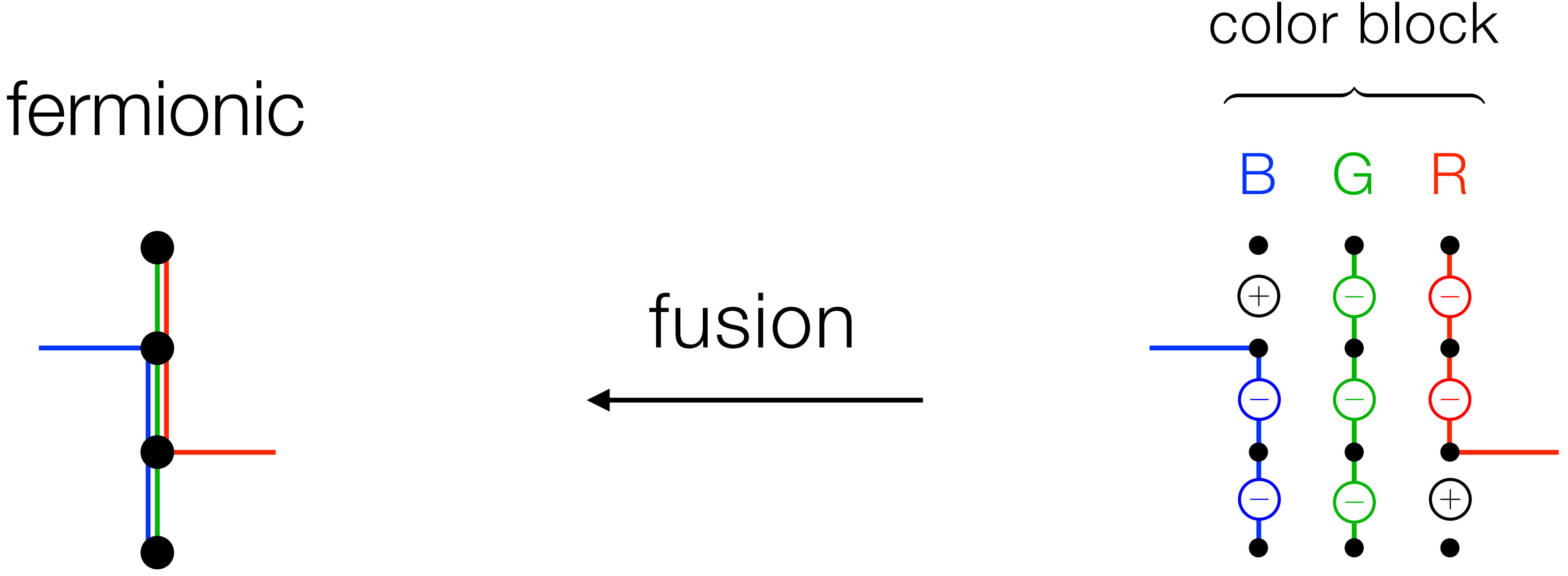
Quantum group connection

Binary representation

Each column can only carry a specific color

Easier to describe

Features



Vertical edges described by subsets of palette

Binary representation

Each column can only carry a specific color

Quantum group connection

Easier to describe

Horizontal edges can carry any single (or no) color in both descriptions

Lattice model setup

$c_1 < \cdots < c_r$

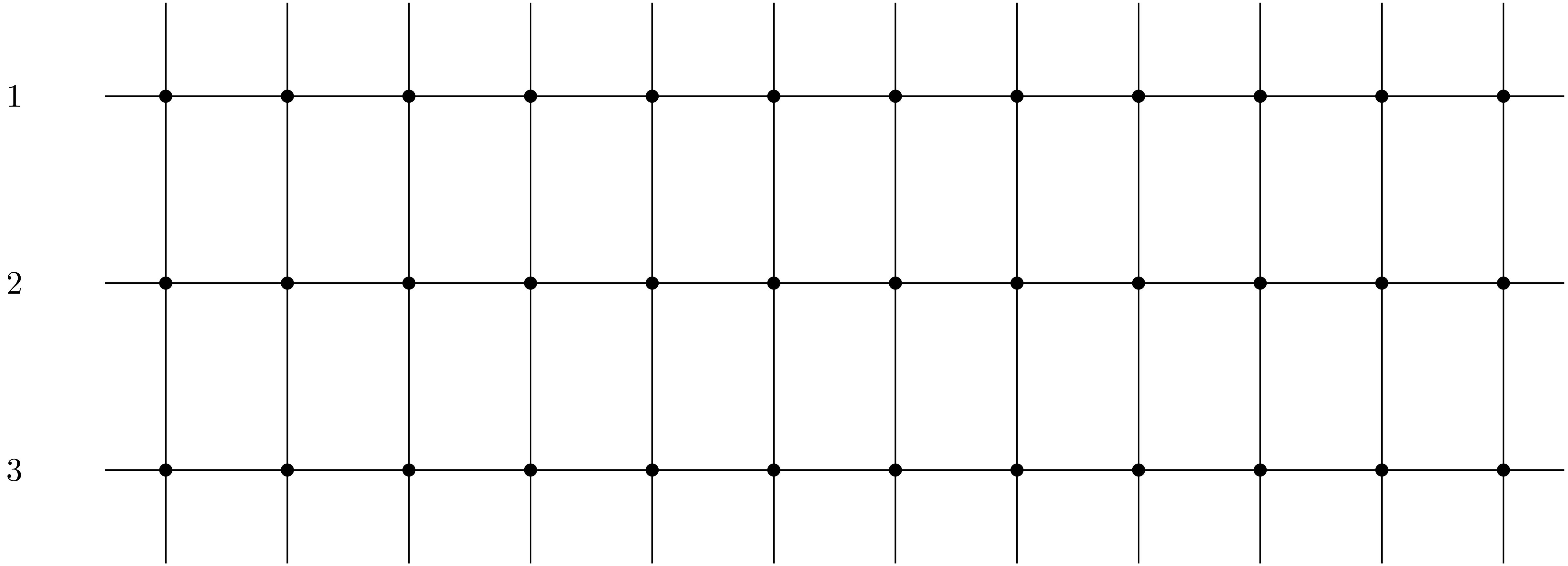
Palette of r colors: B $<$ G $<$ R

Lattice model setup

$c_1 \;<\; \cdots \;<\; c_r$

Palette of r colors: B < G < R

Grid with r rows and sufficiently many columns



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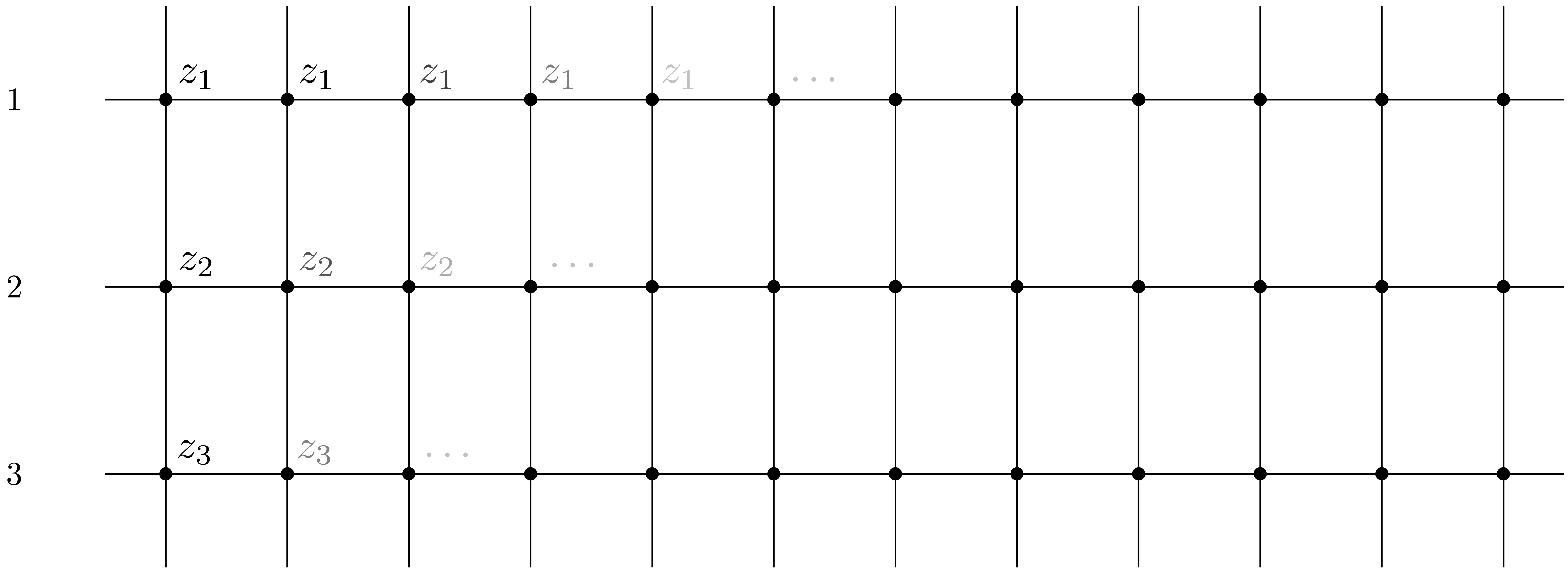
$c_1 < \cdots < c_r$

Palette of r colors: B $<$ G $<$ R

Grid with r rows and sufficiently many columns

Row parameter $z_i \in \mathbb{C}^\times$



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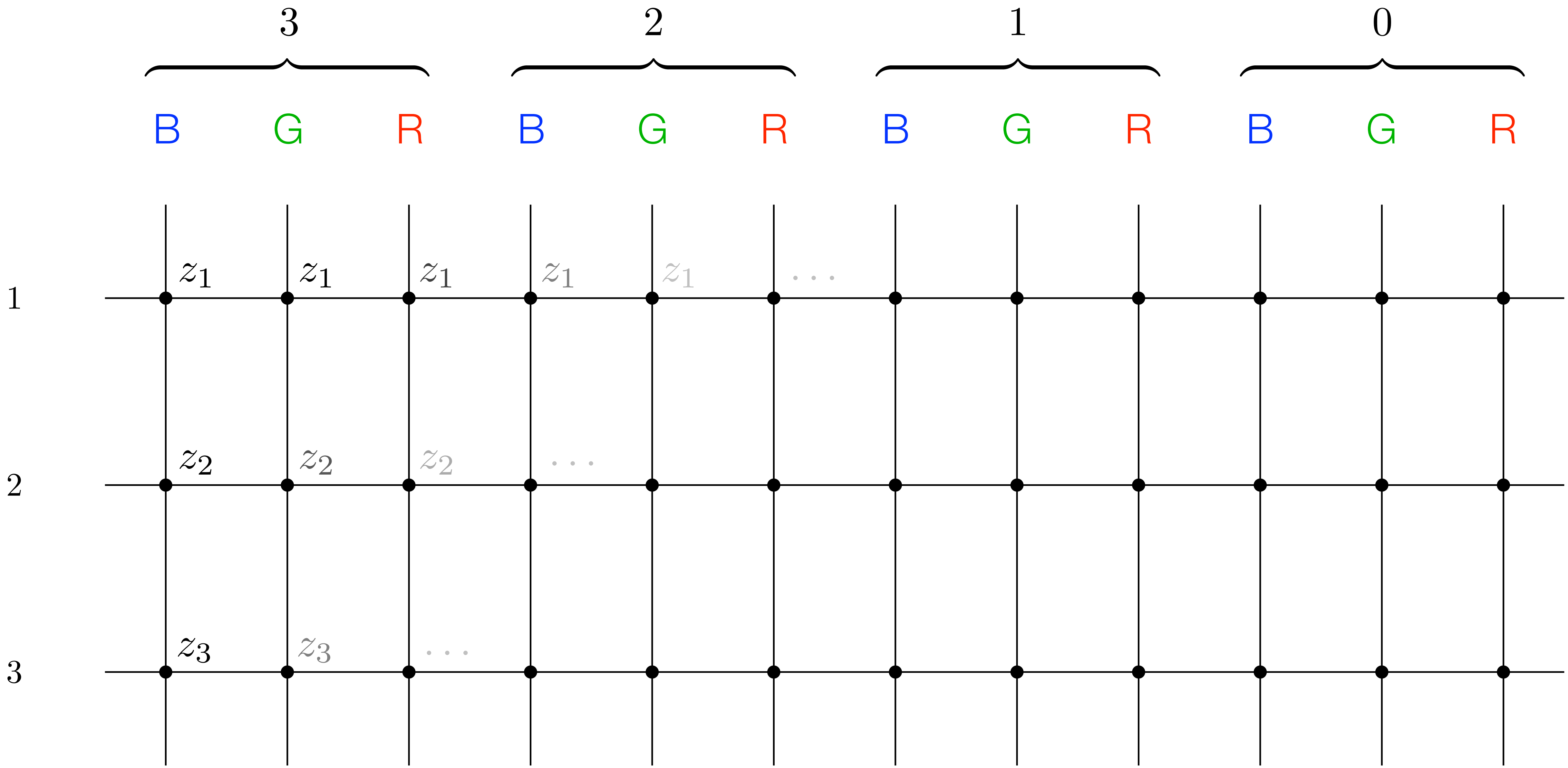
$c_1 < \cdots < c_r$

Palette of r colors: B < G < R

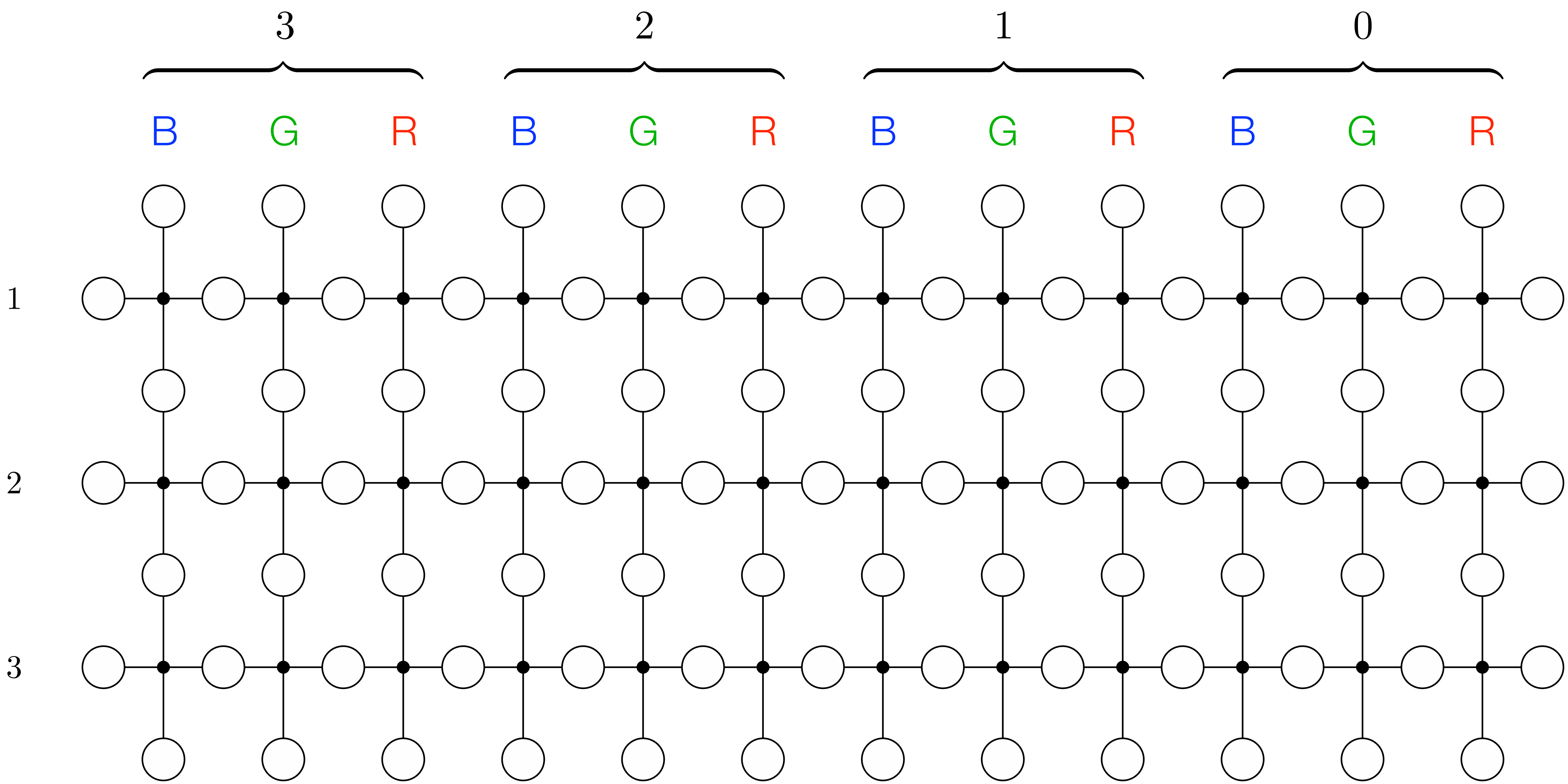
Grid with r rows and sufficiently many columns

Row parameter $z_i \in \mathbb{C}^\times$

Number color blocks from right to left

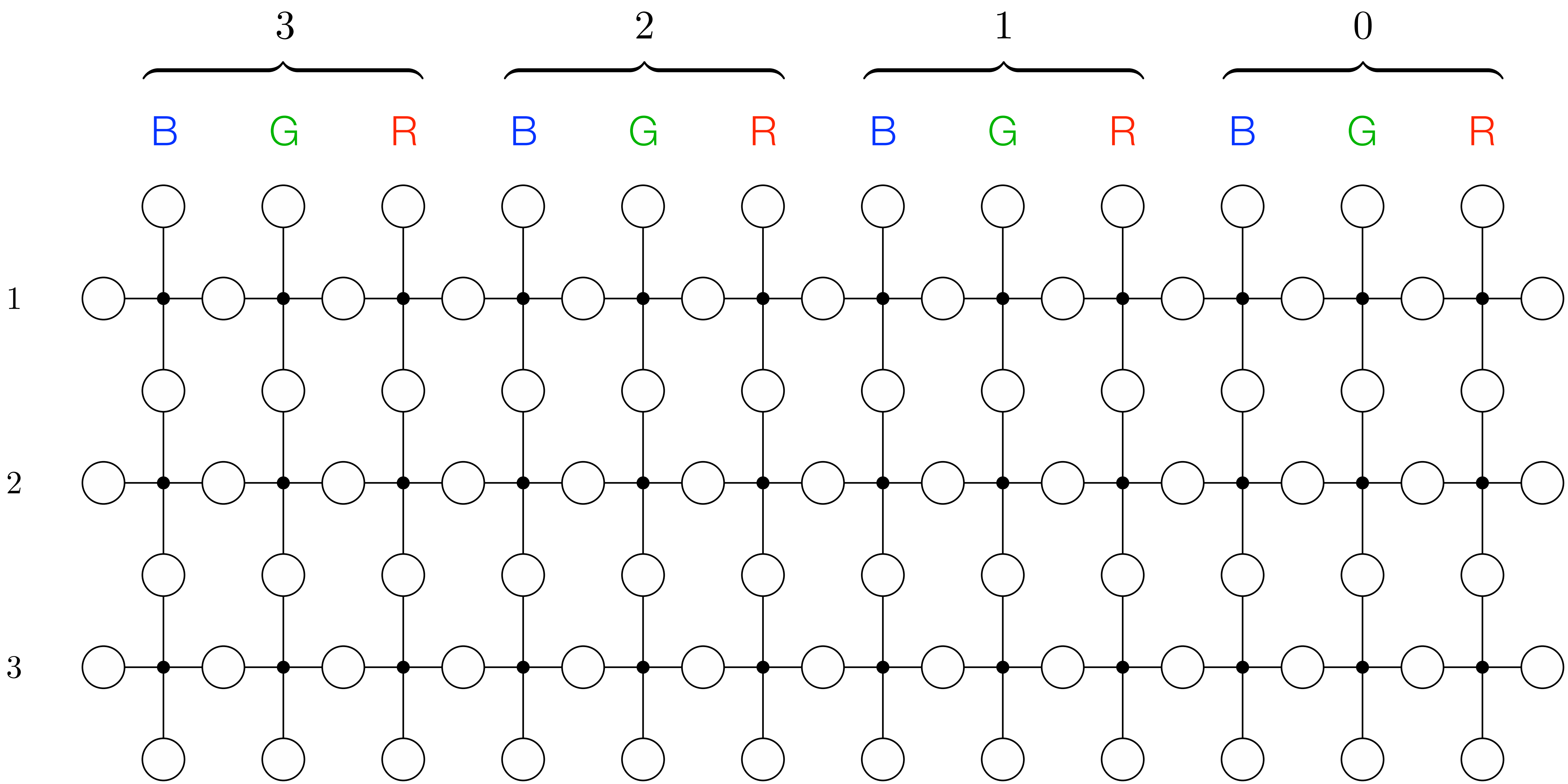


Boundary



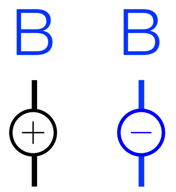
Boundary

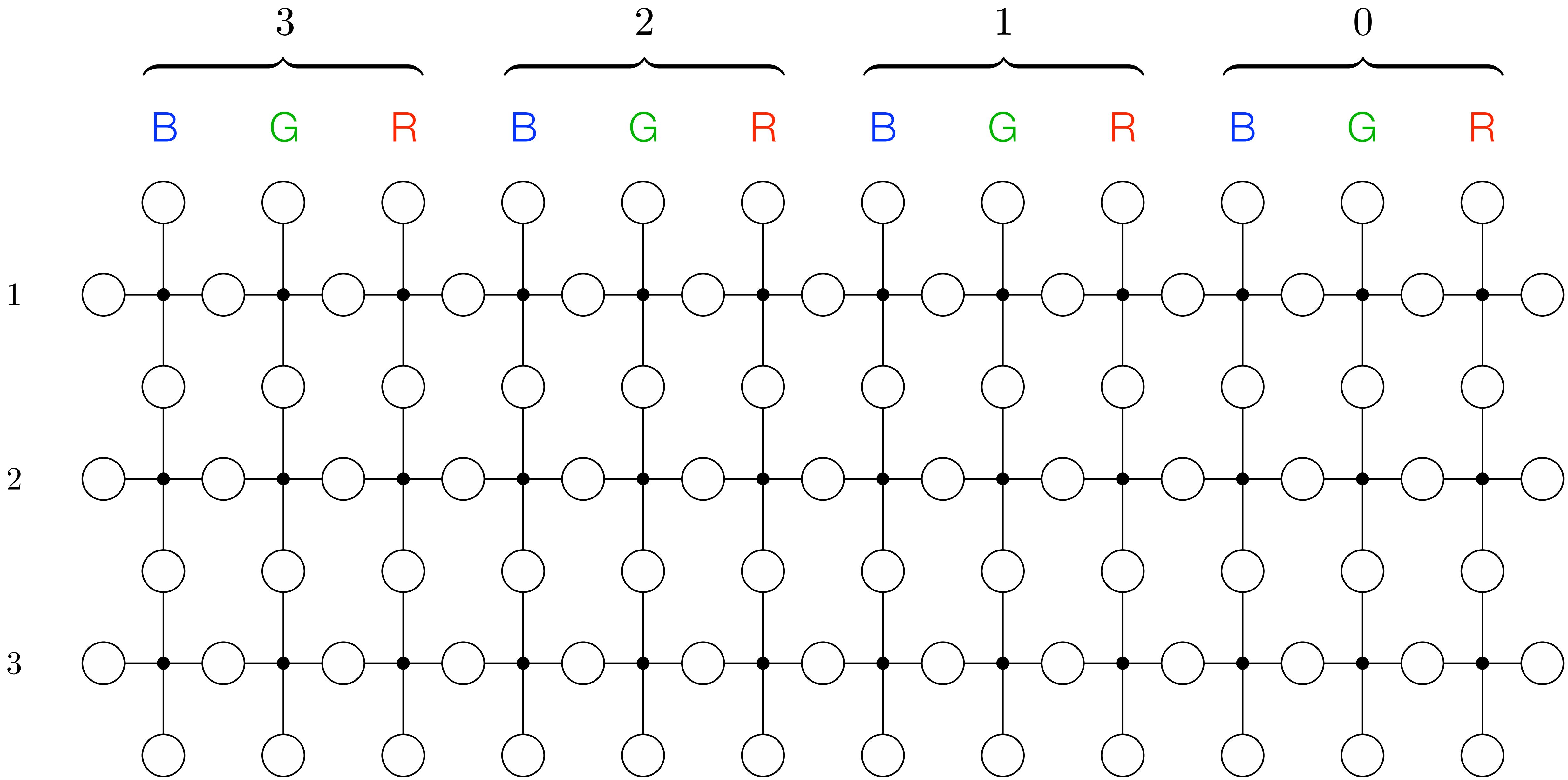
Vertical edges: ⊕ ⊖



Boundary

Vertical edges:

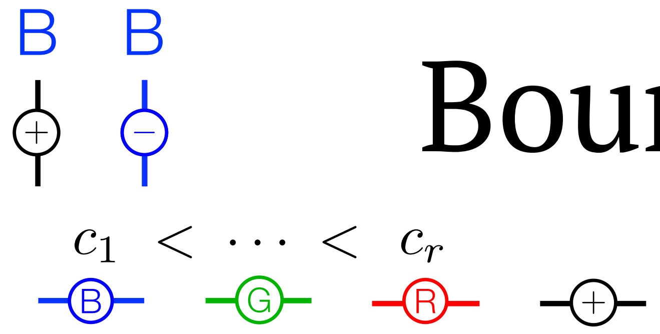


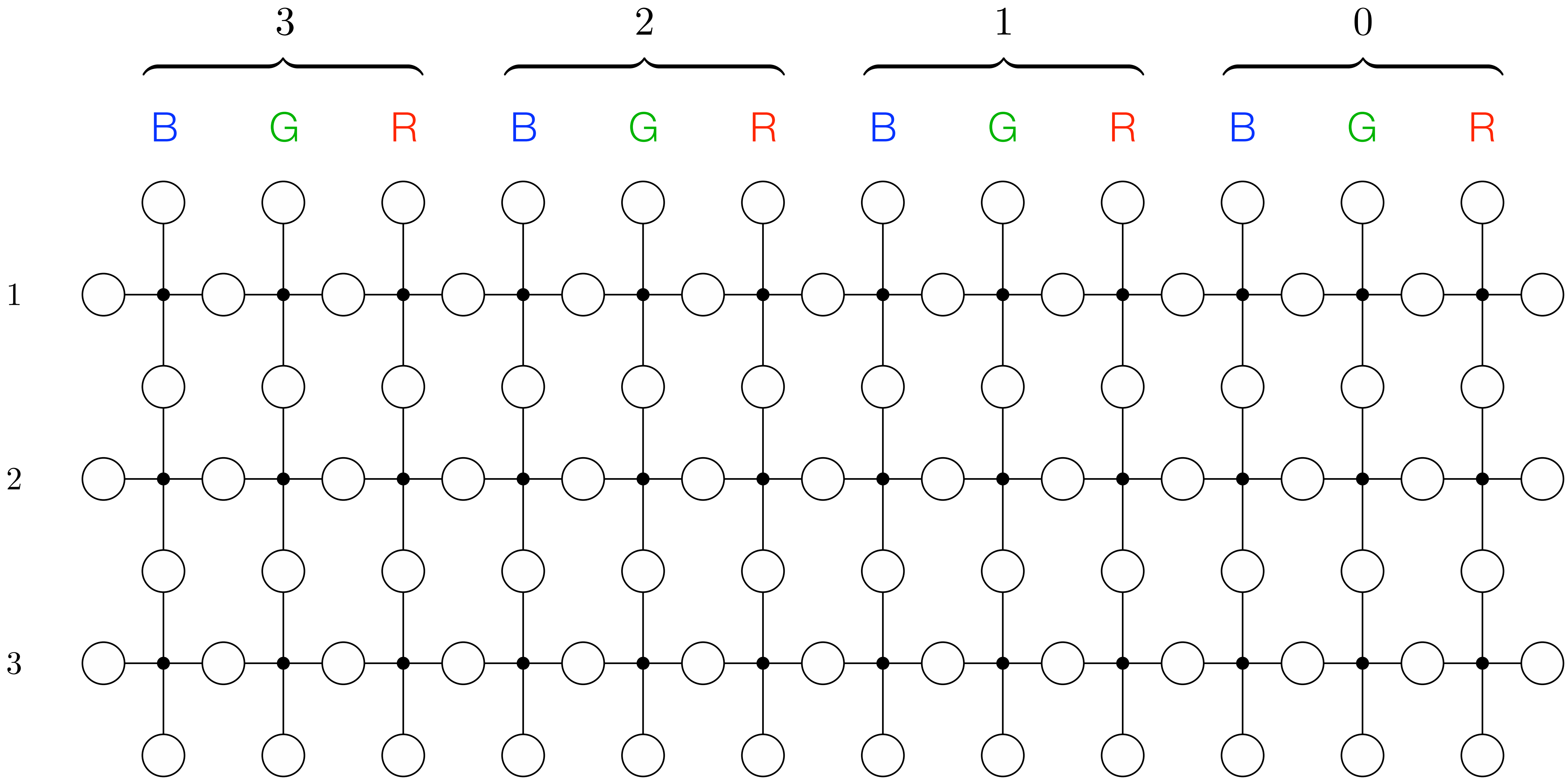


Boundary

Vertical edges:

Horizontal edges:





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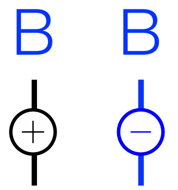
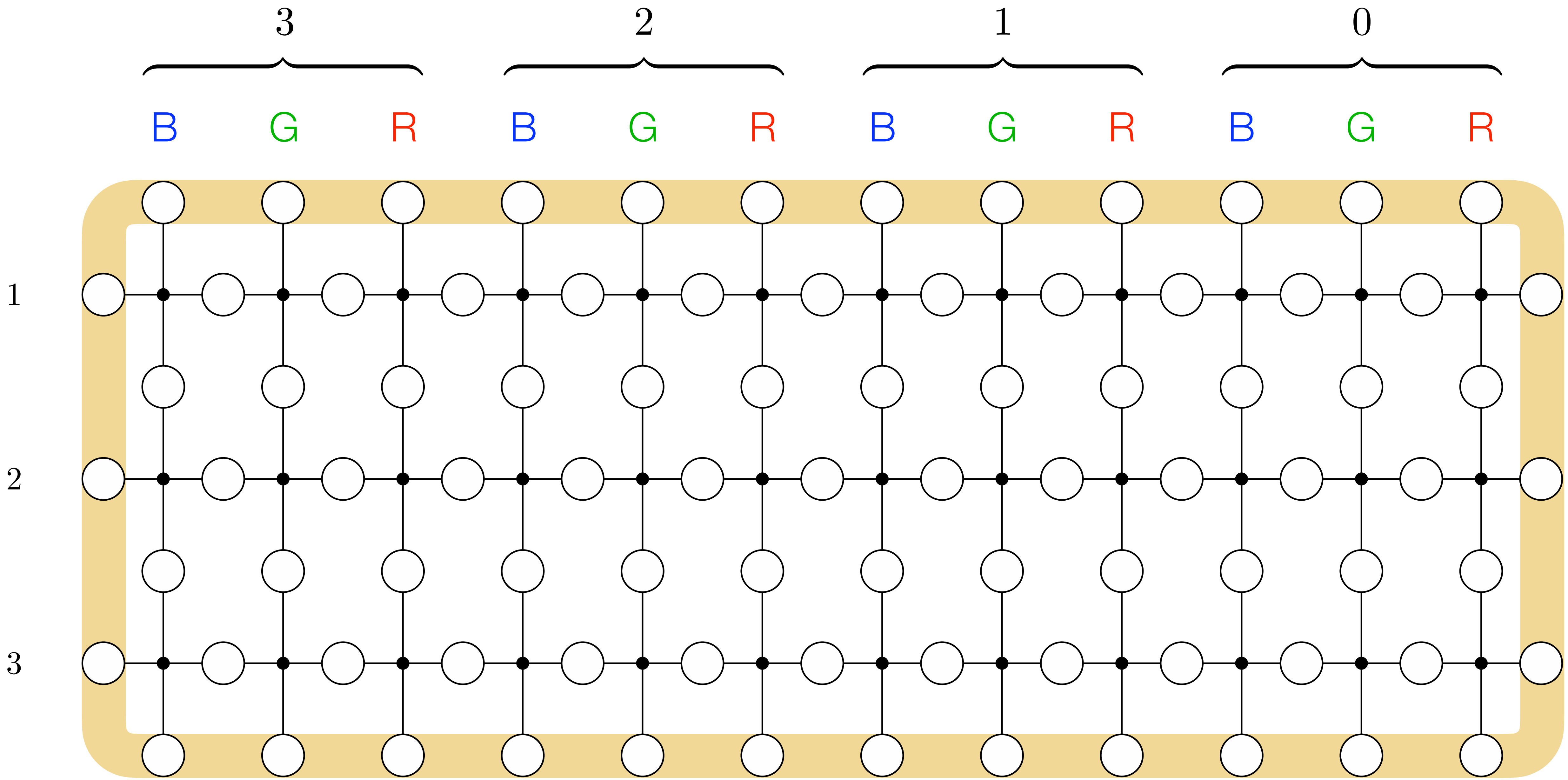
Vertical edges:

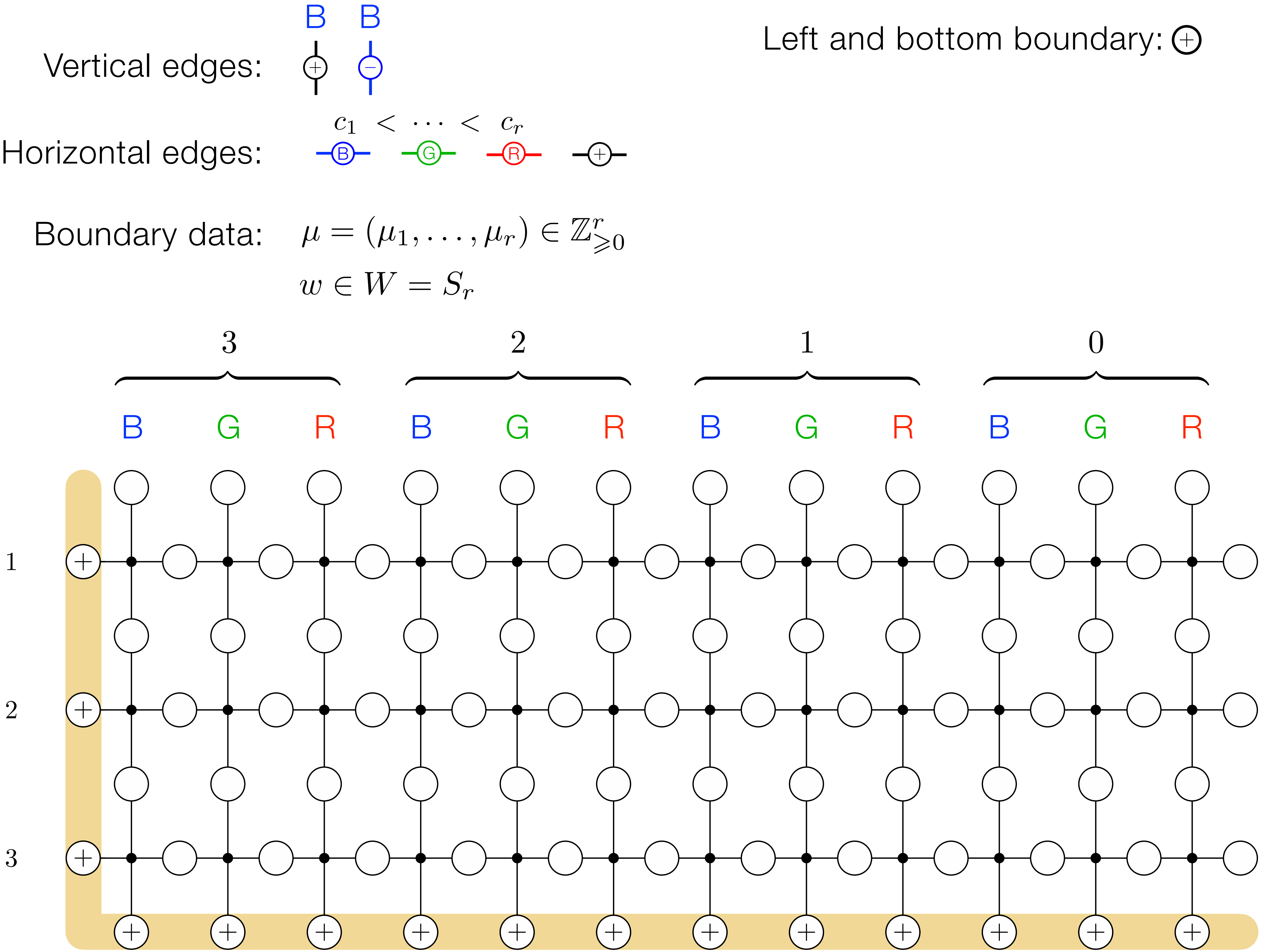


Horizontal edges: $c_1 < \cdots < c_r$

Boundary data: $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{\geqslant 0}$

$w \in W = S_r$




Vertical edges:
B B
+ −
Horizontal edges:
$c_1 < \cdots < c_r$
B G R +
Boundary data:
$\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{\geqslant 0}$
$w \in W = S_r$
Left and bottom boundary: +
3 2 1 0
B G R B G R B G R B G R
1
2
3

Vertical edges: B ⊕ B ⊖

Horizontal edges: $c_1 < \cdots < c_r$ Ⓑ Ⓖ Ⓡ ⊕

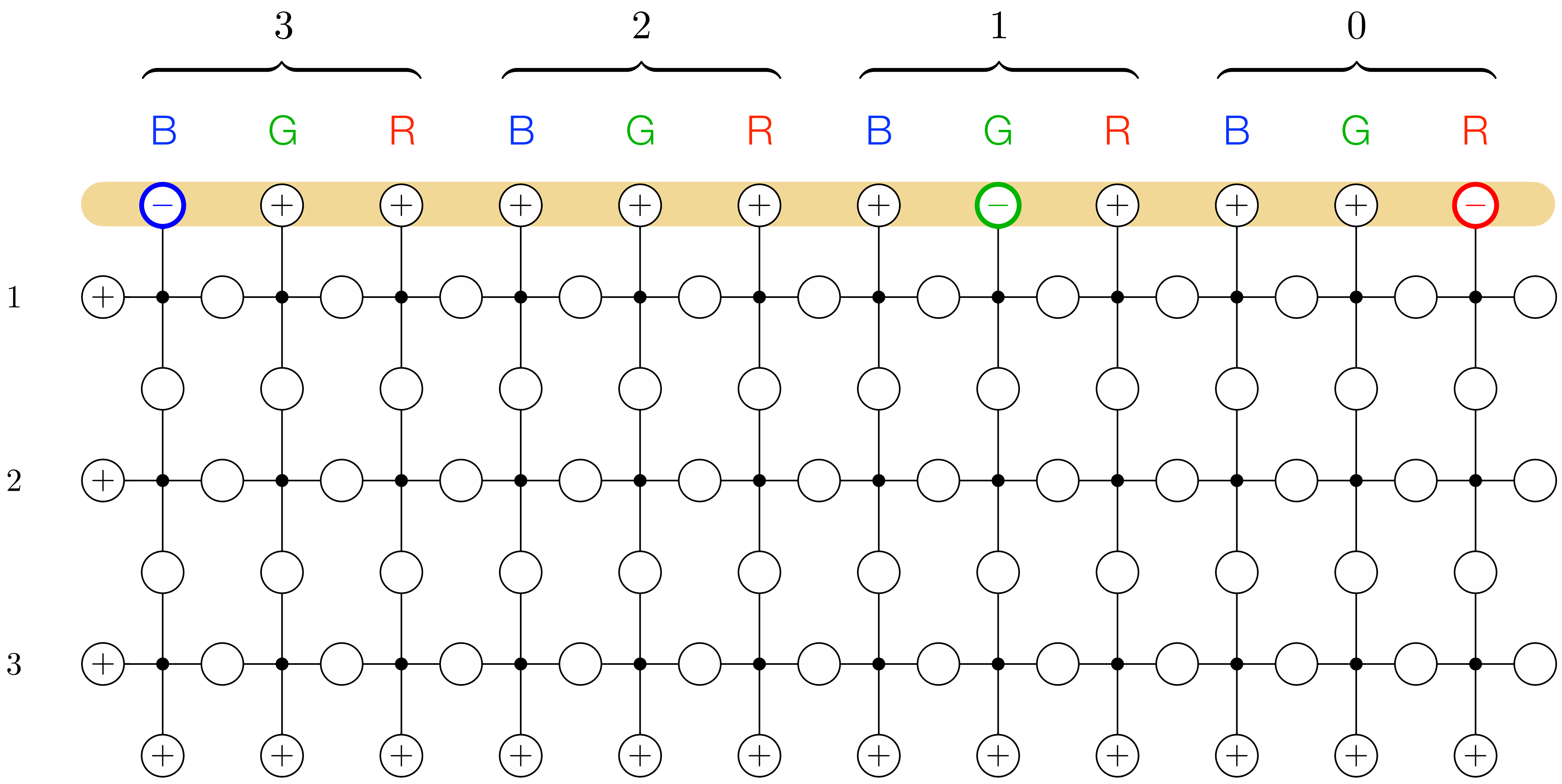
Boundary data: $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}^r_{\geqslant 0}$

$w \in W = S_r$

Left and bottom boundary: ⊕

Top boundary:

⊖ at block μ_j with color c_{r+1-j}



$\mu = (0, 1, 3)$

Vertical edges: $\overset{B}{\oplus}$ $\overset{B}{\ominus}$

Horizontal edges: $c_1 < \cdots < c_r$ Ⓑ Ⓖ Ⓡ ⊕

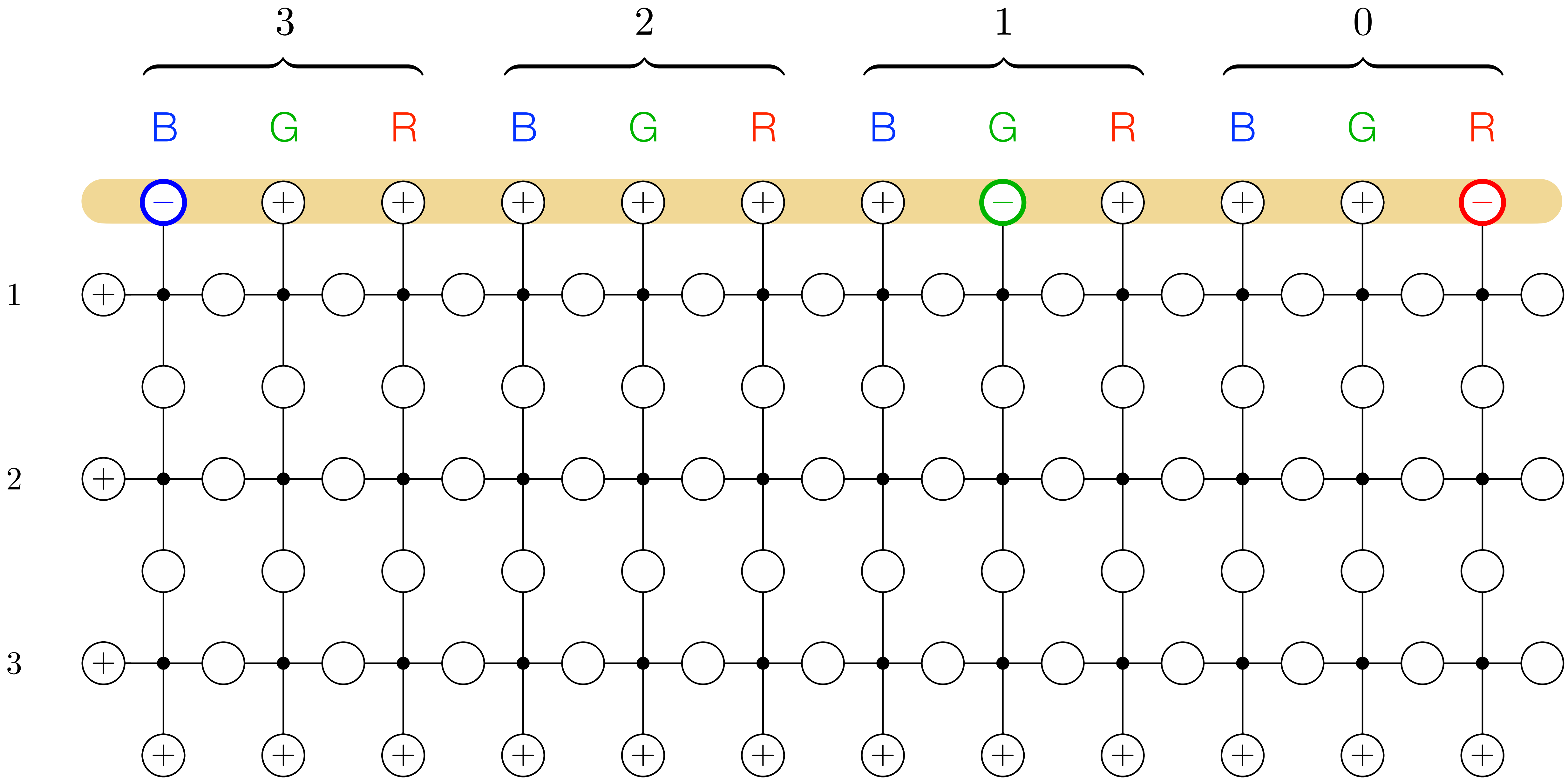
Boundary data: $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{\geqslant 0}$

$w \in W = S_r$

Left and bottom boundary: ⊕

Top boundary: $P = (c_r, \ldots, c_1)$

⊖ at block μ_j with color $c_{r+1-j} = (P)_j$



$\mu = (0, 1, 3)$

Vertical edges: B ⊕ B ⊖

Horizontal edges: $c_1 < \cdots < c_r$ Ⓑ Ⓖ Ⓡ ⊕

Boundary data: $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{\geqslant 0}$

$w \in W = S_r$

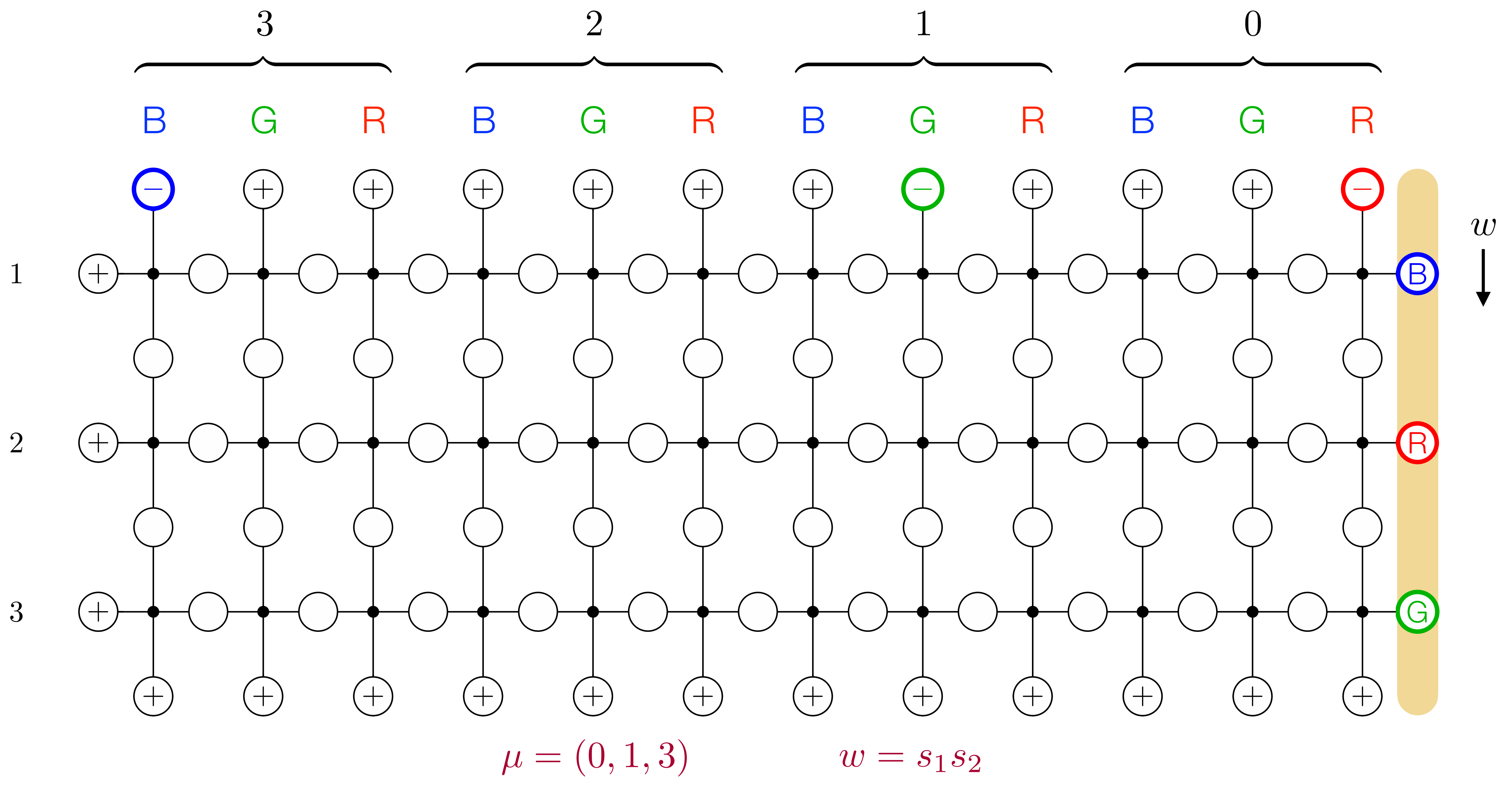
Left and bottom boundary: ⊕

Top boundary: $P = (c_r, \ldots, c_1)$

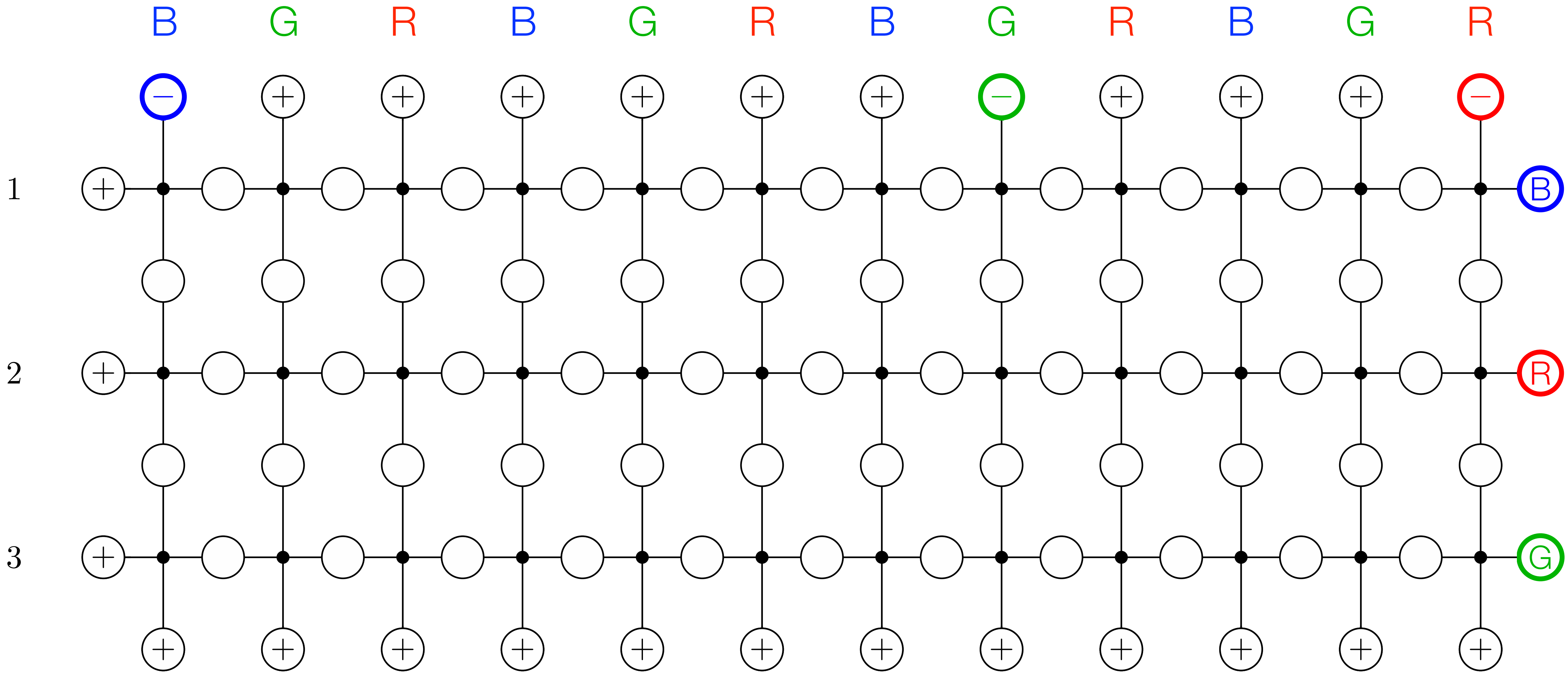
⊖ at block μ_j with color $c_{r+1-j} = (P)_j$

Right boundary:

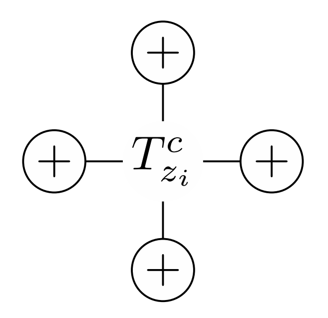
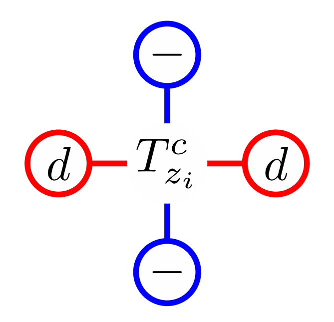
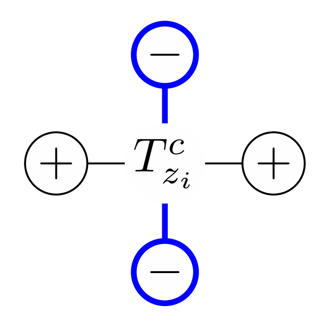
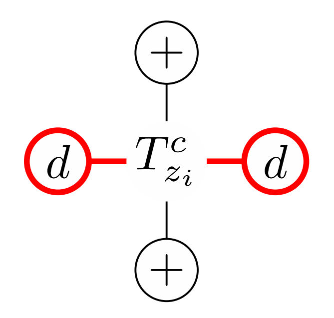
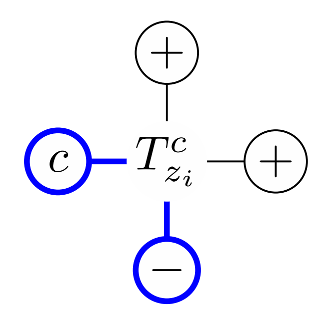
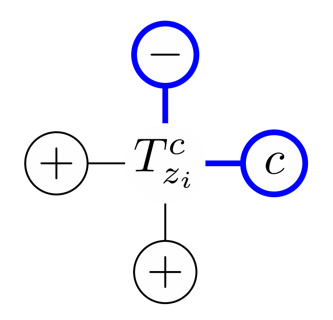
row i with color $c_{r+1-w^{-1}(i)} = (wP)_i$



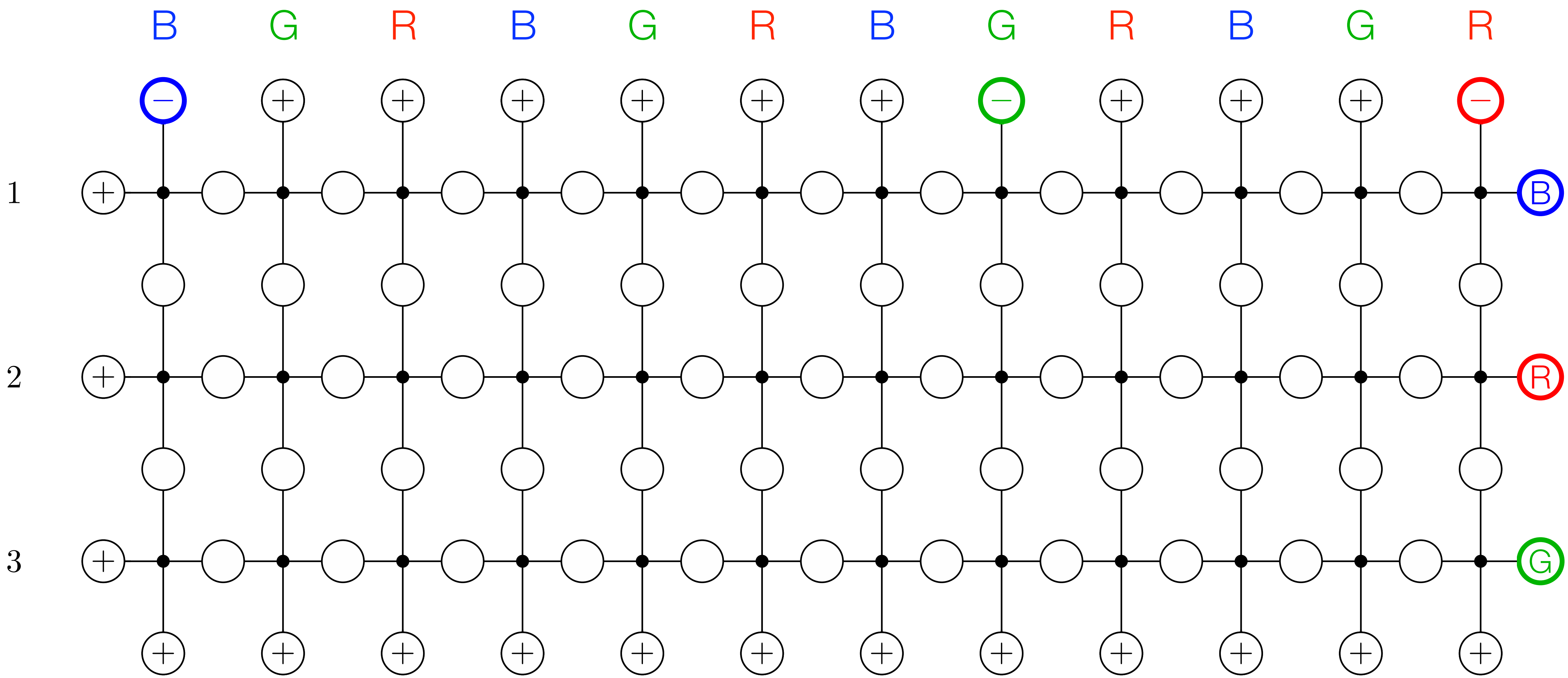
Inner edges from six possible vertex configurations:



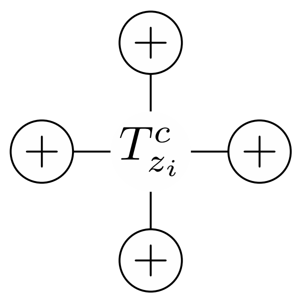
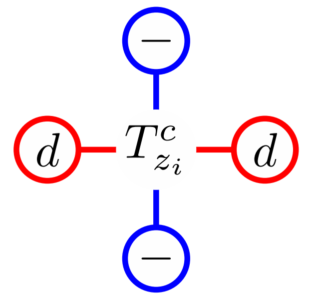
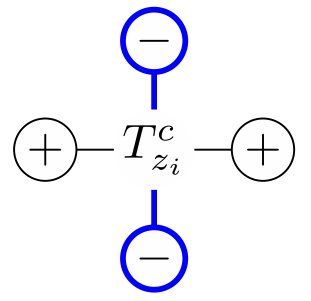
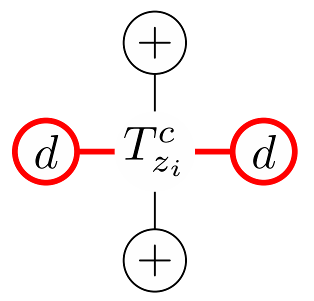
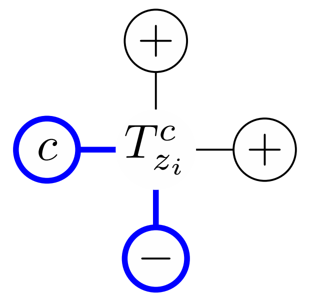
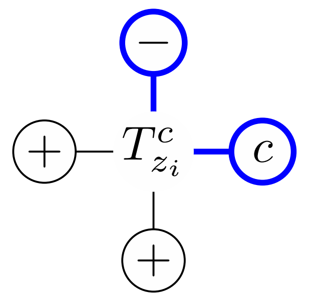
Inner edges from six possible vertex configurations:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
					

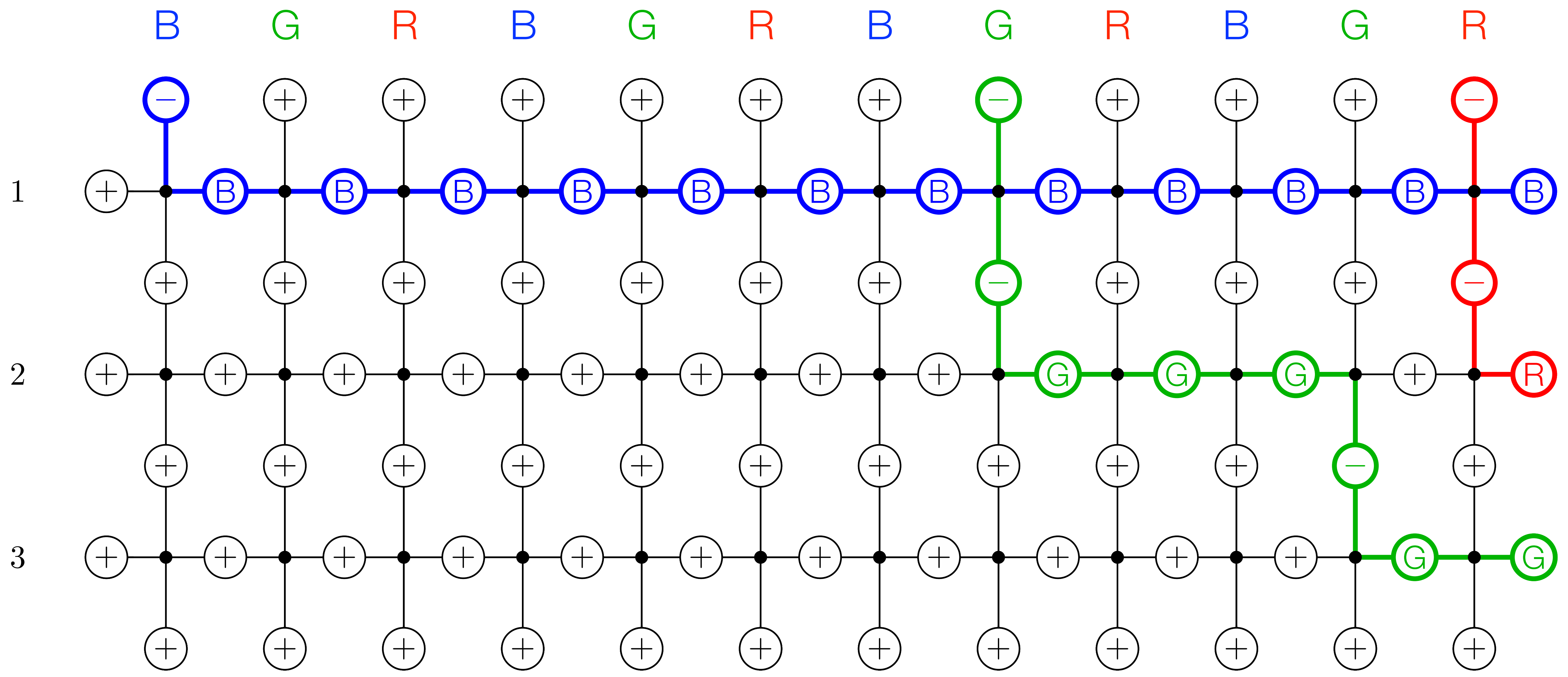
Each vertex is labeled by column color c and parameter z_i for row i denoted $T^c_{z_i}$



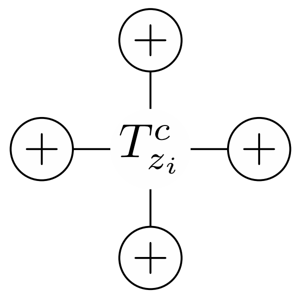
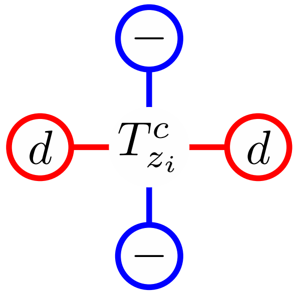
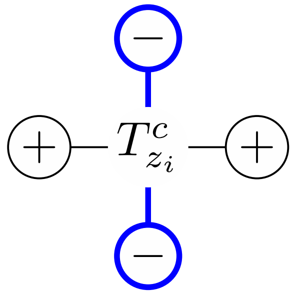
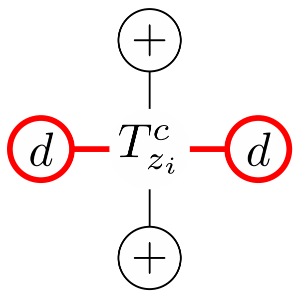
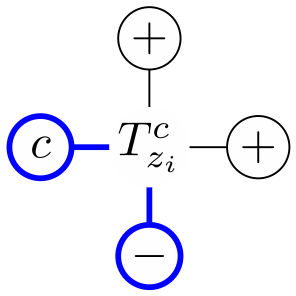
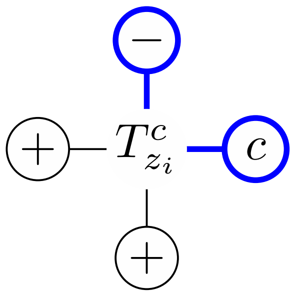
Inner edges from six possible vertex configurations:

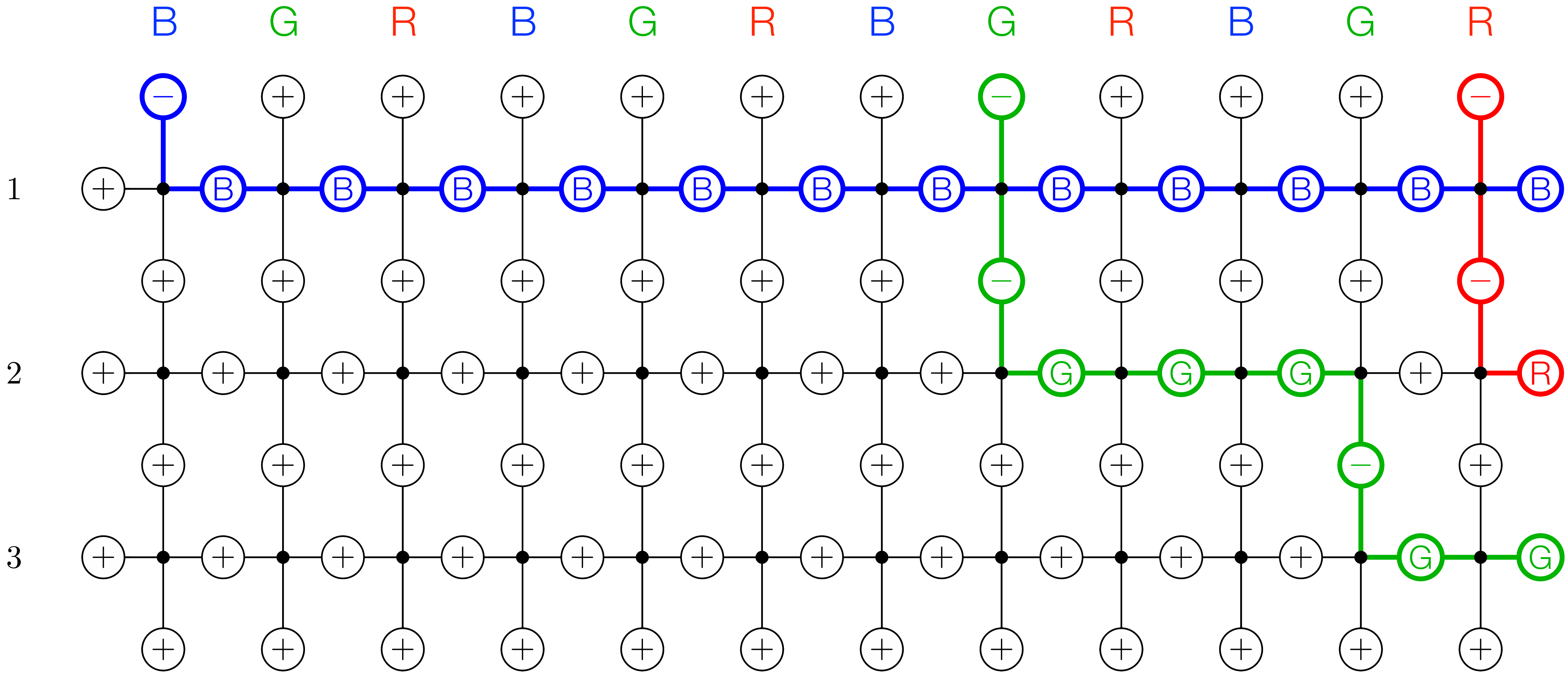
a_1	a_2	b_1	b_2	c_1	c_2
					

Each vertex is labeled by column color c and parameter z_i for row i denoted $T^c_{z_i}$



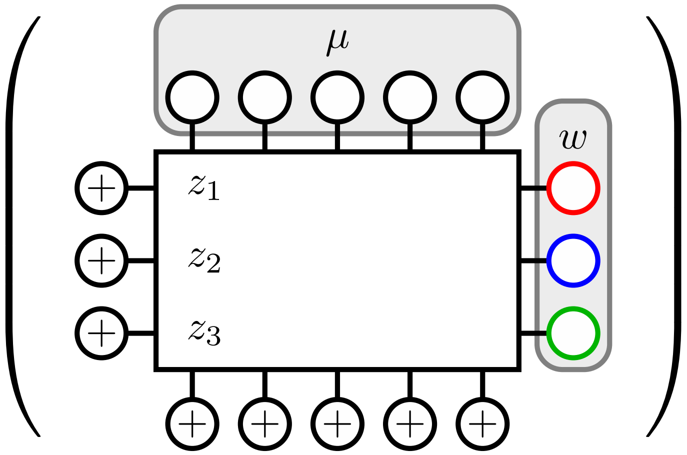
Inner edges from six possible vertex configurations: (parameter $v \in \mathbb{C}$)

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
						
Boltzmann weights	1	v if $c > d$ z_i if $c = d$ 1 if $c < d$	$-v$	z_i if $c = d$, 1 otherwise	$(1-v)z_i$	1



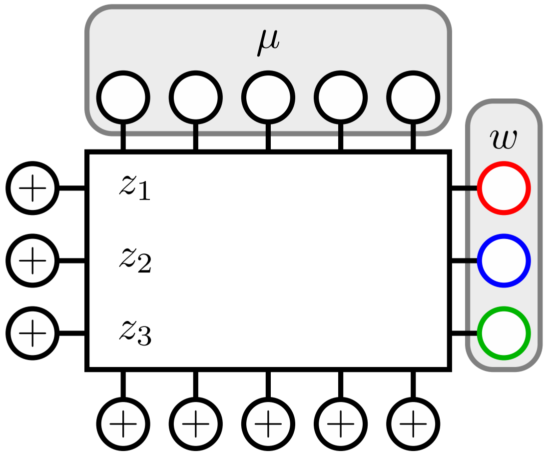
Partition function

$$Z_{\mu,w}(\mathbf{z}) := Z$$



$$= \sum_{\substack{\text{states with}\\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$

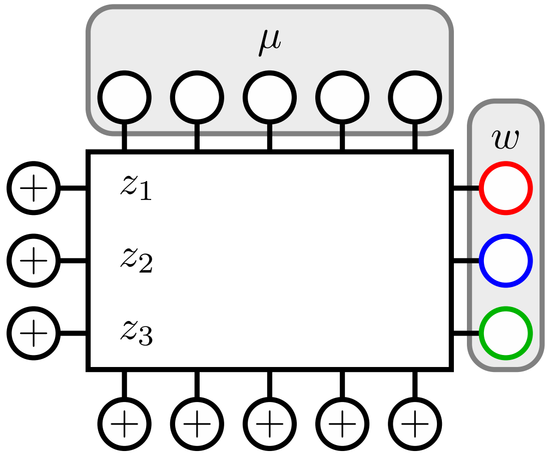
Partition function



$$Z_{\mu,w}(\mathbf{z}) := Z\left(\qquad \right) = \sum_{\substack{\text{states with}\\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$

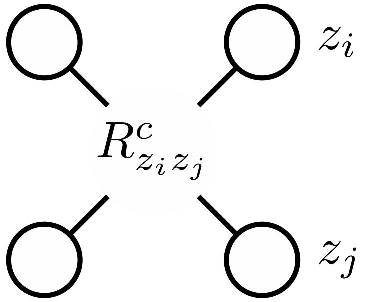
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: –Ⓑ– –Ⓖ– –Ⓡ– –⊕–

Partition function



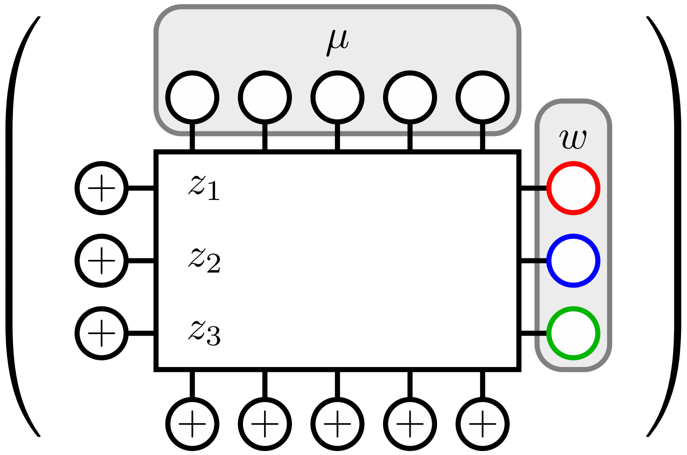
$$Z_{\mu,w}(\mathbf{z}) := Z\left(\quad \right) = \sum_{\substack{\text{states with}\\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: —B— —G— —R— —⊕—



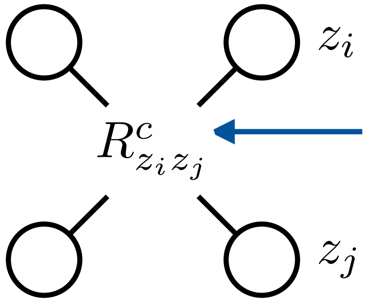
Partition function

$Z_{\mu,w}(\mathbf{z}) := Z$



$= \sum_{\text{states with given boundary}} \prod_{\text{vertices}}$ weights

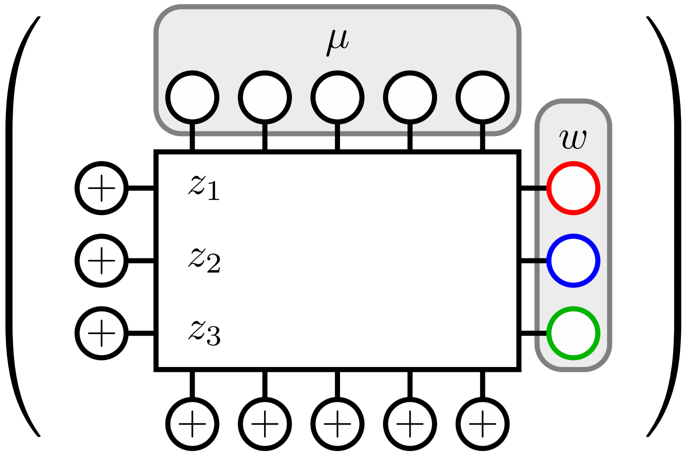
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: B G R ⊕



depends on column color c and row parameters z_i, z_j

Partition function

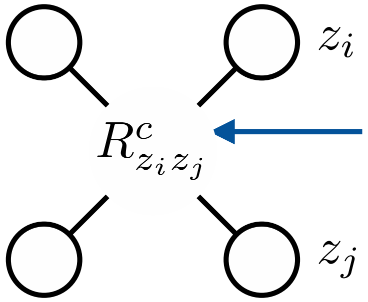
$Z_{\mu,w}(\mathbf{z}) := Z$



$= \sum\limits_{\substack{\text{states with}\\ \text{given boundary}}} \prod\limits_{\text{vertices}}$ weights

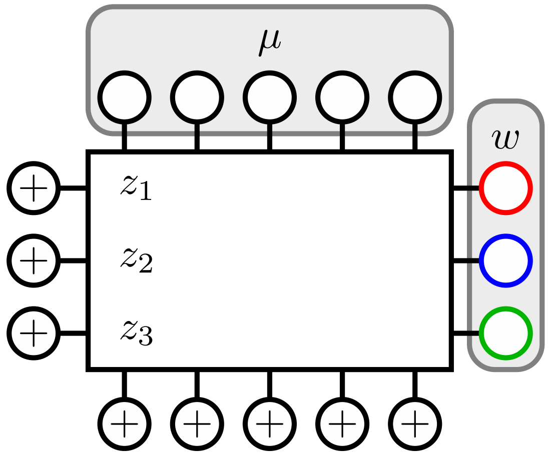
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: B G R ⊕ (table of weights in paper)

[arXiv:1906.04140]



depends on column color c and row parameters z_i, z_j

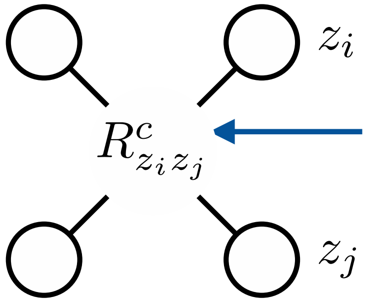
Partition function

$Z_{\mu,w}(\mathbf{z}) := Z\Big($ 

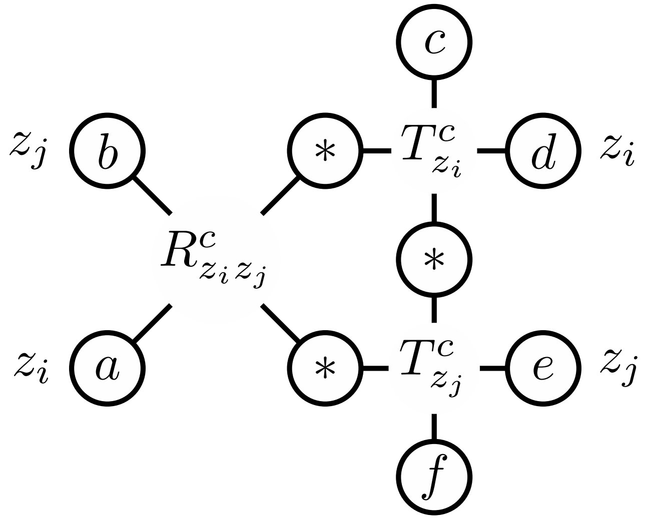
$\Big) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}}$ weights

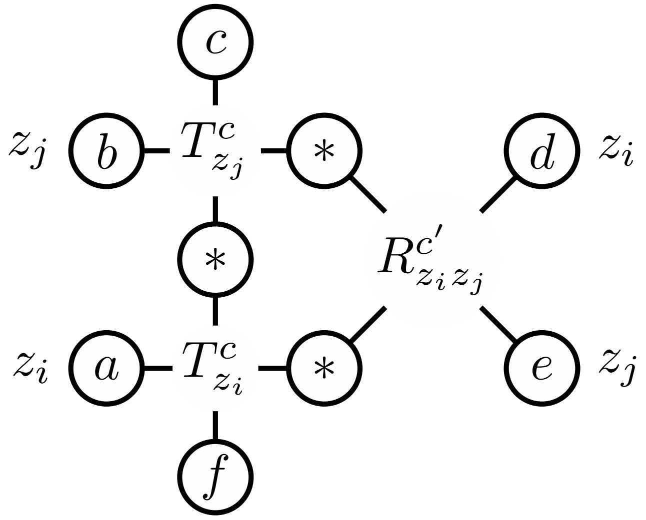
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: B G R + (table of weights in paper)

[arXiv:1906.04140]



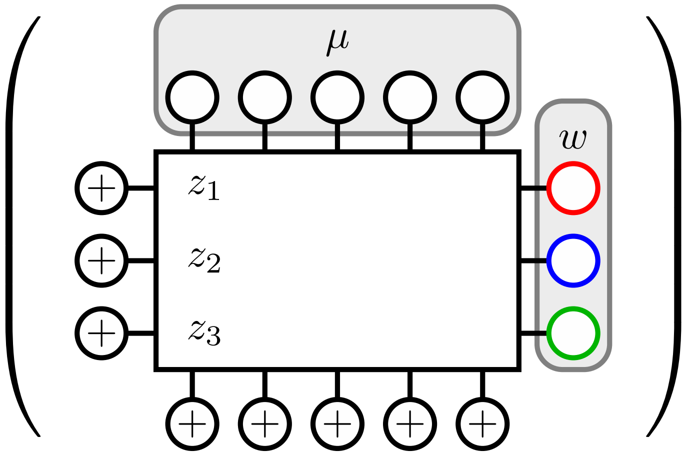
depends on column color c and row parameters z_i, z_j

$Z\Big($ 

$\Big) = Z\Big($ 

$\Big)$

Partition function



$$Z_{\mu,w}(\mathbf{z}) := Z\left(\quad\right) = \sum_{\substack{\text{states with}\\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: –Ⓑ– –Ⓖ– –Ⓡ– –⊕– (table of weights in paper)

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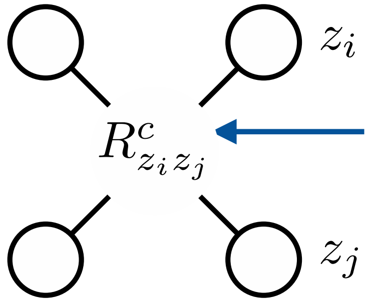
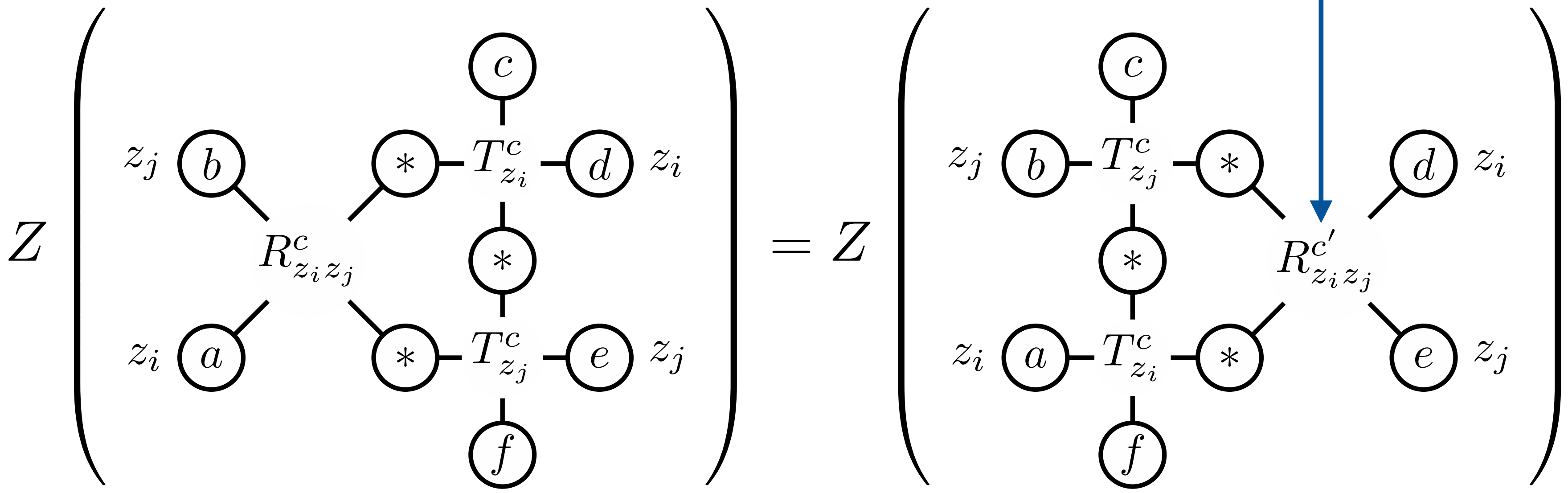


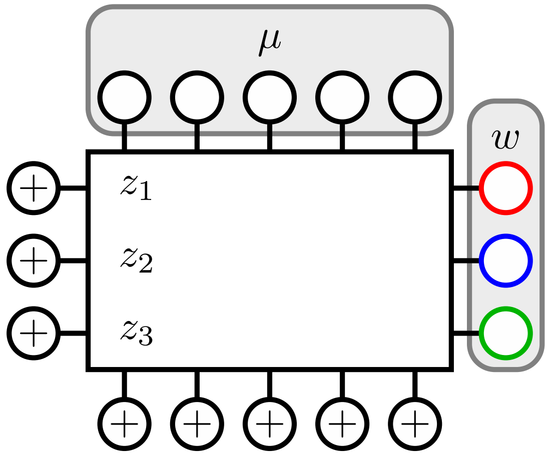
depends on column color c and row parameters z_i, z_j

Next column color (wraps)



Partition function

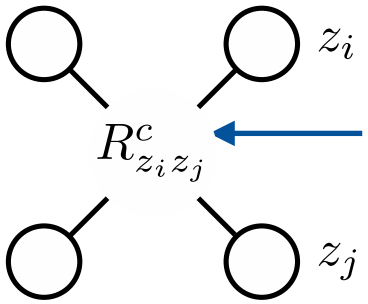
$Z_{\mu,w}(\mathbf{z}) := Z\Bigg($



$\Bigg) = \sum\limits_{\text{states with given boundary}} \prod\limits_{\text{vertices}}$ weights

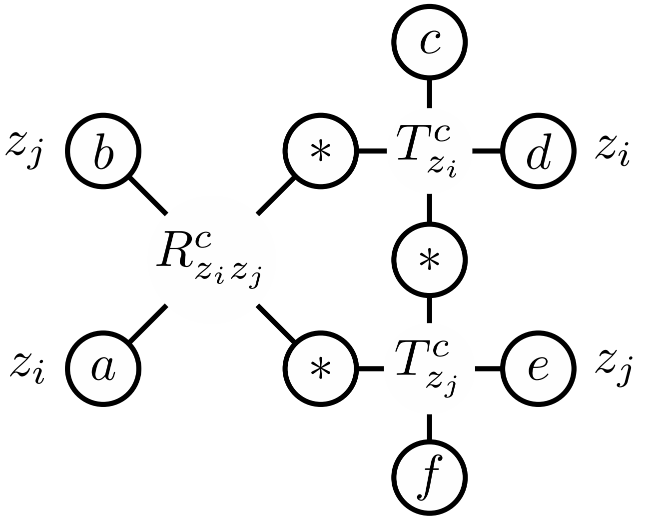
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: B G R + (table of weights in paper)

[arXiv:1906.04140]



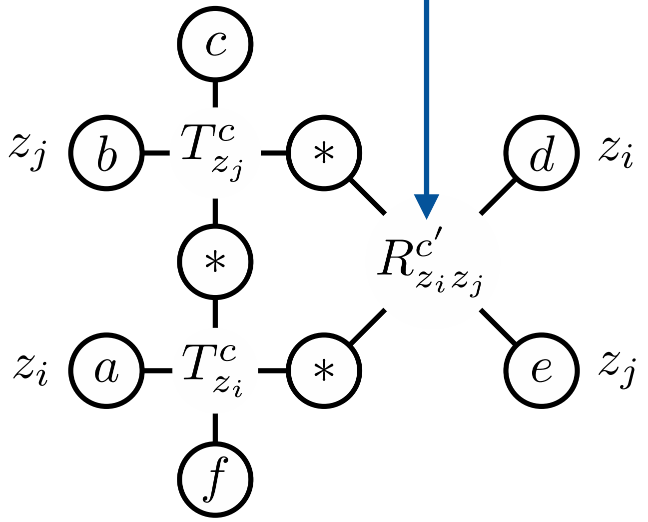
depends on column color c and row parameters z_i, z_j

$Z\Bigg($



$\Bigg) = Z\Bigg($

Next column color (wraps)



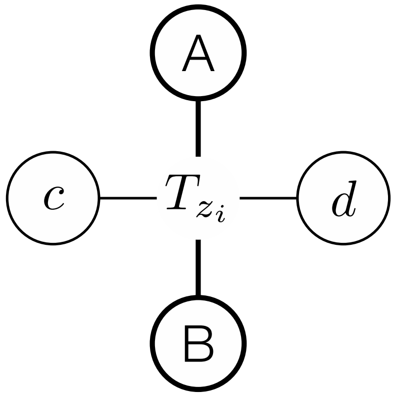
$\Bigg)$

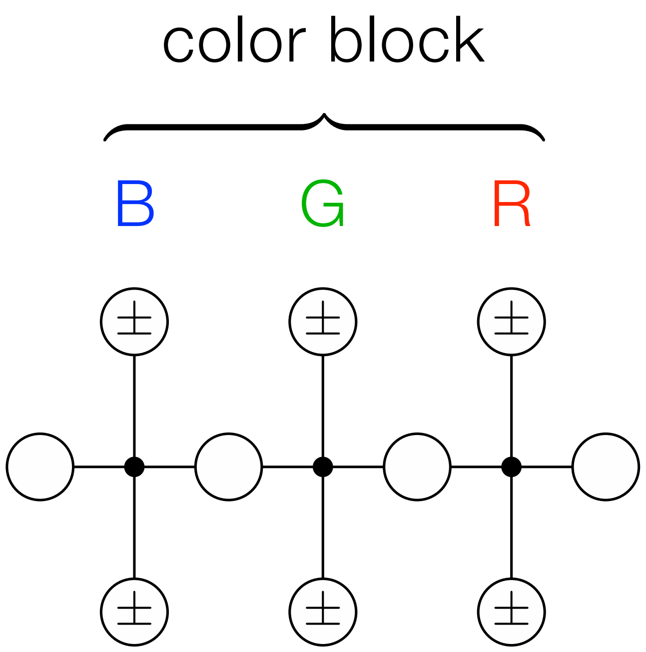
unfused

Fusion

$\mathsf{A}, \mathsf{B} \in \mathcal{P}(\{\mathsf{B}, \mathsf{G}, \mathsf{R}\})$

$c, d \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}, +\}$

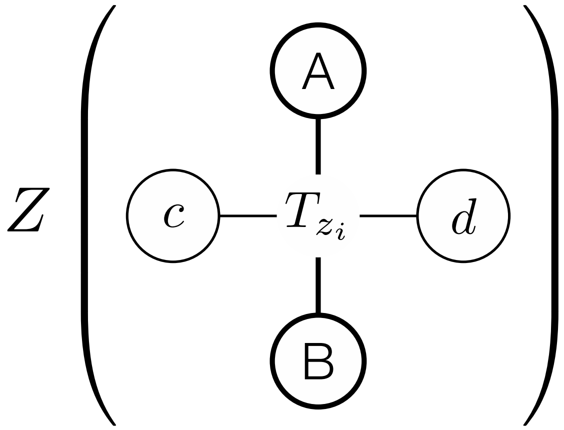




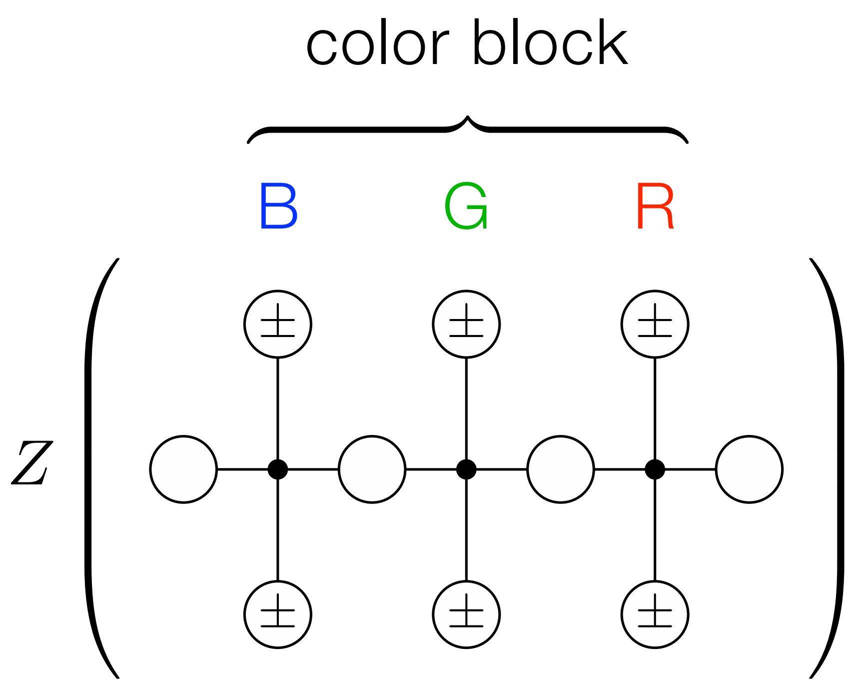
Fusion

$\mathsf{A}, \mathsf{B} \in \mathcal{P}(\{\color{blue}{\mathsf{B}}, \color{green}{\mathsf{G}}, \color{red}{\mathsf{R}}\})$

$c, d \in \{\color{blue}{\mathsf{B}}, \color{green}{\mathsf{G}}, \color{red}{\mathsf{R}}, +\}$



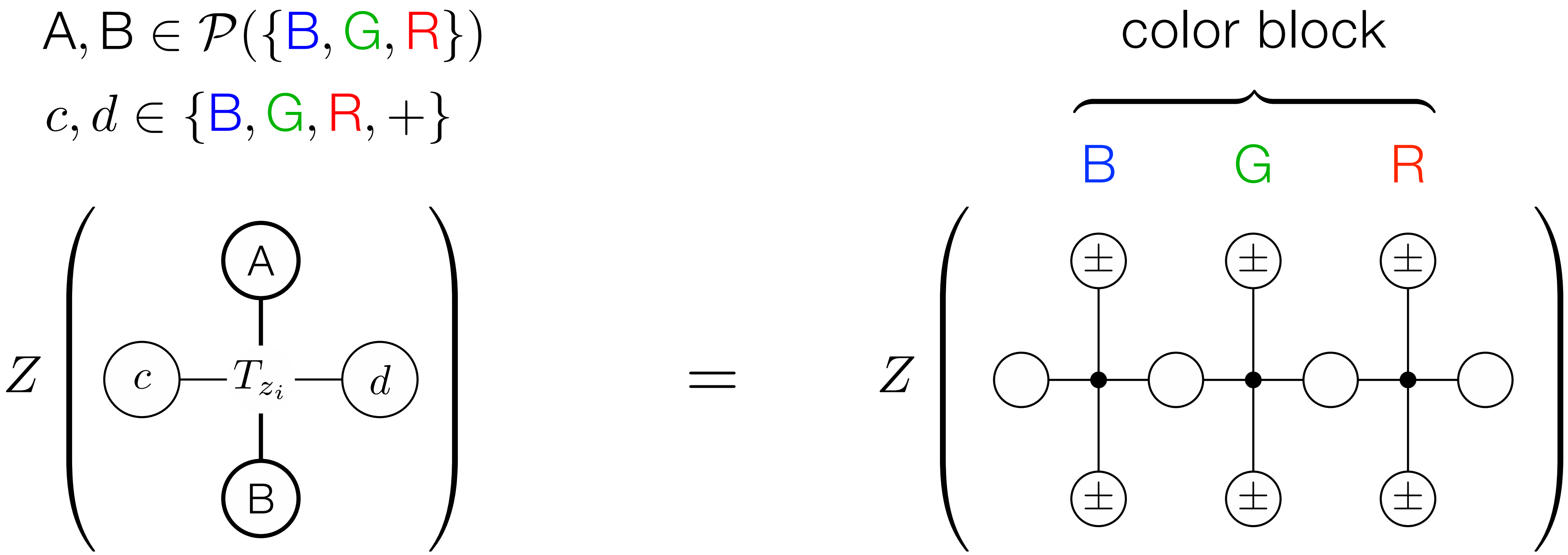
$=$



Fusion

$\mathsf{A}, \mathsf{B} \in \mathcal{P}(\{\mathsf{B}, \mathsf{G}, \mathsf{R}\})$

$c, d \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}, +\}$



Example $r = 2$

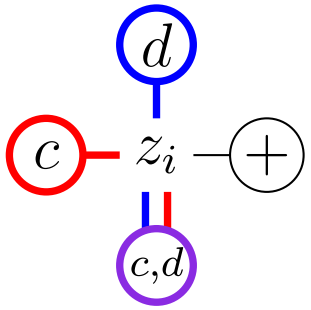
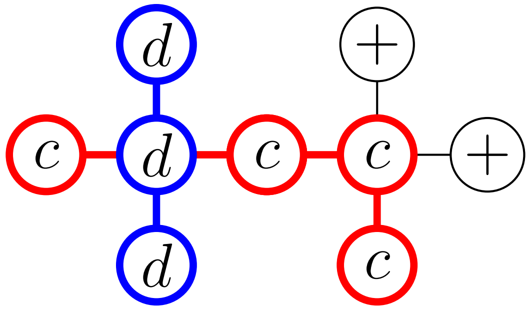
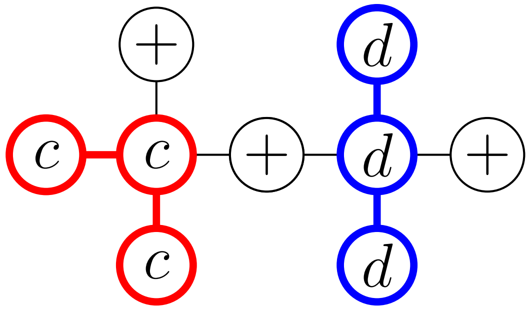
	$c > d$	$c < d$
		
$(1-v)z_i$ if $c > d$, $(-v)(1-v)z_i$ if $c < d$.	left: 1 right: $(1-v)z_i$	left: $(1-v)z_i$ right: $-v$

Figure 8 in arXiv:1906.04140

Fusion

Fused Boltzmann weights: (r colors)

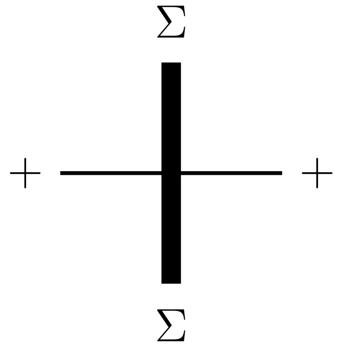
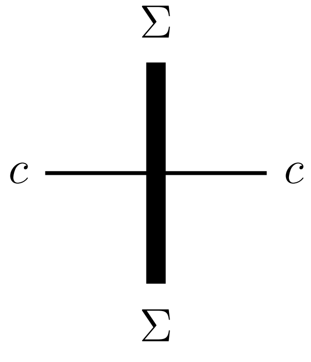
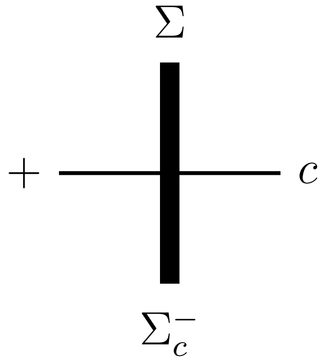
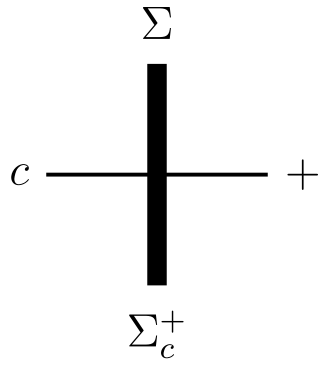
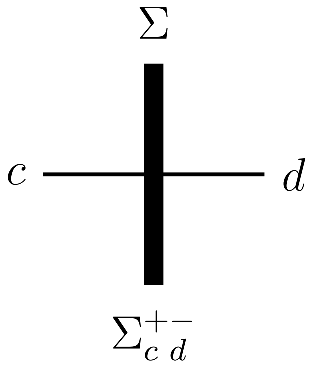
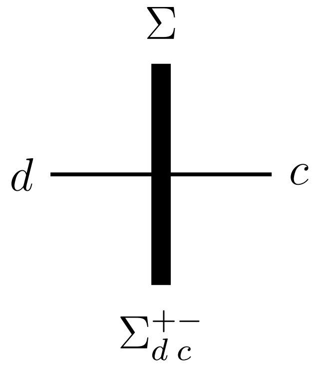
 $(-v)^{\lvert\Sigma_{[1,r]}\rvert}$	 $z_i v^{\lvert\Sigma_{[c+1,r]}\rvert}$	 $(-v)^{\lvert\Sigma_{[1,c-1]}\rvert} v^{\lvert\Sigma_{[c+1,r]}\rvert}$
 $(1-v)z_i(-v)^{\lvert\Sigma_{[c+1,r]}\rvert}$	 $(1-v)z_i(-v)^{\lvert\Sigma_{[c+1,d-1]}\rvert} v^{\lvert\Sigma_{[d+1,r]}\rvert}$	 0

Figure 12 in arXiv:1906.04140

$$\Sigma \in \mathcal{P}(\{c_1, \dots, c_r\})$$

Fusion

Fused Boltzmann weights: (r colors)

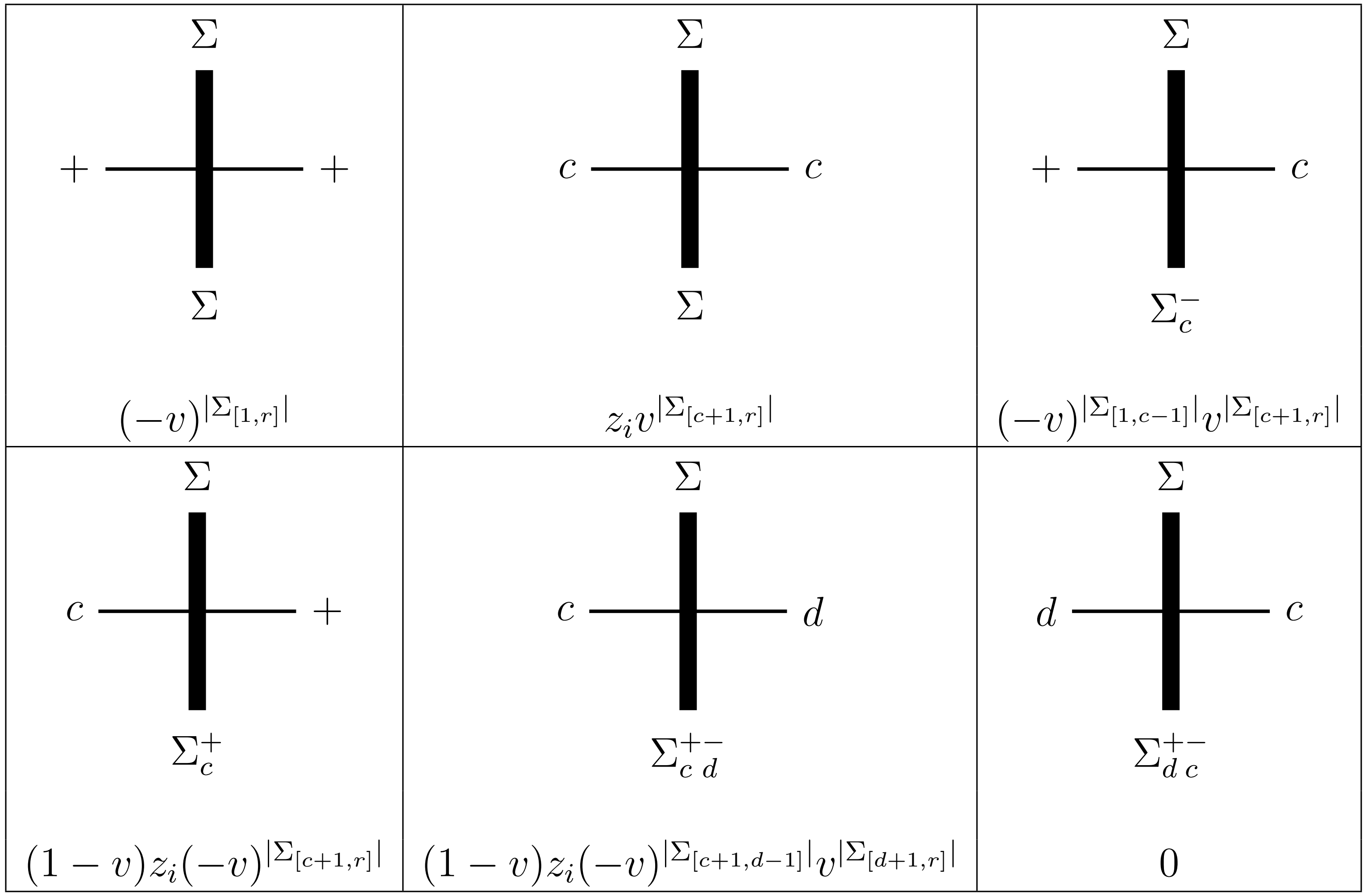


Figure 12 in arXiv:1906.04140

$$\Sigma \in \mathcal{P}(\{c_1, \ldots, c_r\})$$

$\Sigma_a^+ = \Sigma \cup \{a\}$ if $a \notin \Sigma$

$\Sigma_a^- = \Sigma \setminus \{a\}$ if $a \in \Sigma$

$\Sigma_{a\,b}^{+-} = \Sigma \cup \{a\} \setminus \{b\}$ if $a \in \Sigma, b \notin \Sigma$

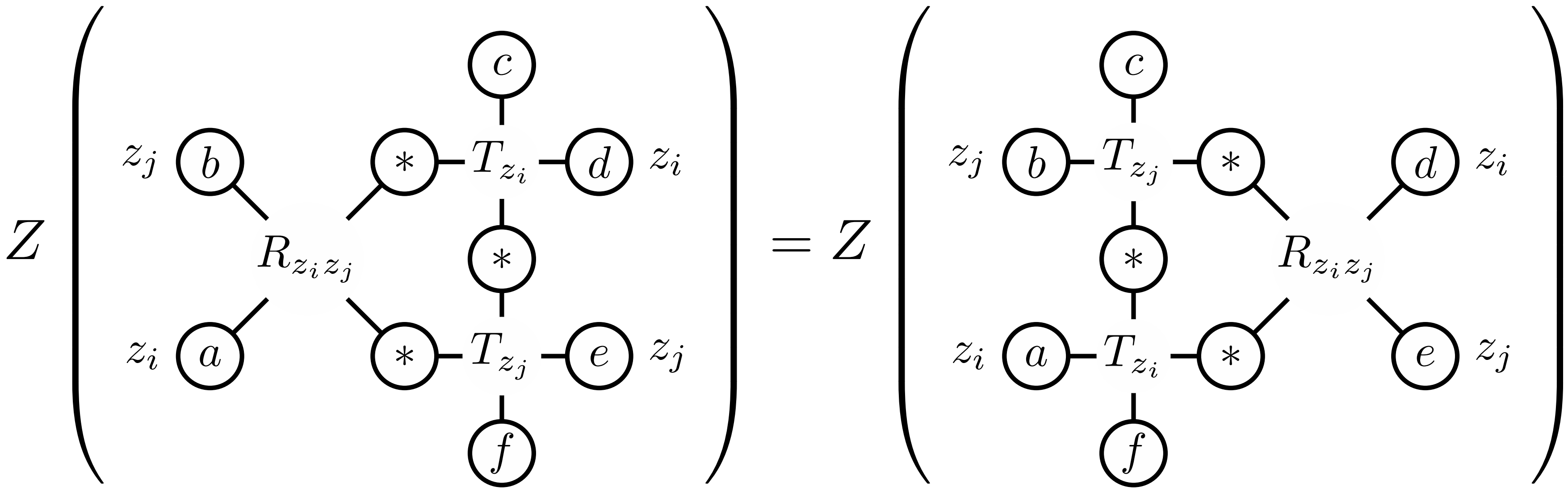
$\Sigma_{[a,b]} = \{c \in \Sigma \mid a \leqslant c \leqslant b\}$

Fusion

The fused lattice model has an ordinary Yang–Baxter equation with R-matrix independent of column which comes from a Drinfeld twist of $U_q\big(\widehat{\mathfrak{gl}}(r|1)\big)$.

Fusion

The fused lattice model has an ordinary Yang–Baxter equation with R-matrix independent of column which comes from a Drinfeld twist of $U_q(\widehat{\mathfrak{gl}}(r|1))$.



Fusion

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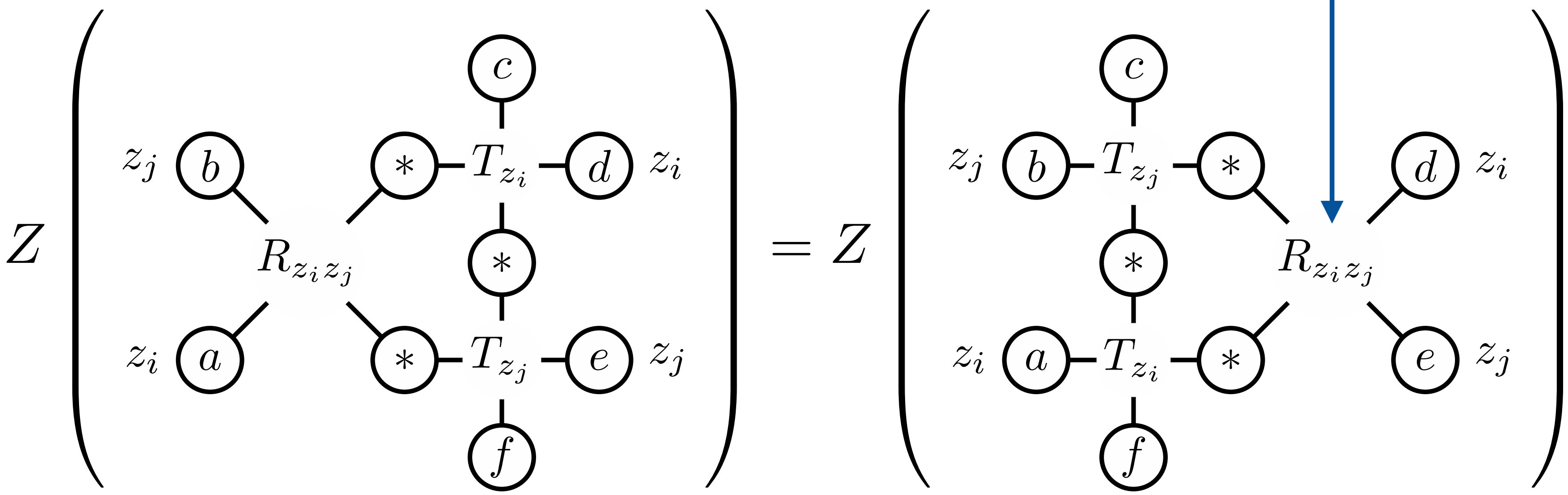
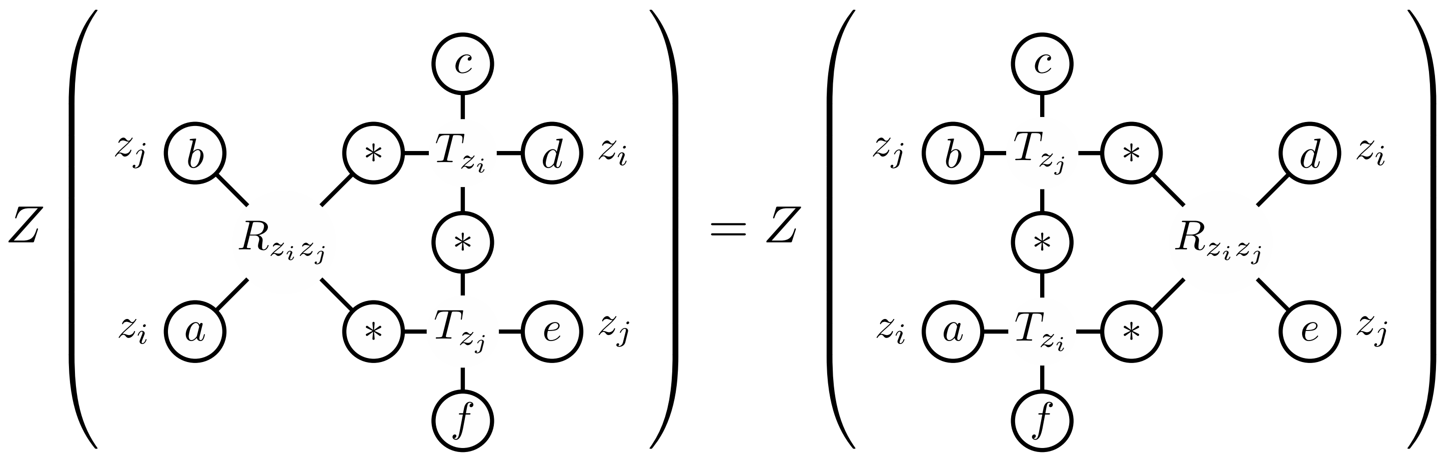
$R_{z_i,z_j} = R^{c_1}_{z_i,z_j}$

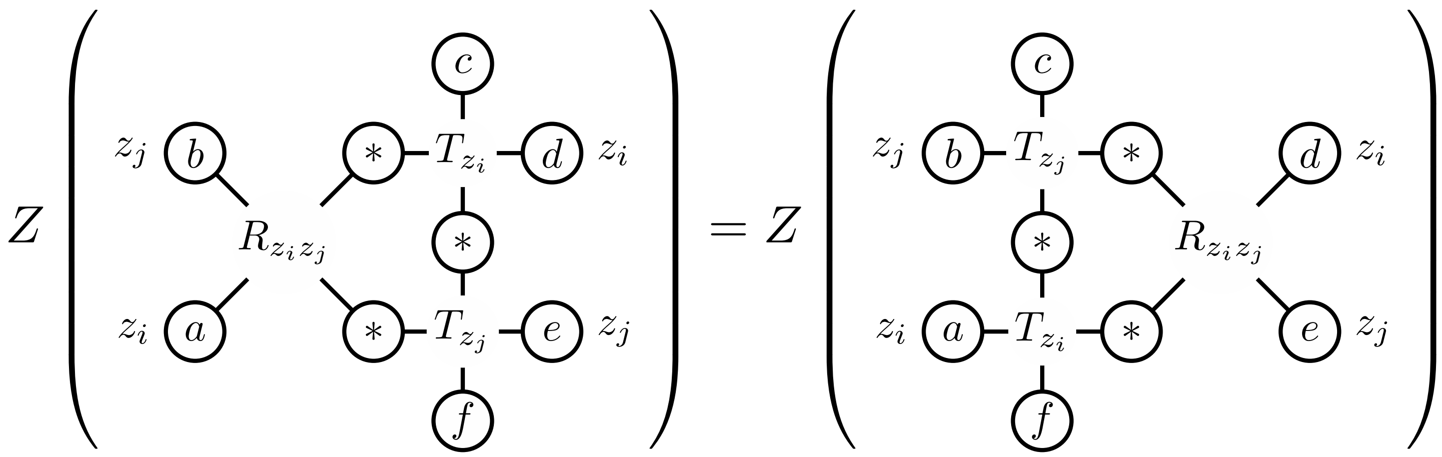


fused

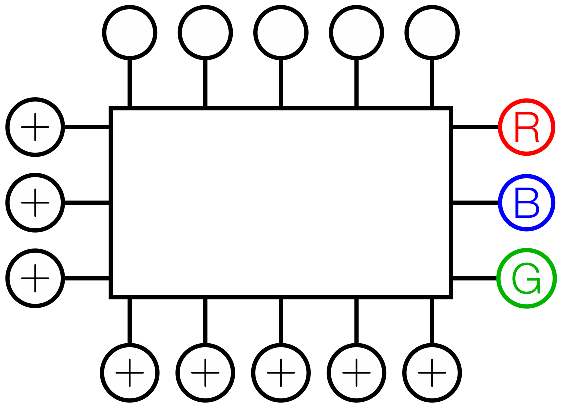
Recursion relations



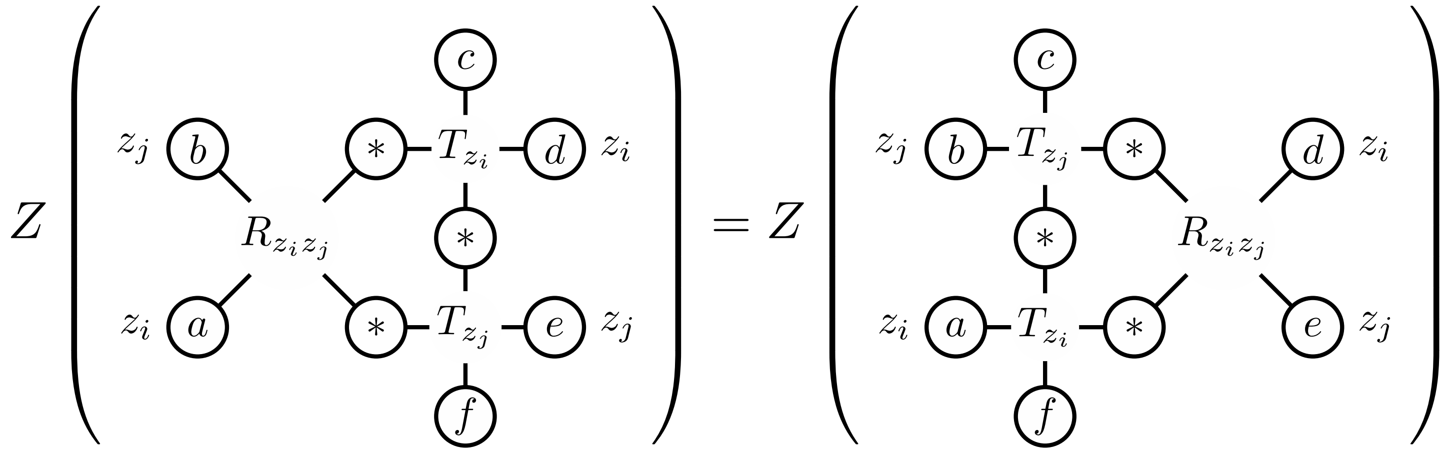
Recursion relations



$Z_{\mu,w}(\mathbf{z}) :=$



Recursion relations



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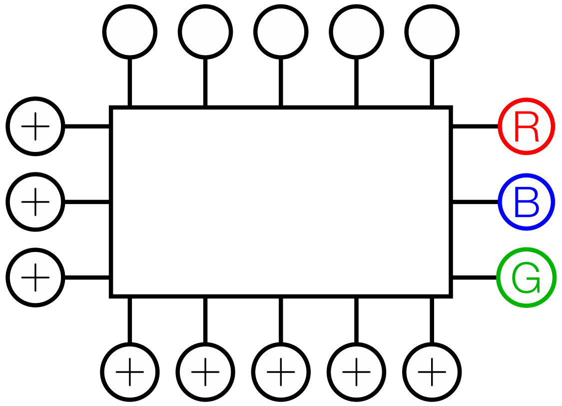
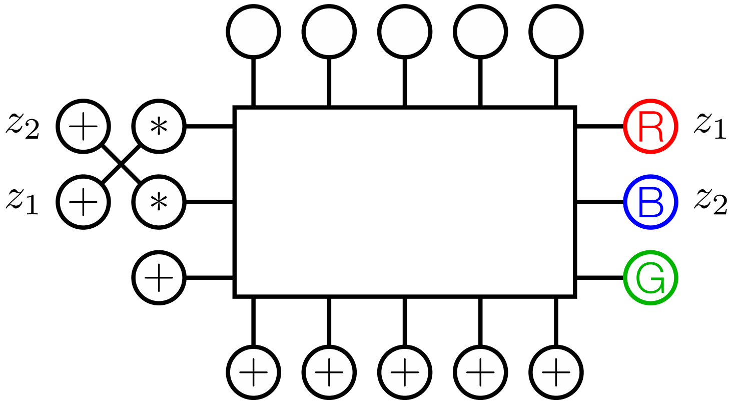
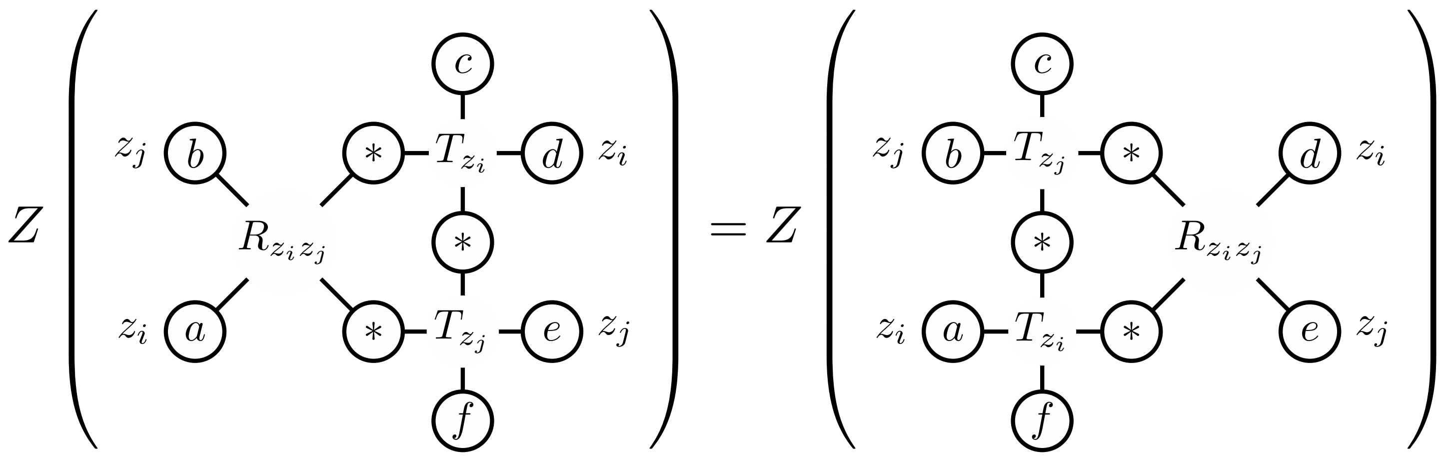


Train argument



Recursion relations



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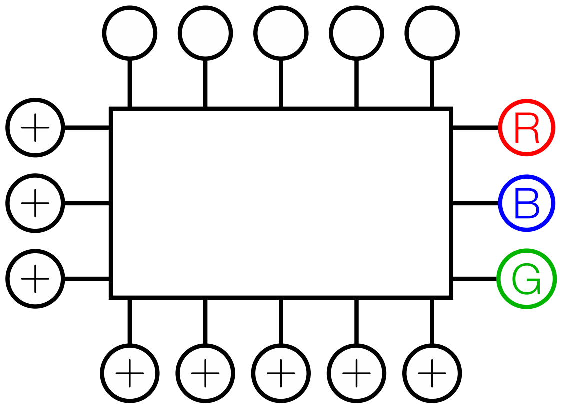
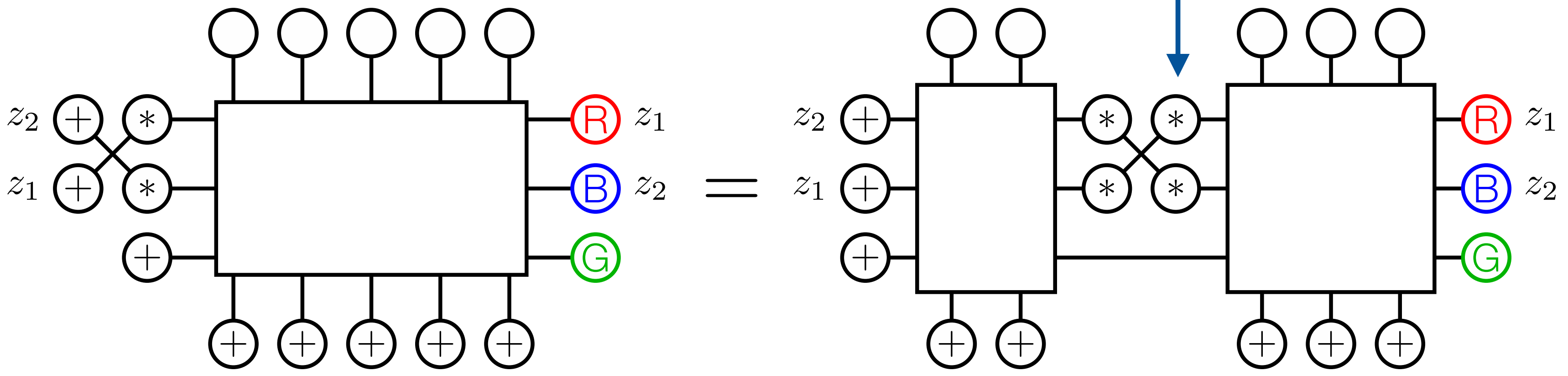
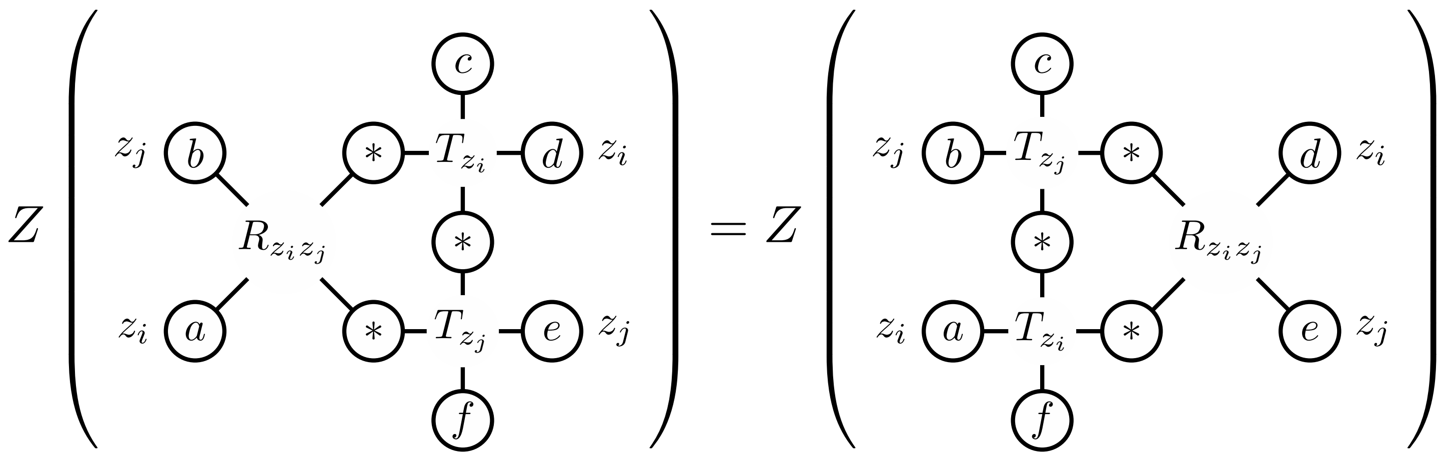


Train argument

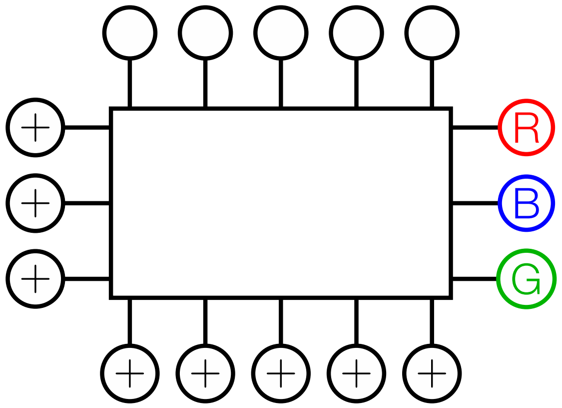


sum over internal states

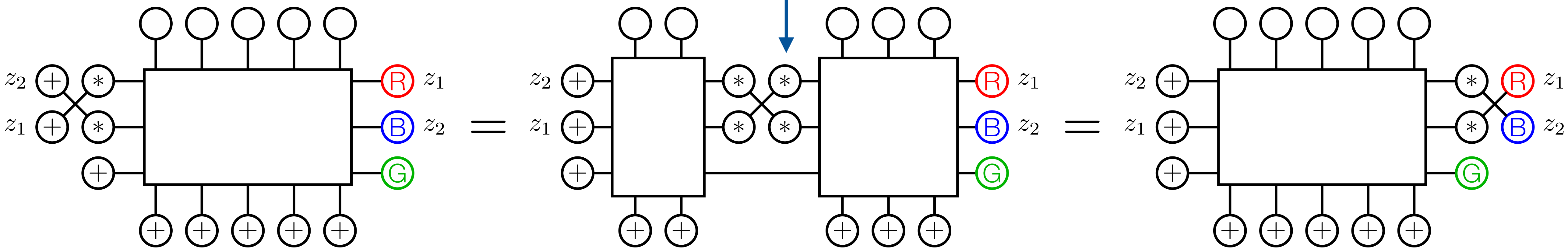
Recursion relations



$Z_{\mu,w}(\mathbf{z}) :=$

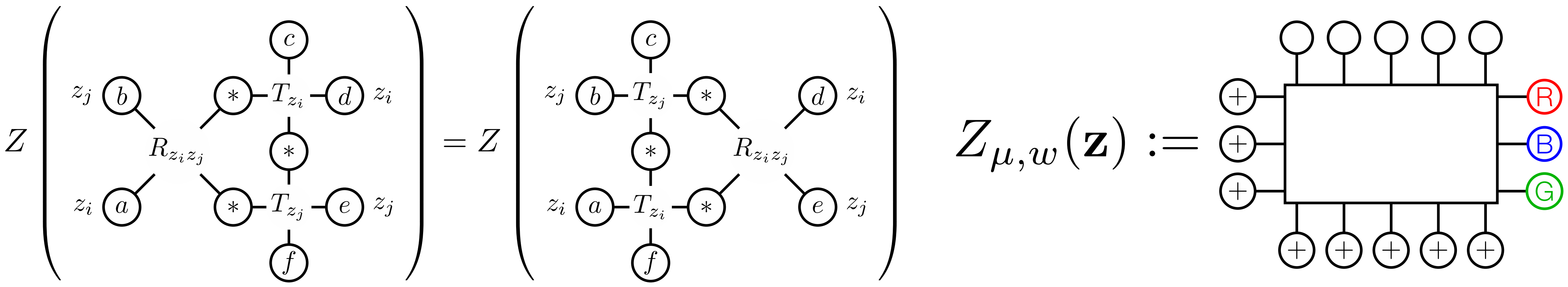


Train argument

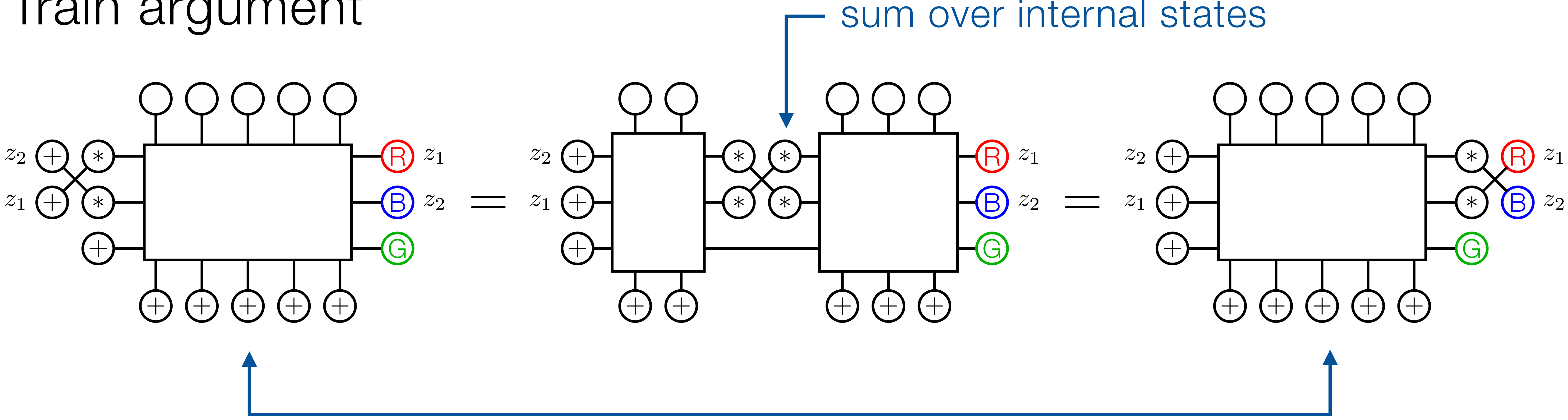


sum over internal states

Recursion relations



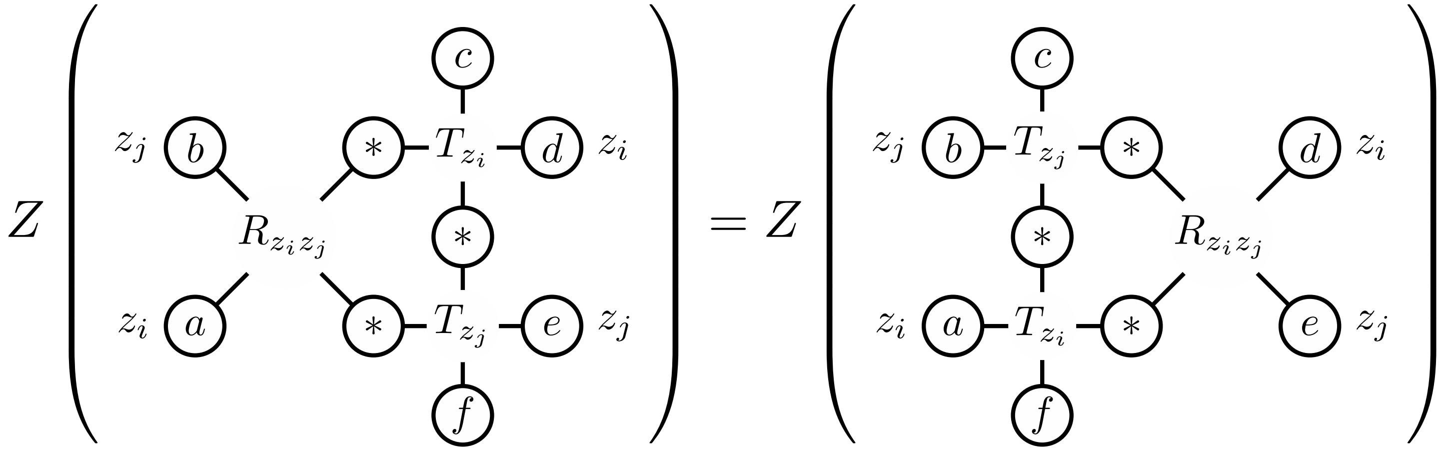
Train argument



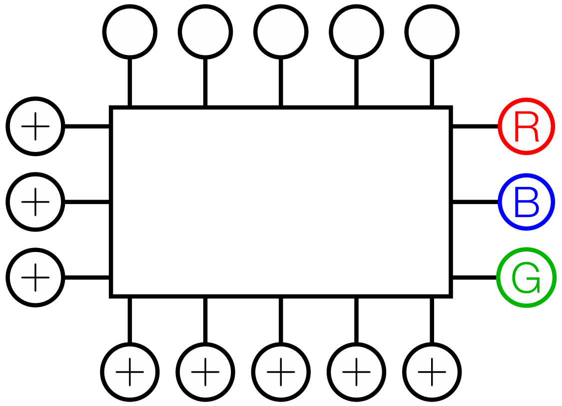
sum over internal states

Insert admissible R-vertices and compare

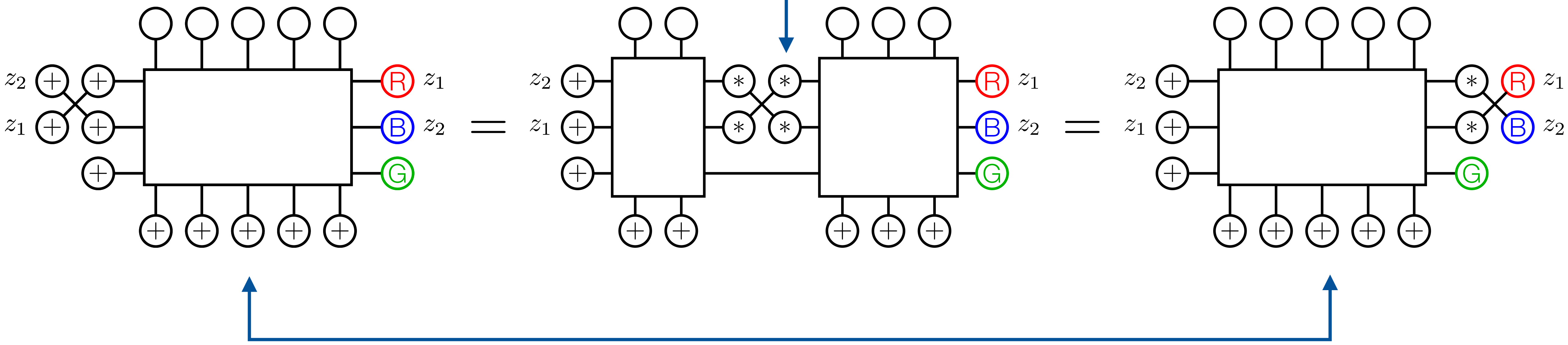
Recursion relations



$Z_{\mu,w}(\mathbf{z}) :=$



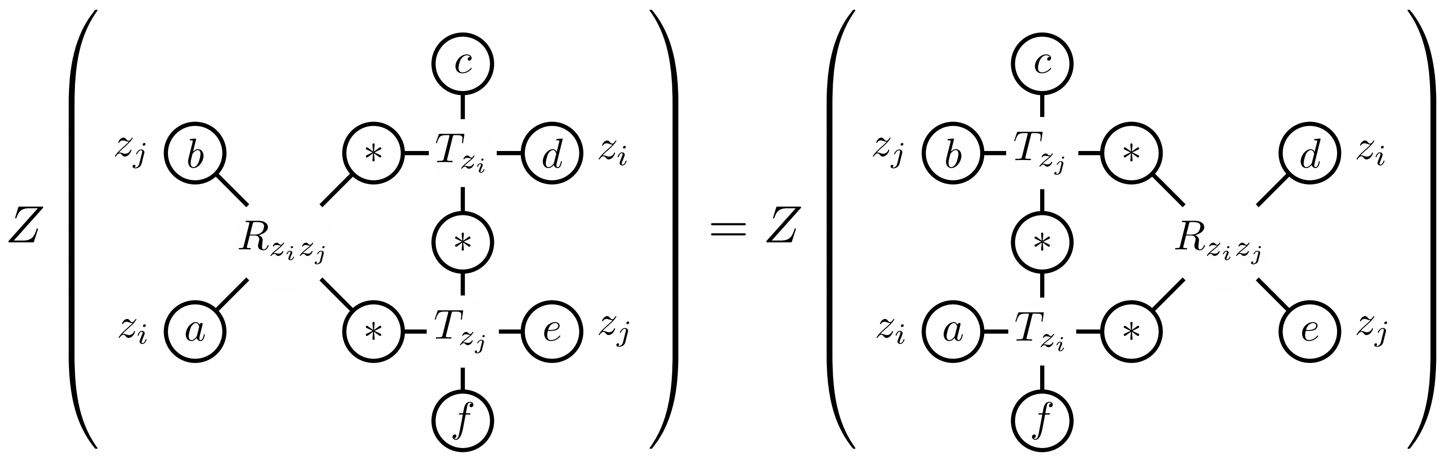
Train argument



sum over internal states

Insert admissible R-vertices and compare

Recursion relations



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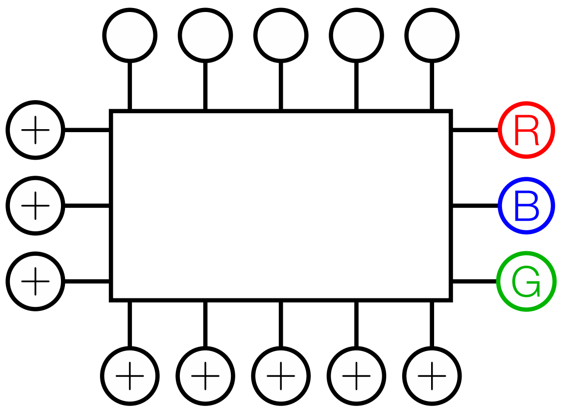
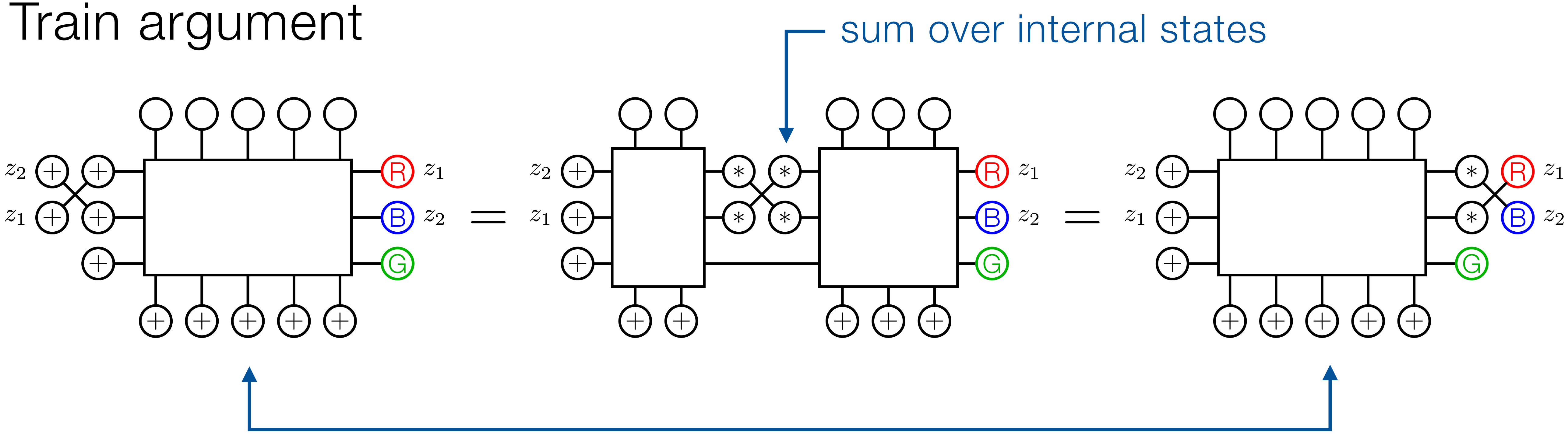


Train argument

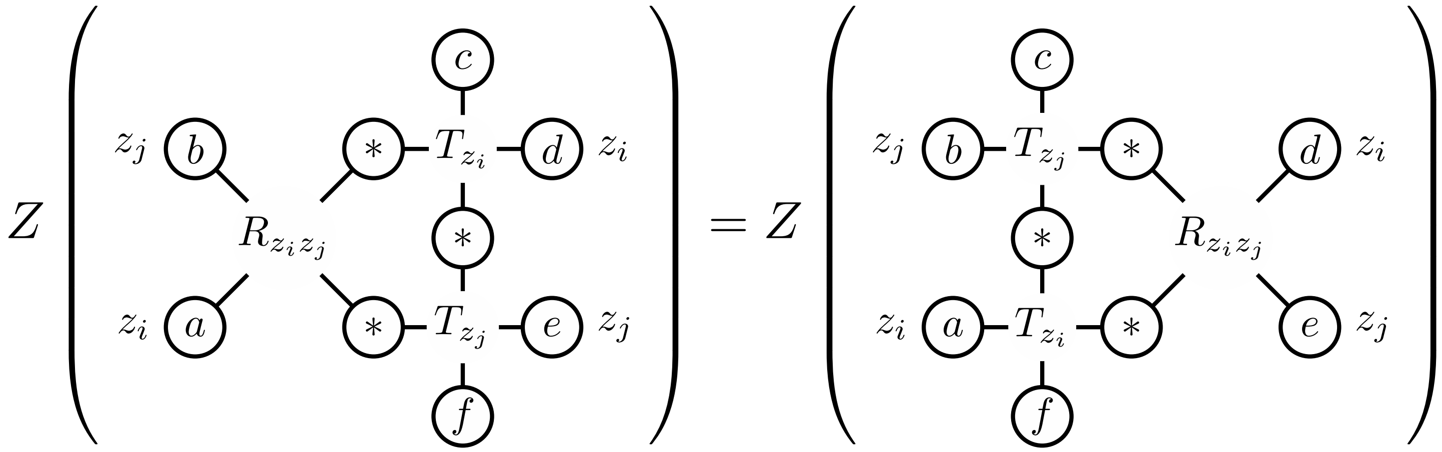


sum over internal states

Insert admissible R-vertices and compare

The odd and even parts of the $U_q(\widehat{\mathfrak{gl}}(r|1))$ only mix in the middle steps

Recursion relations



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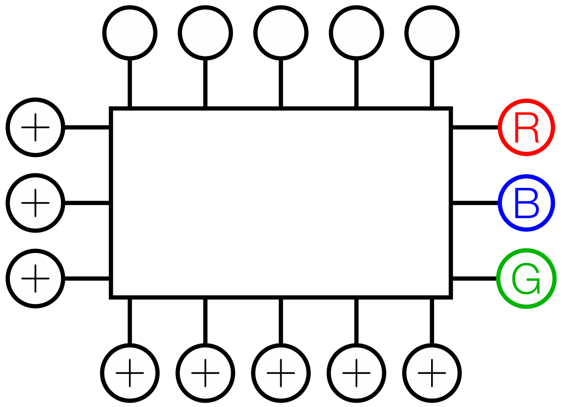
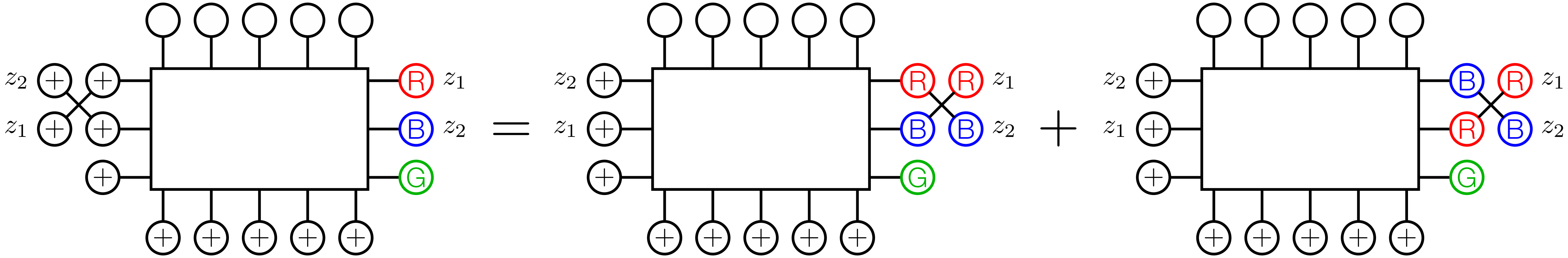
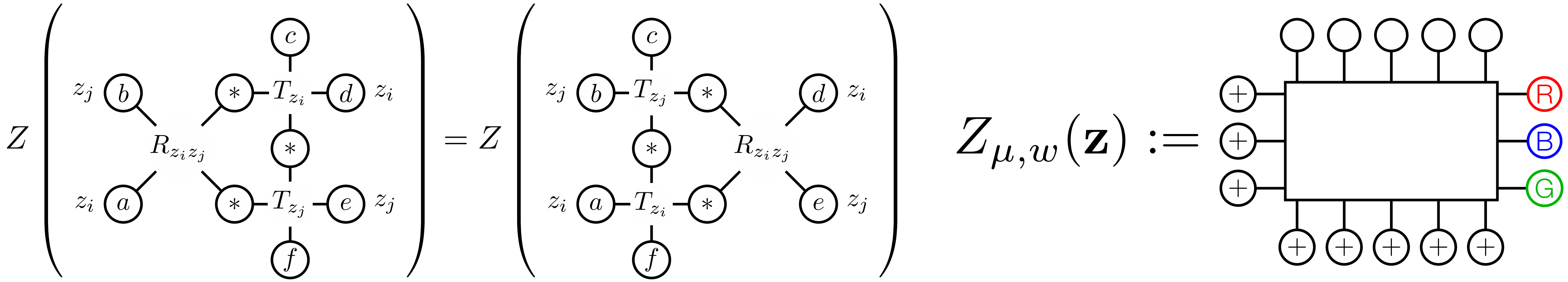


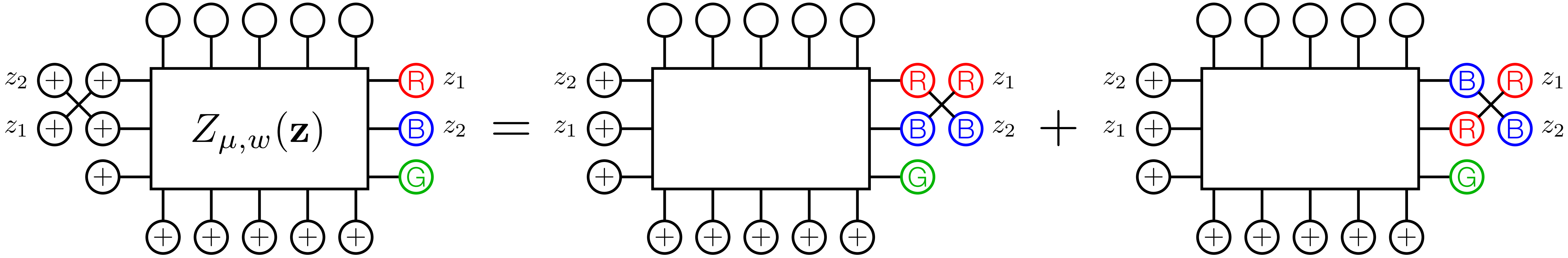
Train argument



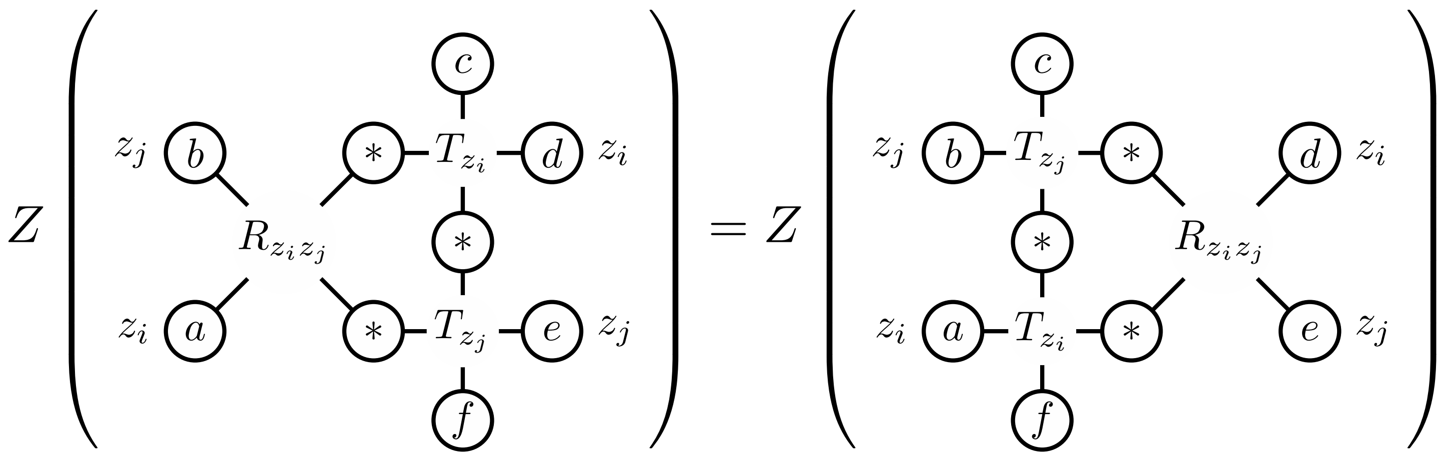
Recursion relations



Train argument



Recursion relations



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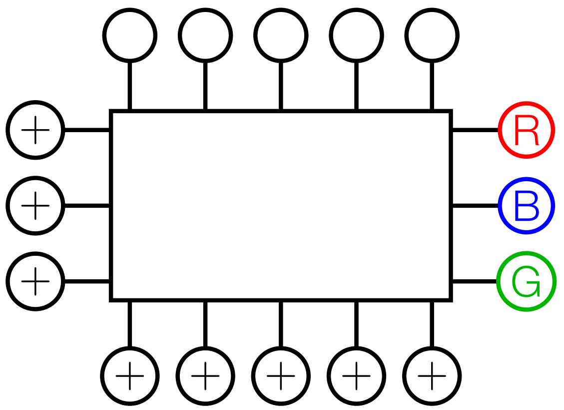
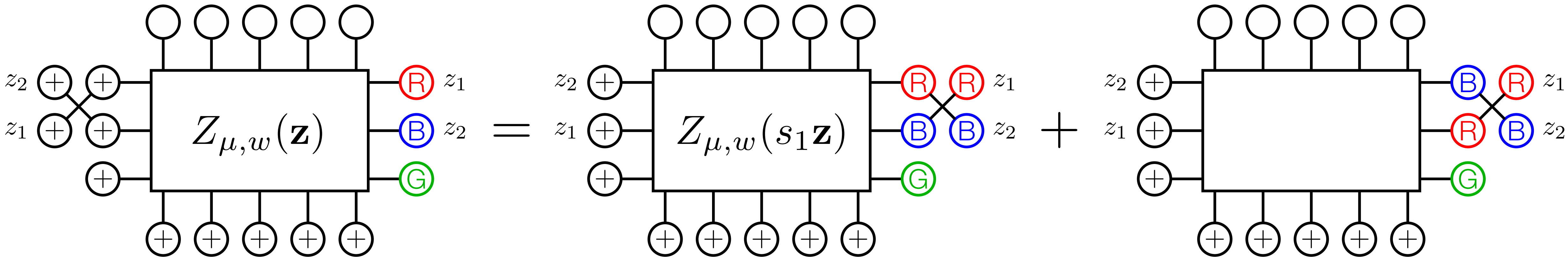
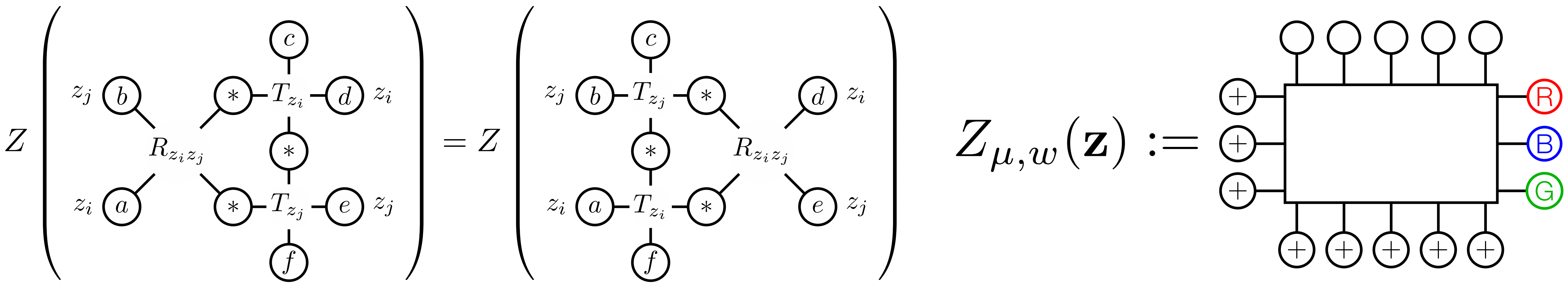


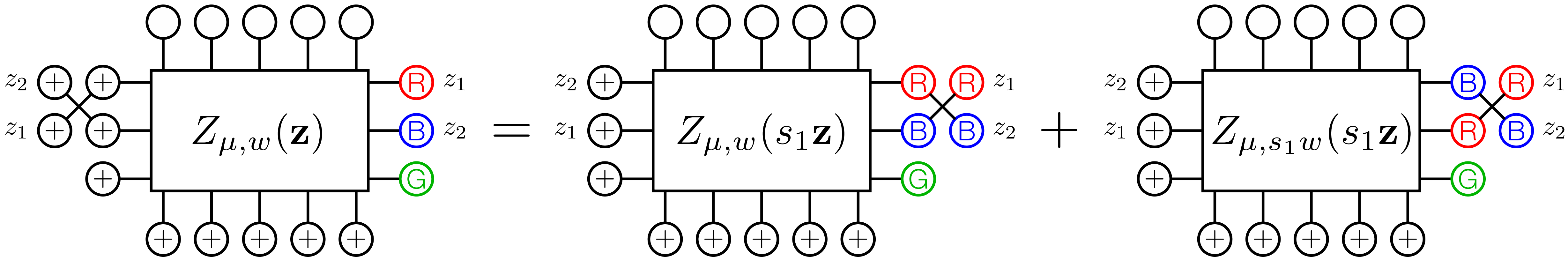
Train argument



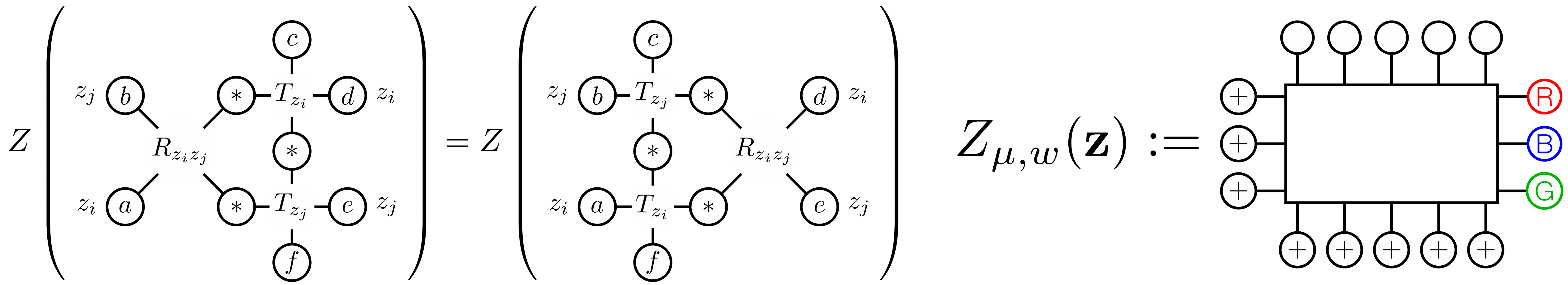
Recursion relations



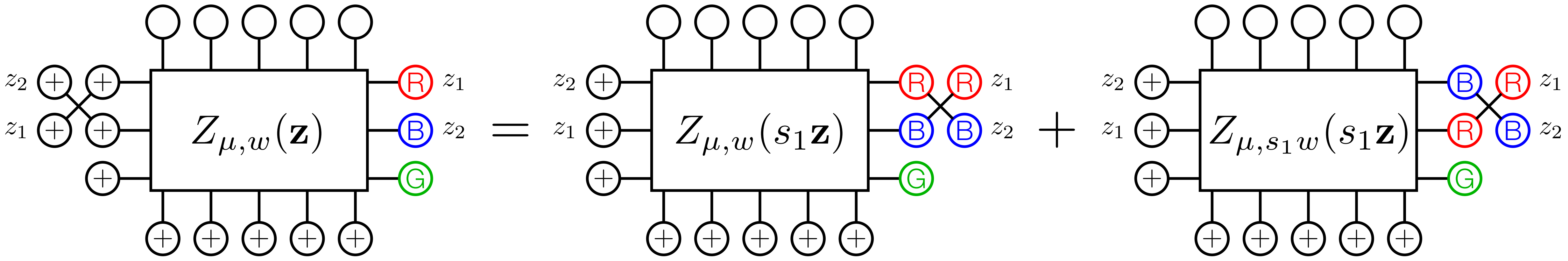
Train argument



Recursion relations



Train argument



Solving for $Z_{\mu,s_1 w}$ gives a recursion relation in w.

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

Demazure operators

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$s_i f(\mathbf{z}) = f(s_i\mathbf{z})$ $\mathbf{z}^\lambda = \prod_i z_i^{\lambda_i}$ $\mathbf{z}^{\alpha_i} = z_i/z_{i+1}$

$T_i \circlearrowright$ polynomials in $\mathbf{z} = (z_1, \ldots, z_r)$

(finite geometric series)

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

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(finite geometric series)

Base case

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

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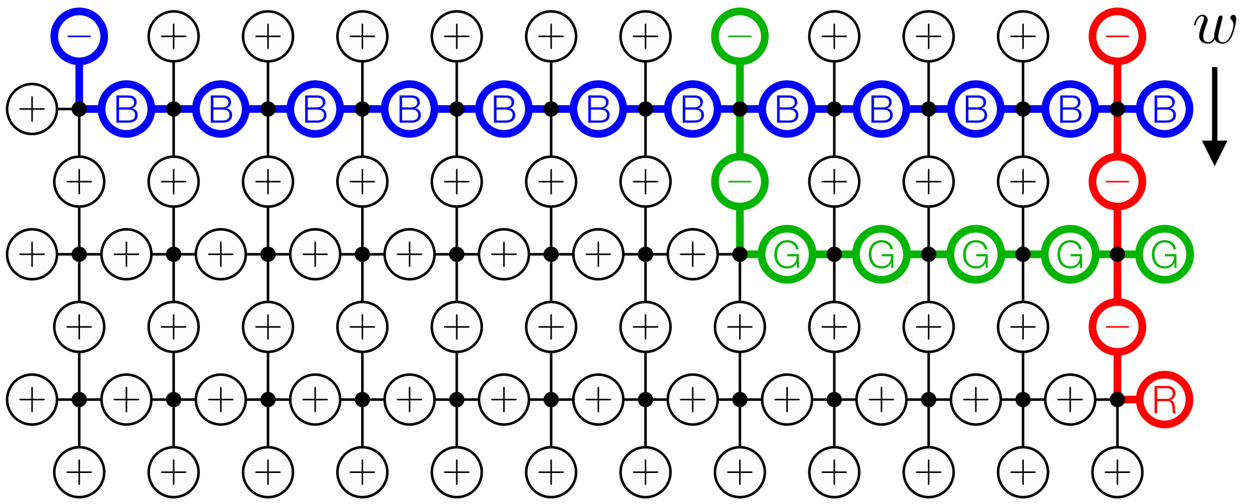
(finite geometric series)

Base case



Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

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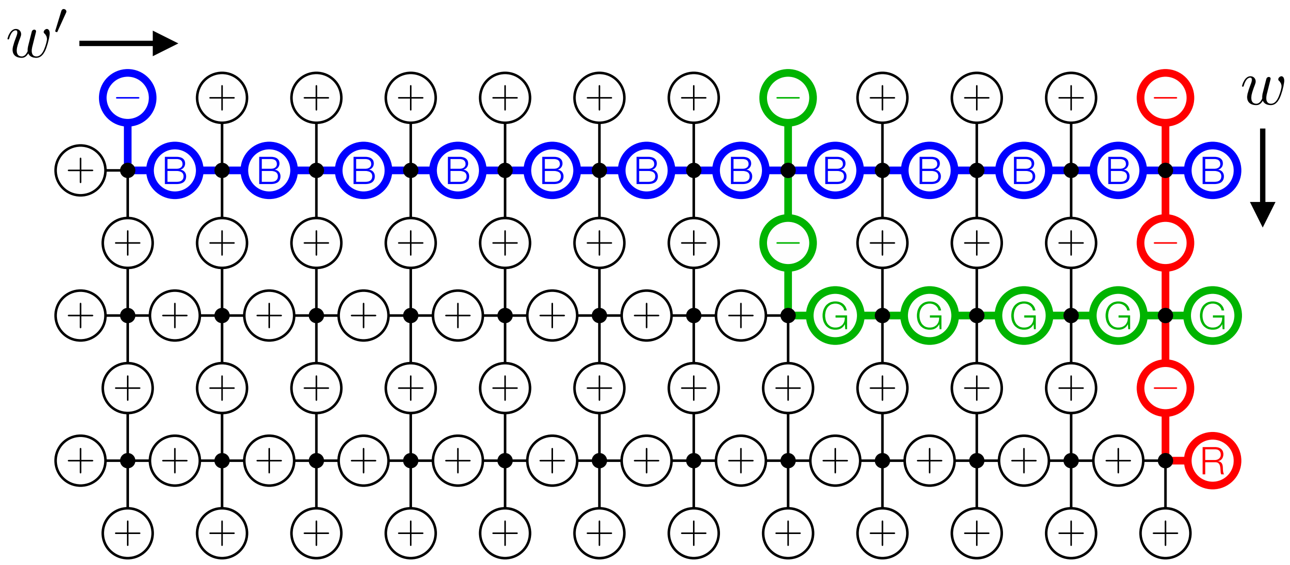
(finite geometric series)

Base case



Recursion relations

$$Z_{\mu,s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

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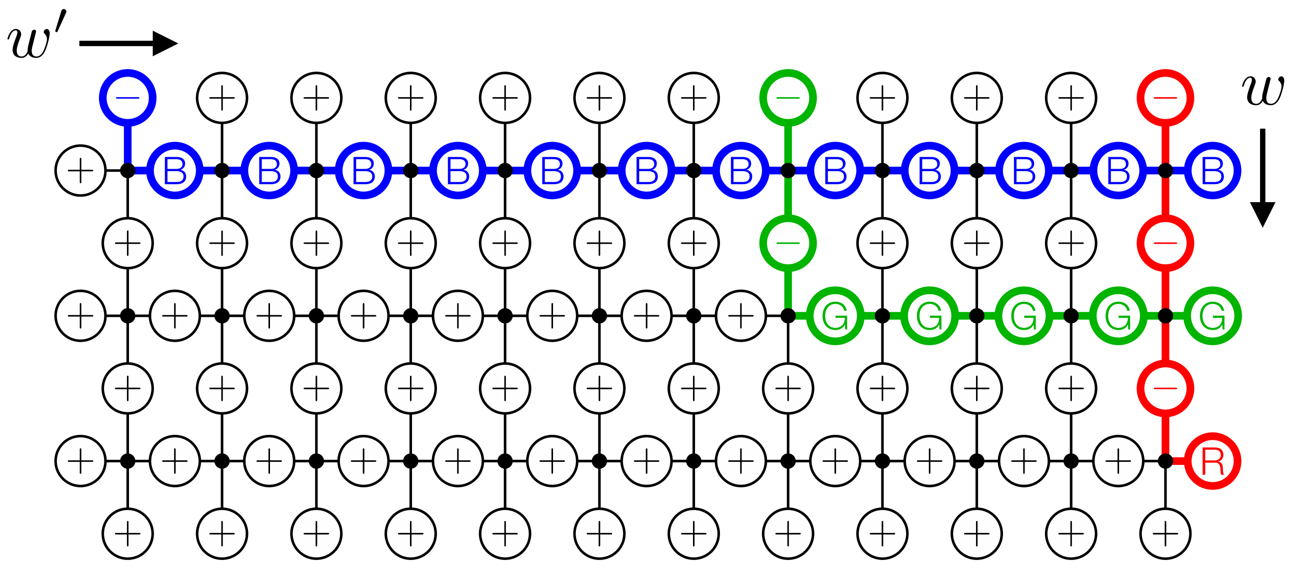
$T_i \circlearrowright$ polynomials in $\mathbf{z} = (z_1, \ldots, z_r)$

(finite geometric series)

Base case

Block number μ_j with color $(P)_j$



Recursion relations

$$Z_{\mu,s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

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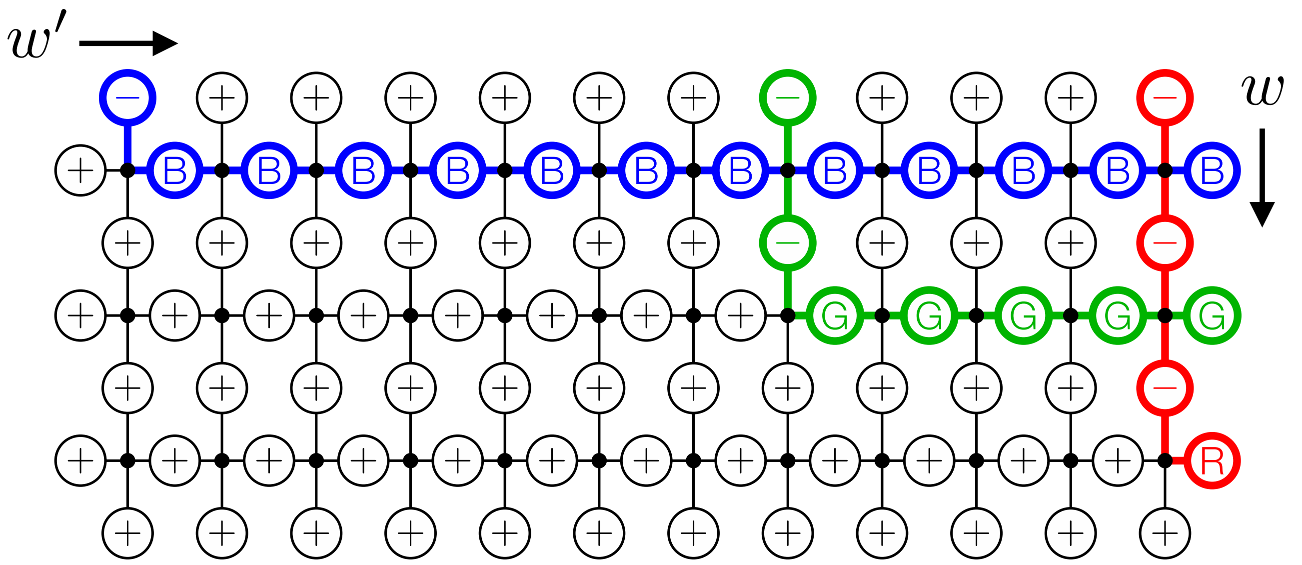
$T_i \circlearrowright$ polynomials in $\mathbf{z} = (z_1, \ldots, z_r)$

(finite geometric series)

Base case

Block number $(w'\mu)_j$ with color $(w'P)_j$

Positions and colors left-to-right



Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

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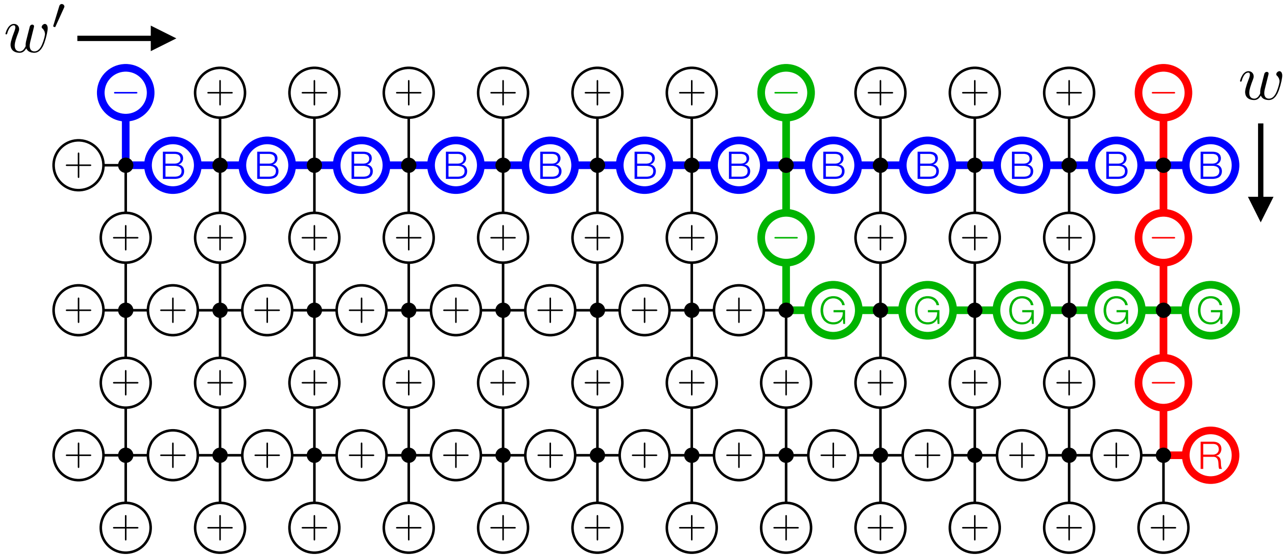
$T_i \circlearrowright$ polynomials in $\mathbf{z} = (z_1, \dots, z_r)$

(finite geometric series)

Base case $w' = w$

Block number $(w'\mu)_j$ with color $(w'P)_j$

Positions and colors left-to-right



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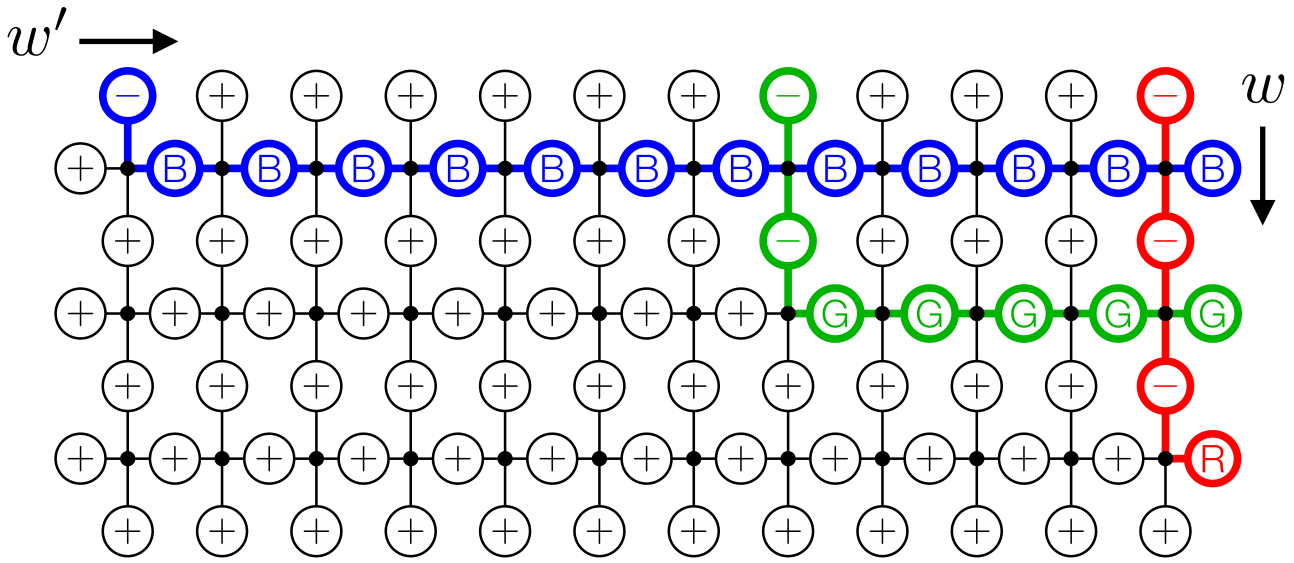
(finite geometric series)

Base case $w' = w$

Block number $(w'\mu)_j$ with color $(w'P)_j$

Positions and colors left-to-right

$$Z_{\mu,w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu}$$ unique state



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$T_i \circlearrowright$ polynomials in $\mathbf{z} = (z_1, \dots, z_r)$

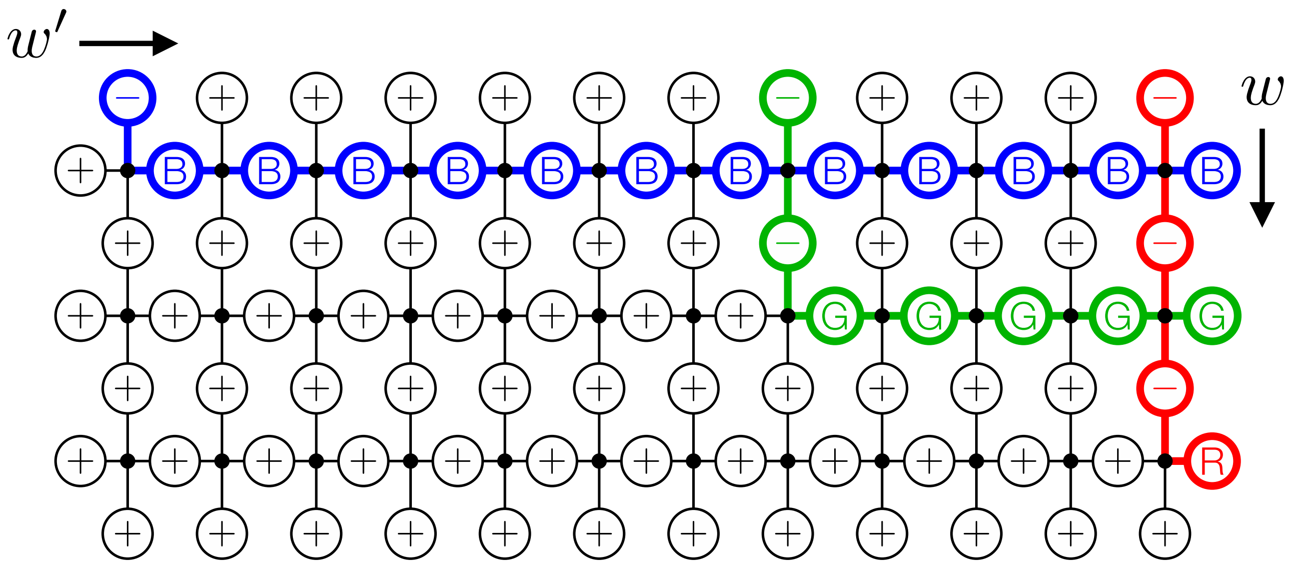
(finite geometric series)

Base case $w' = w$

Block number $(w'\mu)_j$ with color $(w'P)_j$

Positions and colors left-to-right

$$Z_{\mu,w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu}$$ unique state



Corollary: [arXiv:1906.04140]

$$Z_{\mu,w'}(\mathbf{z}) = v^{\ell(w')} T_{i_k}^{e_k} \cdots T_{i_1}^{e_1} \mathbf{z}^{w'\mu}$$

Recursion relations

$$Z_{\mu,s_iw}(\mathbf{z}) := \begin{cases} T_i Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) > \ell(w) \\ T_i^{-1} Z_{\mu,w}(\mathbf{z}) & \text{if } \ell(s_iw) < \ell(w) \end{cases}$$

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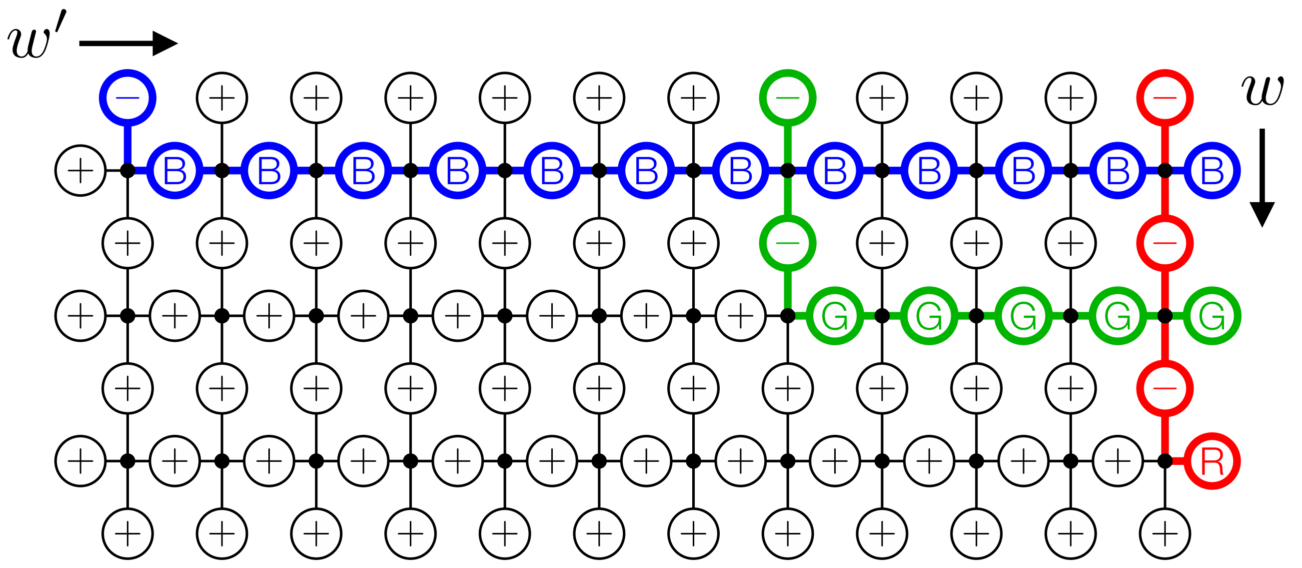
(finite geometric series)

Base case $w' = w$

Block number $(w'\mu)_j$ with color $(w'P)_j$

Positions and colors left-to-right

$Z_{\mu,w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu}$ unique state



Corollary: [arXiv:1906.04140]

$$Z_{\mu,w'}(\mathbf{z}) = v^{\ell(w')} T_{i_k}^{e_k} \cdots T_{i_1}^{e_1} \mathbf{z}^{w'\mu}$$

$w' \to s_{i_1} w' \to \cdots \to s_{i_k} \cdots s_{i_1} w' = w$

$e_{i_m} = \begin{cases} 1 \text{ ascent} \\ -1 \text{ descent} \end{cases}$ at step m

Representation theory

Setup

Details in [arXiv:1906.04140]

Setup

F non-archimedean local field, $\mathfrak{o}$ ring of integers

Details in [arXiv:1906.04140]

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While the following representation-theory-results hold for any split reductive group $\mathbf{G}$, we will restrict to $\mathbf{G} = \mathrm{GL}_r$ for which we can compare with the lattice model just described.

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Let $\mathbf{G} = \mathrm{GL}_r$, $G = \mathbf{G}(F)$. Standard maximal split torus $\mathbf{T}$.

Details in [arXiv:1906.04140]

Principal series representation

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Upper triangular ⟶ $\mathbf{B}$

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Modular quasicharacter

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Iwahori subgroup $J \subset \mathbf{G}(\mathfrak{o})$ of lower triangular matrices $\bmod \mathfrak{p}$

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$$\Phi_w^{\mathbf{z}}(bw'k) = \begin{cases} \delta^{1/2}\tau_{\mathbf{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases} \qquad b \in B, w' \in W, k \in J$$

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Spherical vector right-invariant under $K = \mathbf{G}(\mathfrak{o})$

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Parabolic (block-triangular) mod $\mathfrak{p}$

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$\mathbf{B}_-$ opposite Borel (lower triangular), unipotent radical $\mathbf{N}_-$

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Base case

$\phi_w(\mathbf{z}; g)$ is determined by its values on $g = \varpi^{-\lambda} w'$ with $\lambda \in \mathbb{Z}^r$ and $w' \in W = S_r$ such that

$$\lambda_i - \lambda_{i+1} \geqslant \begin{cases} 0 & \text{if } (w')^{-1}\alpha_i \in \Delta^+ \\ -1 & \text{if } (w')^{-1}\alpha_i \in \Delta^- \end{cases}$$

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Bijection between data determining the values for Iwahori Whittaker functions and the boundary data for the lattice model.

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$$\phi_{w'}(\mathbf{z}; \varpi^{-\lambda} w') = v^{\ell(w')} \mathbf{z}^{\lambda}$$

Compare: $Z_{\mu, w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu} = v^{\ell(w')} \mathbf{z}^{\lambda + \rho}$

Recursion relations

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$$\begin{tikzcd}
\Omega_{\mathbf{z}} & \mathcal{A}_{s_i}^{\mathbf{z}} \arrow[r, bend left] & \Phi_w^{\mathbf{z}}
\end{tikzcd}$$

[Casselman 80,
Brubaker–Bump–Licata 15]

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[Casselman–Shalika 80,
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$$\begin{tikzcd}
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Intertwining integral $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z}) \to I(w\mathbf{z})$ $\quad \mathcal{A}_w^{\mathbf{z}}\Phi(g) = \int_{N \cap wN_- w^{-1}} \Phi(w^{-1}ng)\, dn$

[Casselman–Shalika 80,
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$$\begin{tikzcd}
\Omega_{\mathbf{z}} & \mathcal{A}_{s_i}^{\mathbf{z}} \arrow[l] \arrow[r] & \Phi_w^{\mathbf{z}}
\end{tikzcd}$$

[Casselman 80,
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Leads to recursion relations

$$\mathbf{z}^\rho \phi_{s_i w}(\mathbf{z}; g) = \begin{cases} T_i \; \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) > \ell(w), \\ T_i^{-1} \; \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

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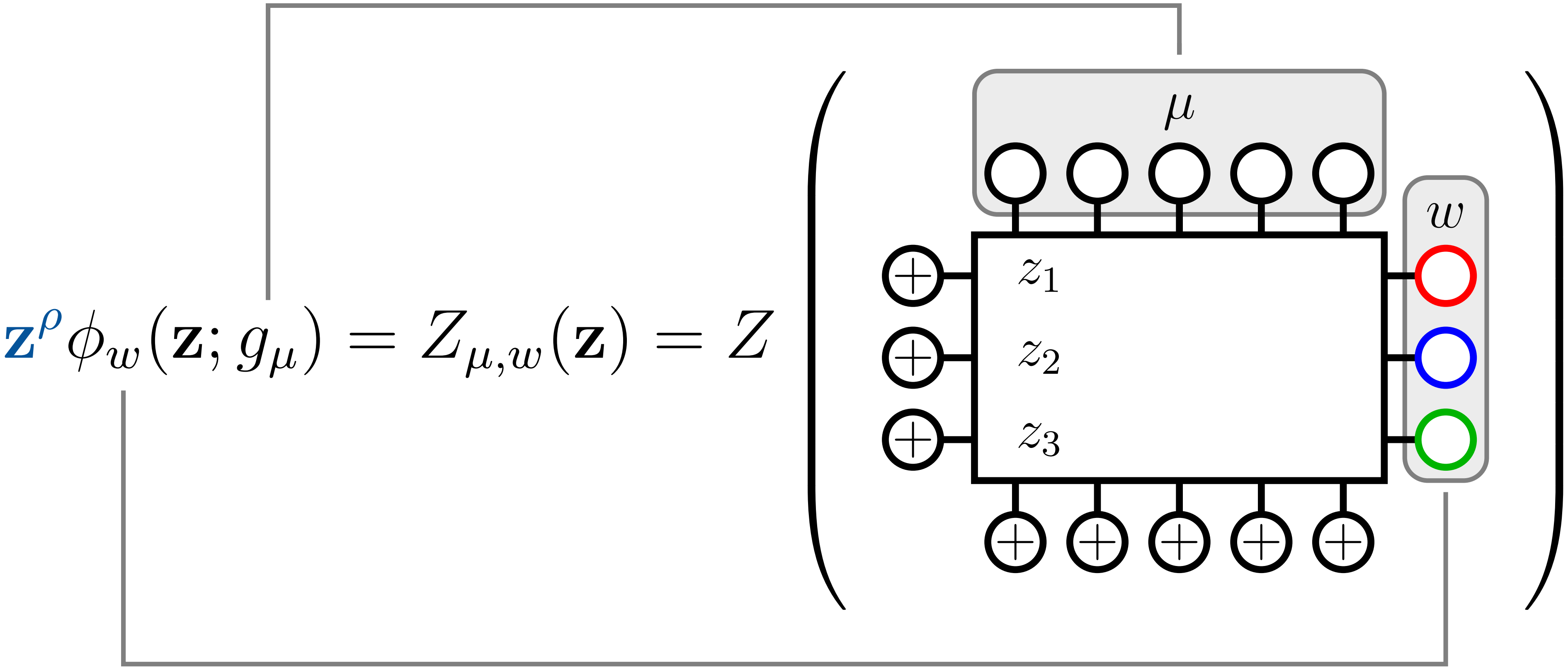
Same Demazure operators as before

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

Main theorem

$$\mathbf{z}^{\rho}\phi_w(\mathbf{z}; g_\mu) = Z_{\mu,w}(\mathbf{z}) = Z$$



$$g_\mu = \varpi^{-\lambda} w' \qquad w'\mu = \lambda + \rho$$

arXiv:1906.04140

Non-distinct colors

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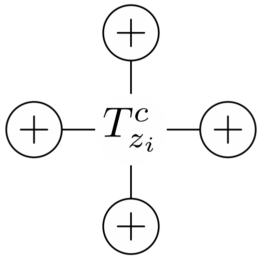	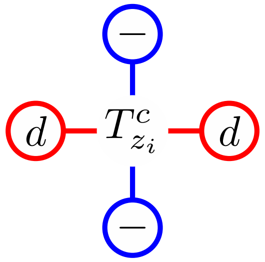	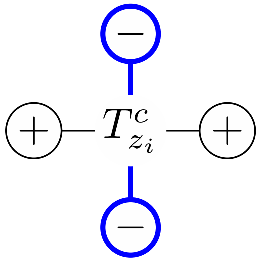	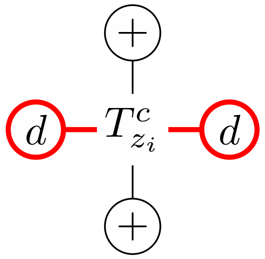	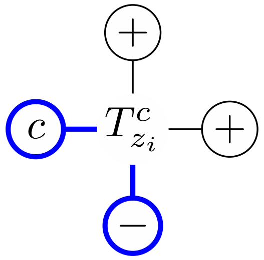	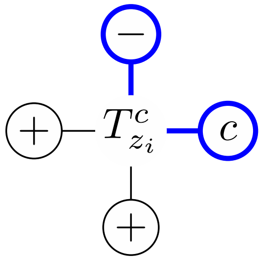
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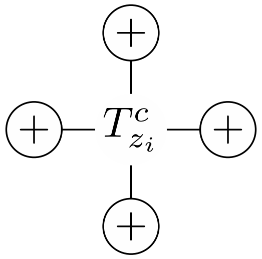	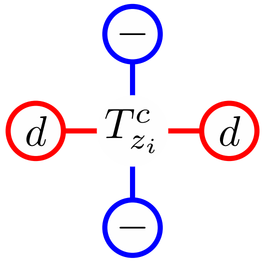	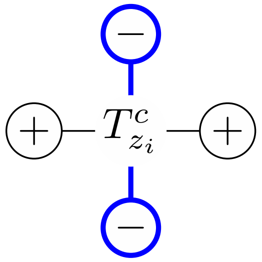	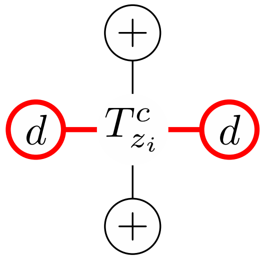	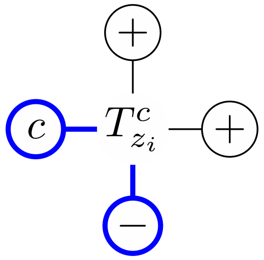	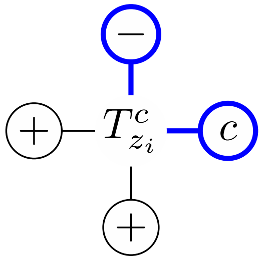
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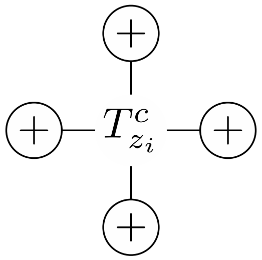	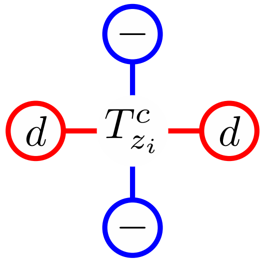	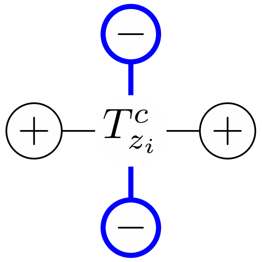	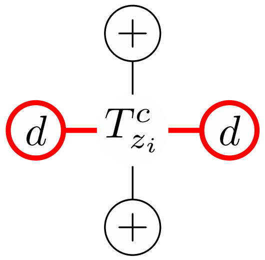	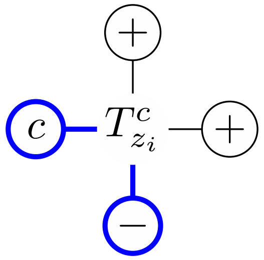	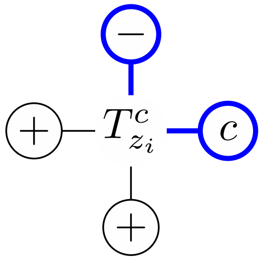
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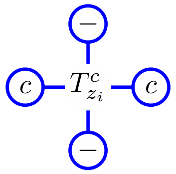

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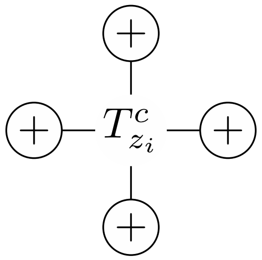	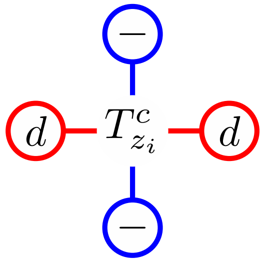	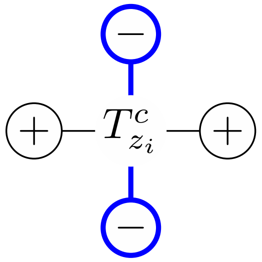	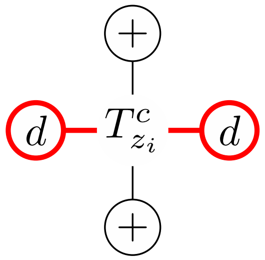	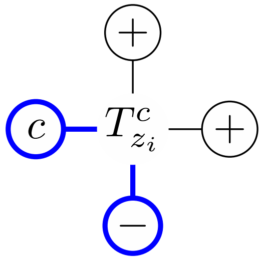	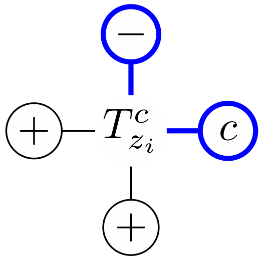
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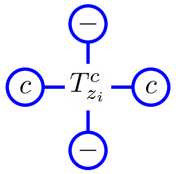

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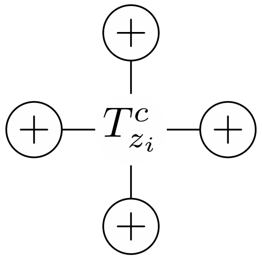	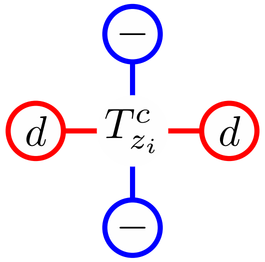	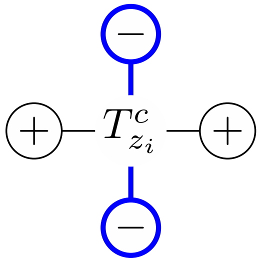	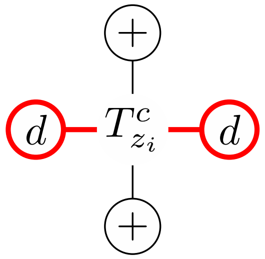	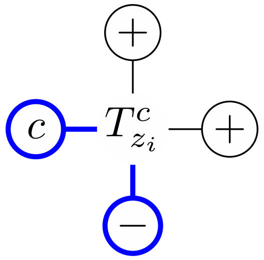	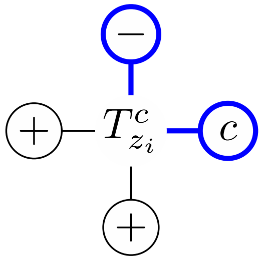
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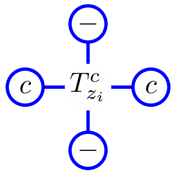

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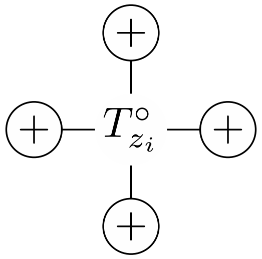	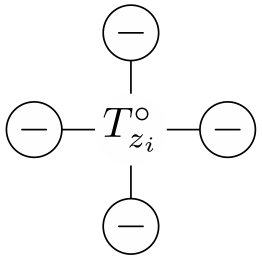	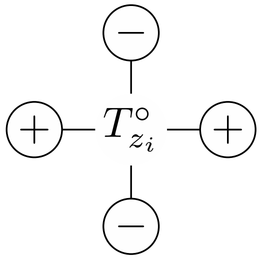	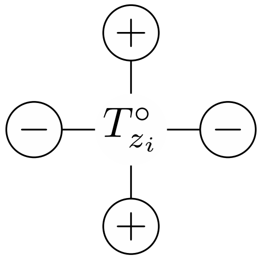	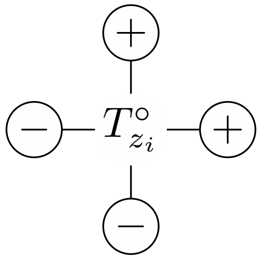	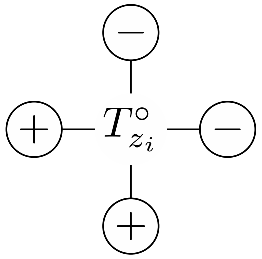
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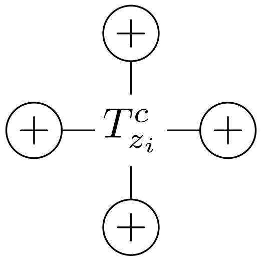	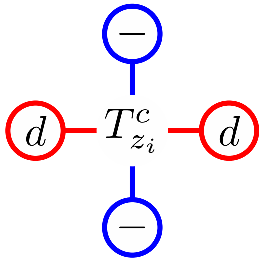	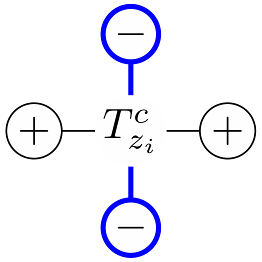	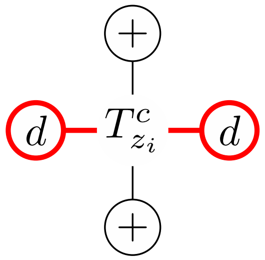	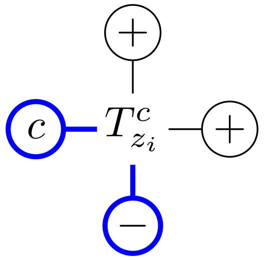	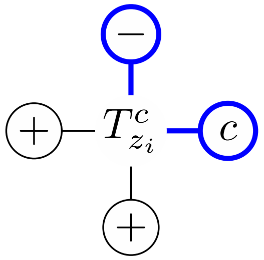
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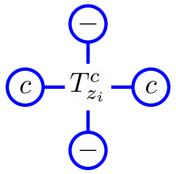

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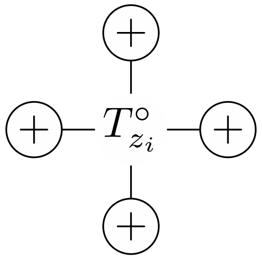	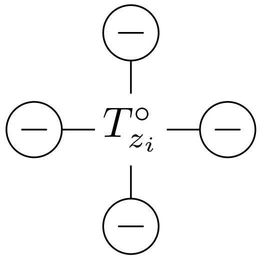	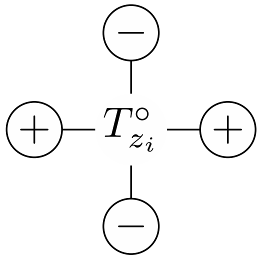	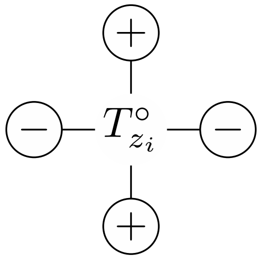	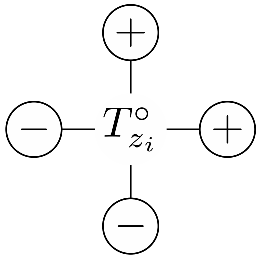	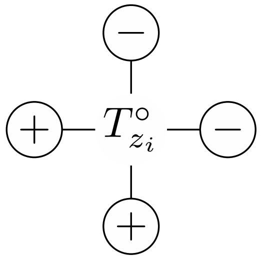
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Non-distinct colors

Non-trivial feature:

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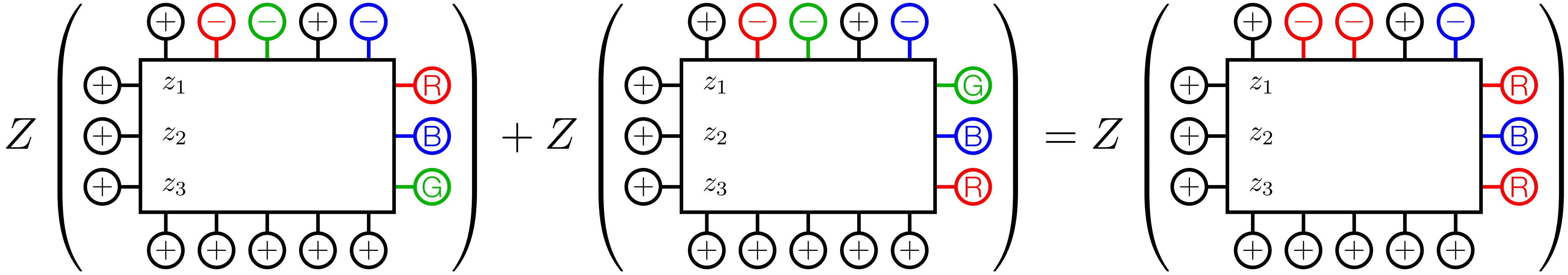

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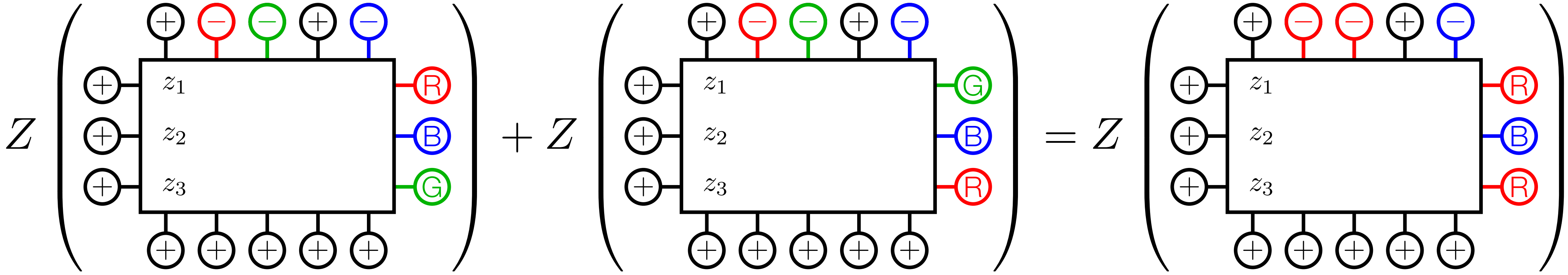

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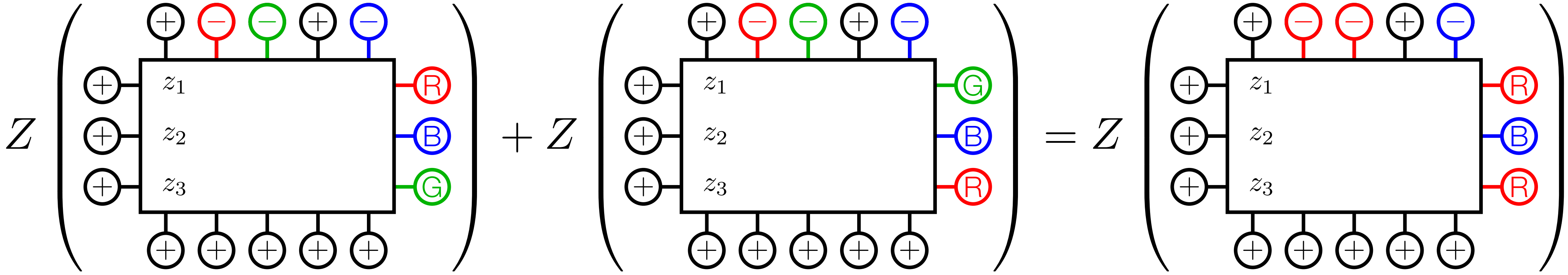

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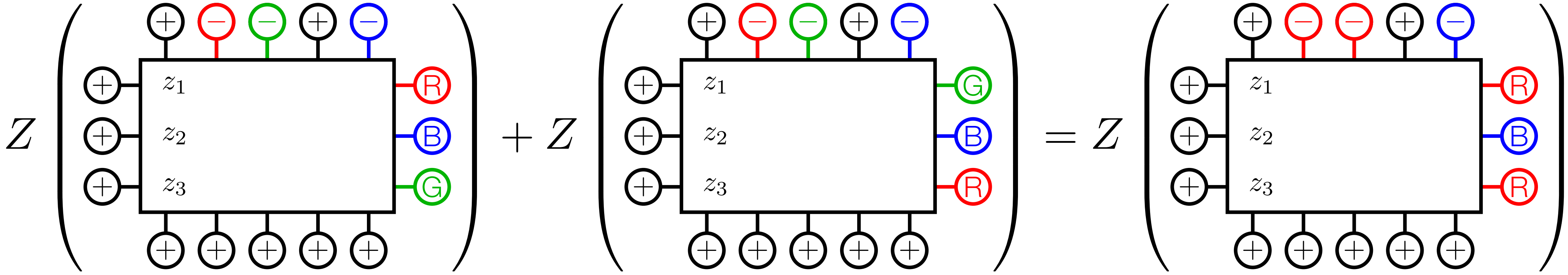

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block-triangular

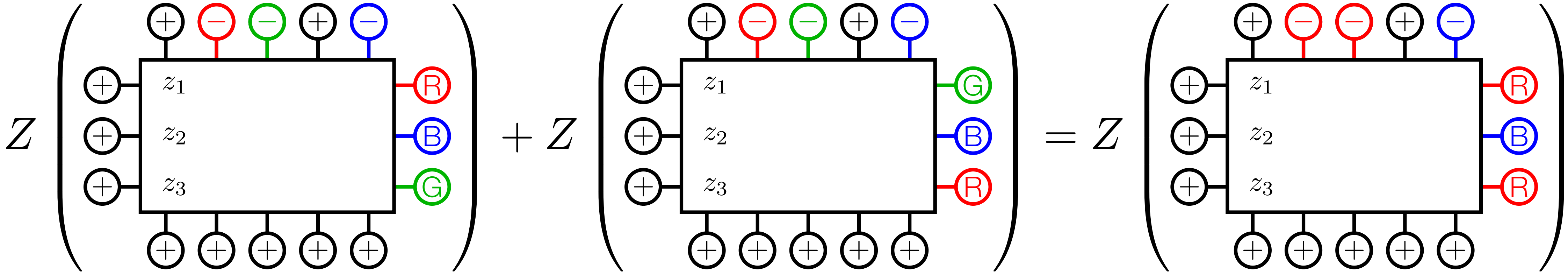
$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

Iwahori $\subseteq$ parahoric $\subseteq$ spherical

arXiv:1906.04140

Non-distinct colors

Non-trivial feature: (fused systems)



In particular, we have a cancellation of states where red and green vertical edges overlap.

These models compute parahoric Whittaker functions $\sum_{\tilde{w}\in W_P} \phi_{w\tilde{w}}$

block-triangular

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

Same model
different boundary conditions

Iwahori $\subseteq$ parahoric $\subseteq$ spherical

arXiv:1906.04140

Special polynomials

TABLE 2. Relations between different Whittaker functions and associated special polynomials.

Whittaker function		**Special polynomial**
Spherical Whittaker function $\sum_{w\in W}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Schur polynomial $\prod_{\alpha\in\Delta^+}(1-v\mathbf{z}^{-\alpha})s_\lambda(\mathbf{z})$
Li's Whittaker function $\sum_{w\in W}(-v)^{-\ell(w)}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Hall-Littlewood polynomial $\mathbf{z}^{-\rho}P_{\lambda+\rho}(\mathbf{z},v^{-1})$
Iwahori Whittaker function $\phi_{w_1}(\mathbf{z};\varpi^{-\lambda})$	$=$	Non-symmetric Macdonald polynomial $(-v)^{\ell(w)}\mathbf{z}^{-\rho}w_0E_{w_0w(\lambda+\rho)}(\mathbf{z};\infty,v)$
Parahoric Whittaker function $\psi_1^{\mathbf{J}}(\mathbf{z};\varpi^{-\lambda})$	$=$	Macdonald polynomial with prescribed symmetry $\mathbf{z}^{-\rho}S_{\lambda+\rho}^{(\emptyset,\mathbf{J})}(\mathbf{z};0,v^{-1})a_{\lambda+\rho}^{(\emptyset,\mathbf{J})}$

References for each row is found in Table 1 in the same paper: arXiv:1906.04140

Quantum groups

	$\mathrm{GL}_r(F)$
spherical	$U_q(\widehat{\mathfrak{gl}}(1\|1))$ [Brubaker–Bump–Friedberg 09]
Iwahori	$U_q(\widehat{\mathfrak{gl}}(r\|1))$ [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]

Quantum groups

	$\mathrm{GL}_r(F)$	metaplectic n-cover of $\mathrm{GL}_r(F)$
spherical	$U_q\big(\widehat{\mathfrak{gl}}(1\vert 1)\big)$ [Brubaker–Bump–Friedberg 09]	$U_q\big(\widehat{\mathfrak{gl}}(1\vert n)\big)$ [Brubaker–Buciumas–Bump 19]
Iwahori	$U_q\big(\widehat{\mathfrak{gl}}(r\vert 1)\big)$ [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]	

Quantum groups

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Iwahori	$U_q\big(\widehat{\mathfrak{gl}}(r\|1)\big)$ [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]	$U_q\big(\widehat{\mathfrak{gl}}(r\|n)\big)$ [Brubaker–Buciumas–Bump–HG arXiv:2012.15778]

Metaplectic cover

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

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Where we assume group of n-th roots of unity $\mu_n \subset F$

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The preimage $\widetilde{T}$ of the maximal torus T is non-abelian.

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The irreducible representations of $\widetilde{T}$ are n^r-dimensional.

See for example [Savin 04, McNamara 16]

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Basis of Whittaker functionals enumerated by

$$\underset{\uparrow\ \text{Weight lattice}}{\theta \in \Lambda} / n\Lambda \simeq (\mathbb{Z}/n\mathbb{Z})^r$$

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Iwahori: $n^r \cdot r!$

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Space of Whittaker functions is of dimension:

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Iwahori: $n^r \cdot \underset{\downarrow\,|W|}{r!}$

Metaplectic version

Horizontal edges: $\underbrace{\text{B}, \text{G}, \text{R}}^{c_1 < \cdots < c_r}, \underbrace{0, \ldots, n-1}^{x \in \mathbb{Z}/n\mathbb{Z}}$

arXiv:2012.15778

Metaplectic version

Horizontal edges: $\overbrace{\text{B}, \text{G}, \text{R}}^{c_1 < \cdots < c_r}, \overbrace{0, \ldots, n-1}^{x \in \mathbb{Z}/n\mathbb{Z}}$

color and it's superpartner "scolor"

B G R

0 1 2

arXiv:2012.15778

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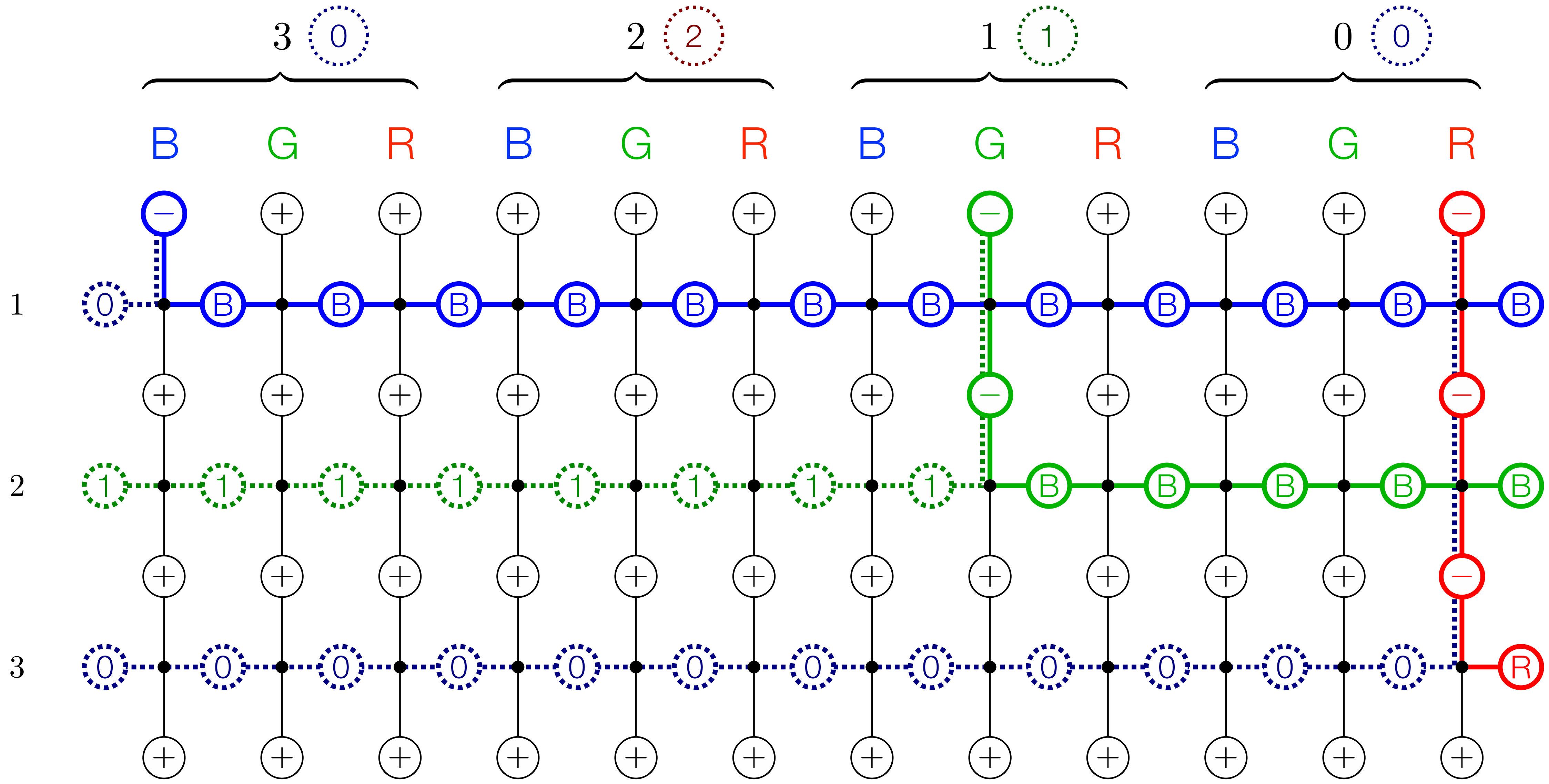

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Ⓑ Ⓖ Ⓡ ⓪ ① ②

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arXiv:2012.15778

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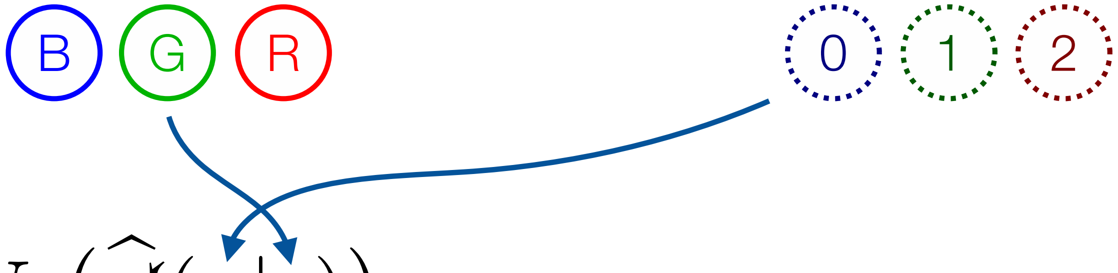

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[Kazhdan–Patterson 84, McNamara 16 (more generally)]

$\Omega_{\mathbf{z}}^{\theta}$ $\mathcal{A}_{s_i}^{\mathbf{z}}$ $\Phi_{w}^{\mathbf{z}}$

Similar to [Lemma 6.3, Patnaik–Puskás 17]

arXiv:2012.15778

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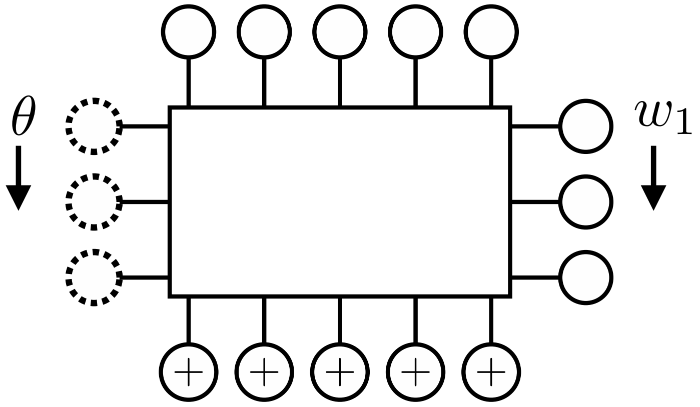

arXiv:2012.15778

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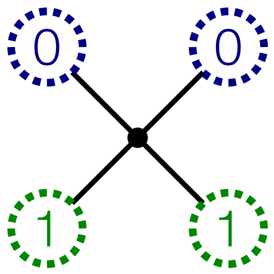
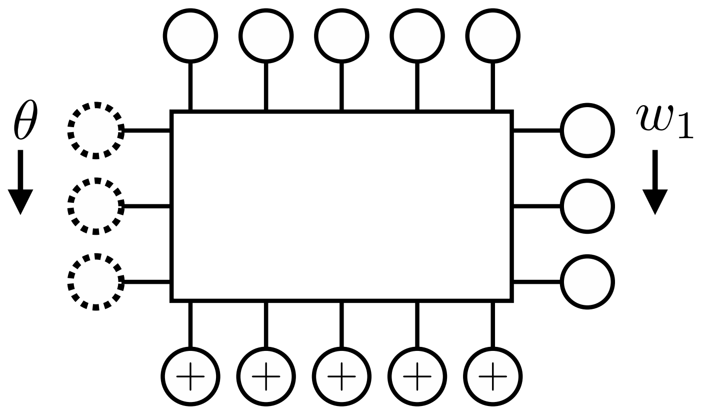


$U_q\left(\widehat{\mathfrak{gl}}(n)\right)$



arXiv:2012.15778

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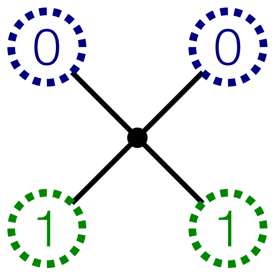
$$\begin{tikzcd}
\Omega^{\theta}_{\mathbf{z}} & \mathcal{A}^{\mathbf{z}}_{s_i} \arrow[l] \arrow[r] & \Phi^{\mathbf{z}}_{w}
\end{tikzcd}$$

Similar to [Lemma 6.3, Patnaik–Puskás 17]



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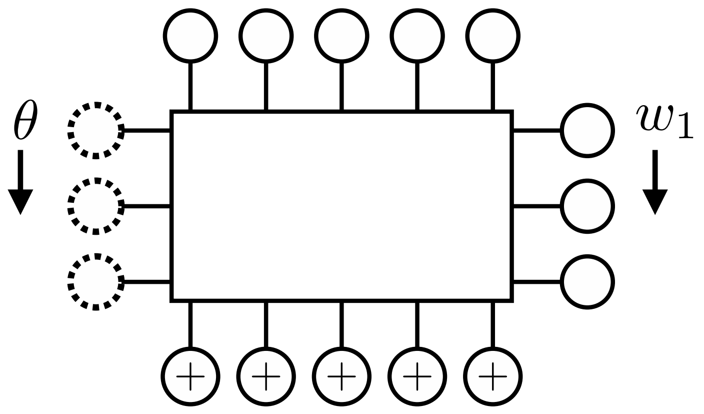

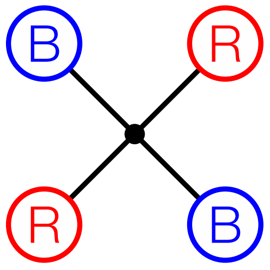

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Thank you!

Slides will be made available at

https://hgustafsson.se