

A vertex model for Iwahori Whittaker functions

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Solvable Lattice Models Seminar – Stanford University

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Papers

Joint work with Ben Brubaker, Valentin Buciumas and Daniel Bump

Colored five-vertex models and Demazure atoms

Journal of Combinatorial Theory, Series A 178 (Feb, 2021)

arXiv:1902.01795

Colored vertex models and Iwahori Whittaker functions

arXiv:1906.04140

Metaplectic Iwahori Whittaker functions and supersymmetric lattice models

arXiv:2012.15778

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Outline

Iwahori Whittaker functions of unramified principal series
on $\mathrm{GL}_r(F)$

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non-archimedean local field

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Solvable vertex model represented by colored lines and
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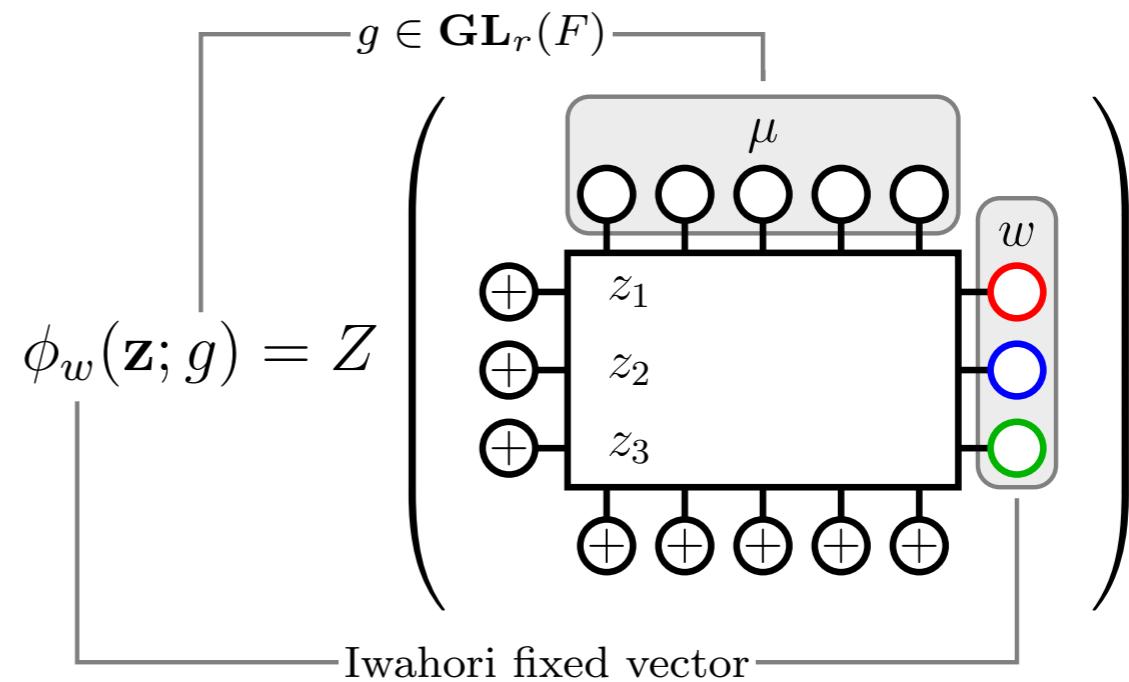
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The lattice model

Features

Described by [colored lines](#) moving right- and downwards on a grid from the top boundary to the right boundary.

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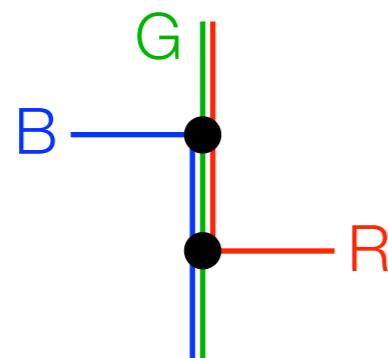
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Vertical edges described
by subsets of palette

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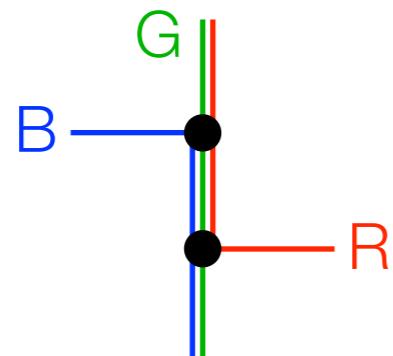
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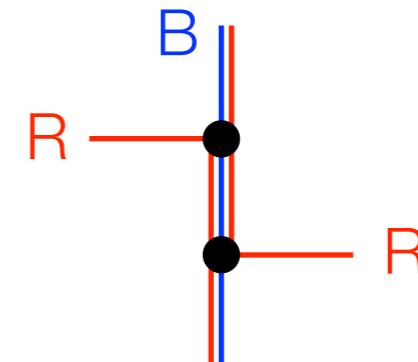
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subsets with multiplicities

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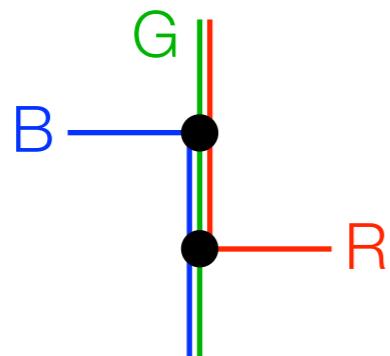
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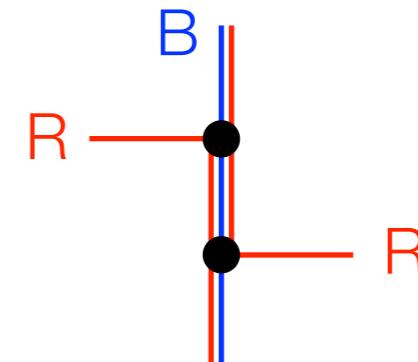
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fermionic



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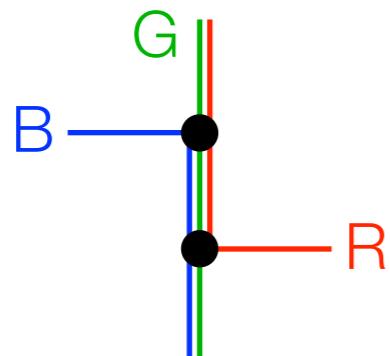
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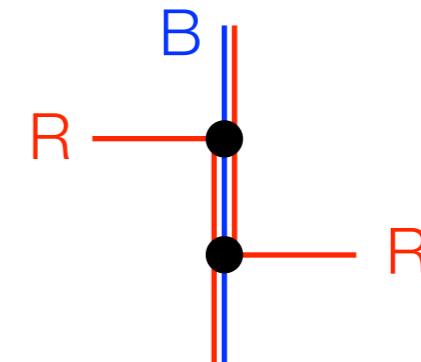
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bosonic



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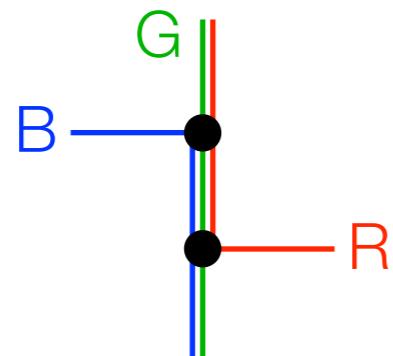
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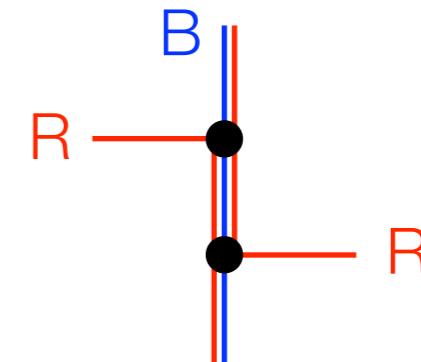
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See above papers and

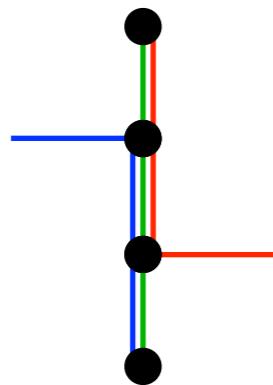
[Aggarwal–Borodin–Wheeler arXiv:2101.01605]

See for example

[Borodin–Wheeler arXiv:1808.01866]

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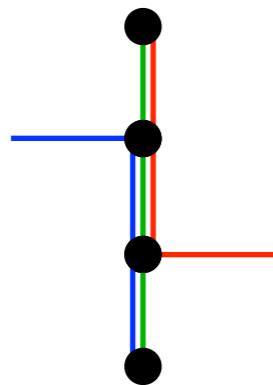
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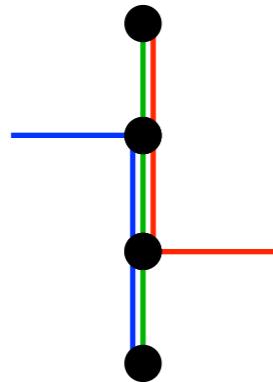


Vertical edges described
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Binary representation

Features

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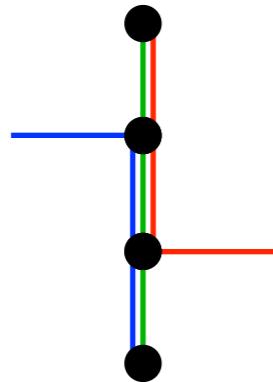
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B G R
⊕ ⊖ ⊖

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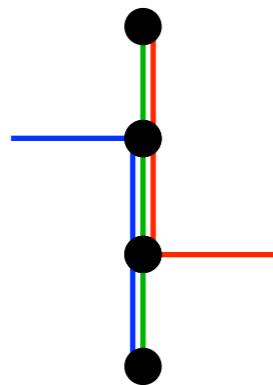
Vertical edges described
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B	G	R
⊕	⊖	⊖
⊖	⊖	⊖

Binary representation

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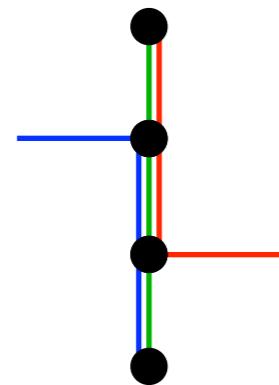
Vertical edges described
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B	G	R
⊕	⊖	⊖
⊖	⊖	⊖
⊖	⊖	⊕

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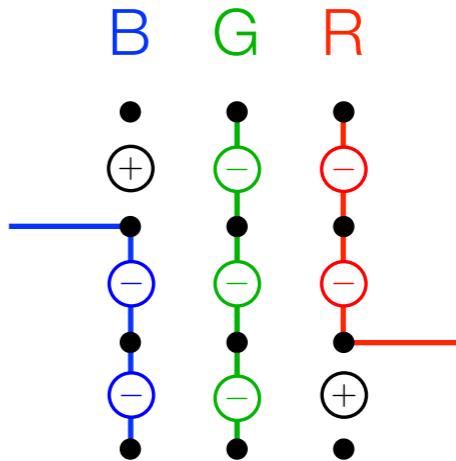
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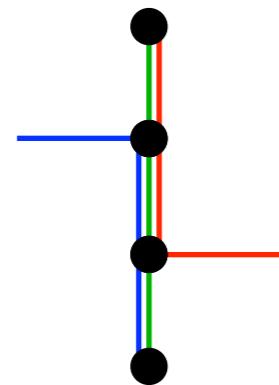
color block



Binary representation

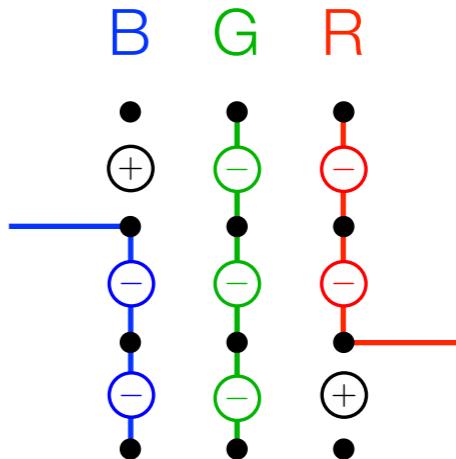
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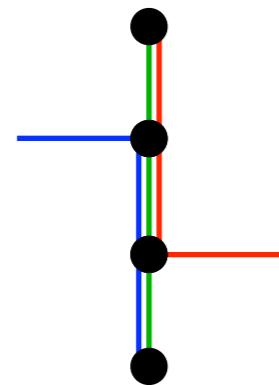


Binary representation

Each column can only
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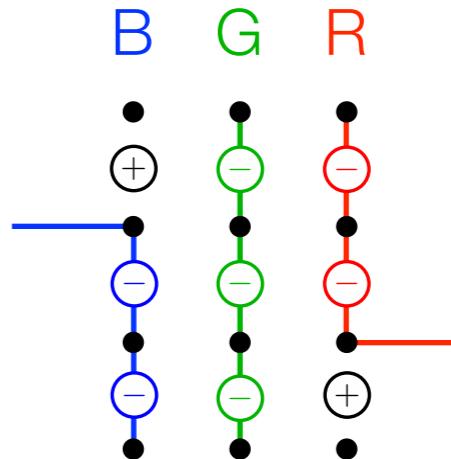
fermionic



Vertical edges described
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fusion

color block

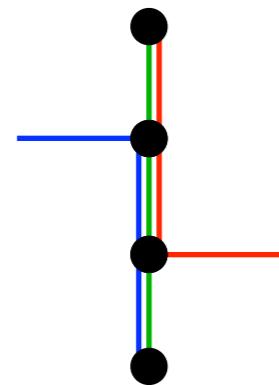


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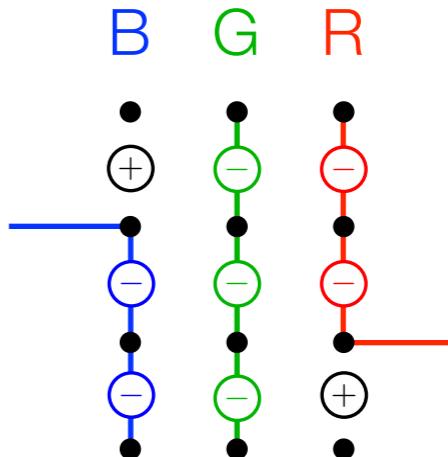
fermionic



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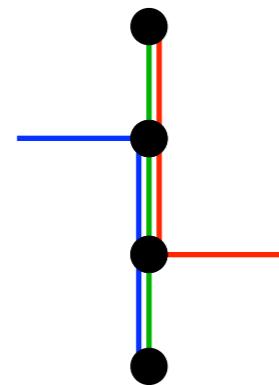
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Easier to describe

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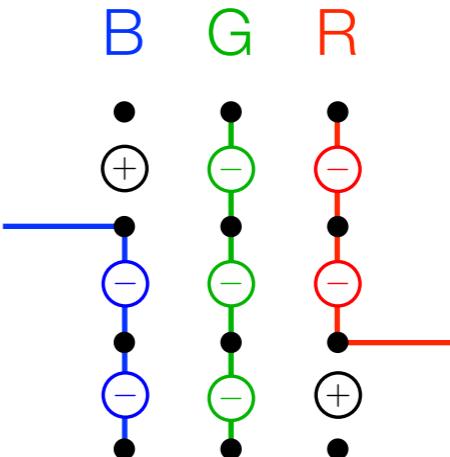


Vertical edges described
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Quantum group connection

fusion

color block



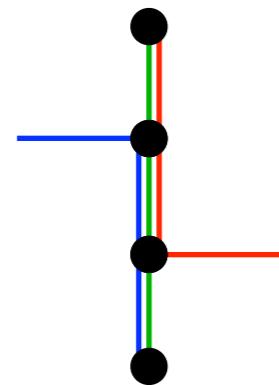
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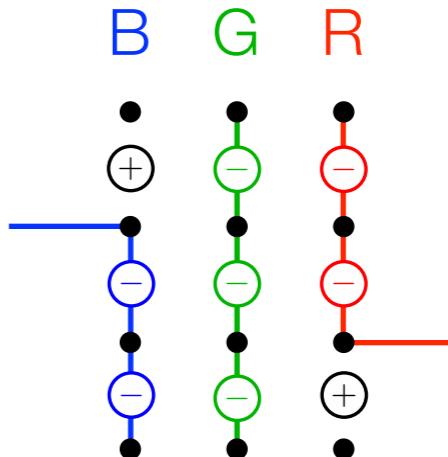


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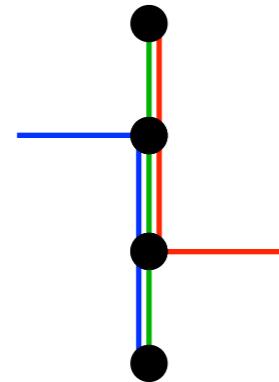
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Easier to describe

.....

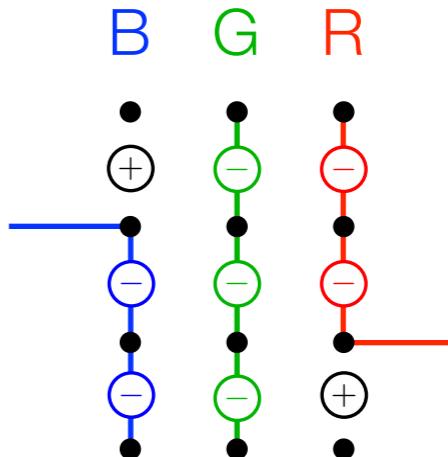
Features

fermionic



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Quantum group connection

Easier to describe

.....

Horizontal edges can carry any single (or no) color in both descriptions



Lattice model setup

$$c_1 < \dots < c_r$$

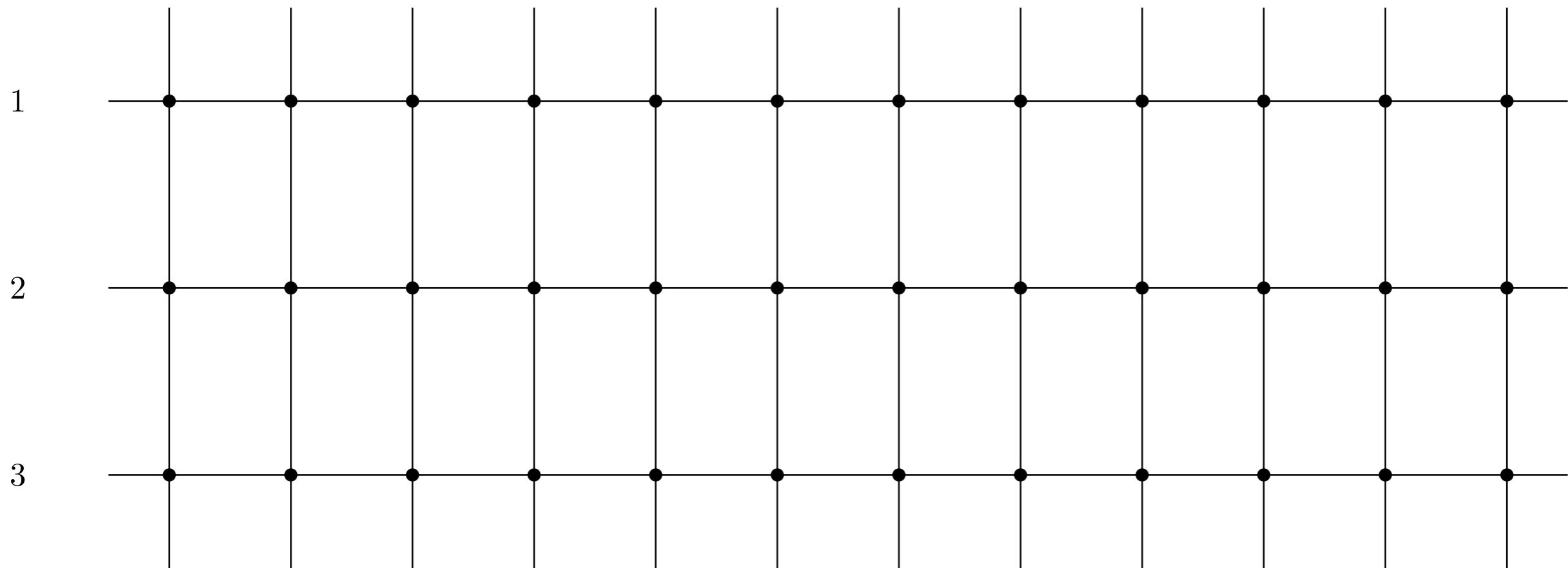
Palette of r colors: B < G < R

Lattice model setup

$$c_1 < \dots < c_r$$

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Grid with r rows and sufficiently many columns



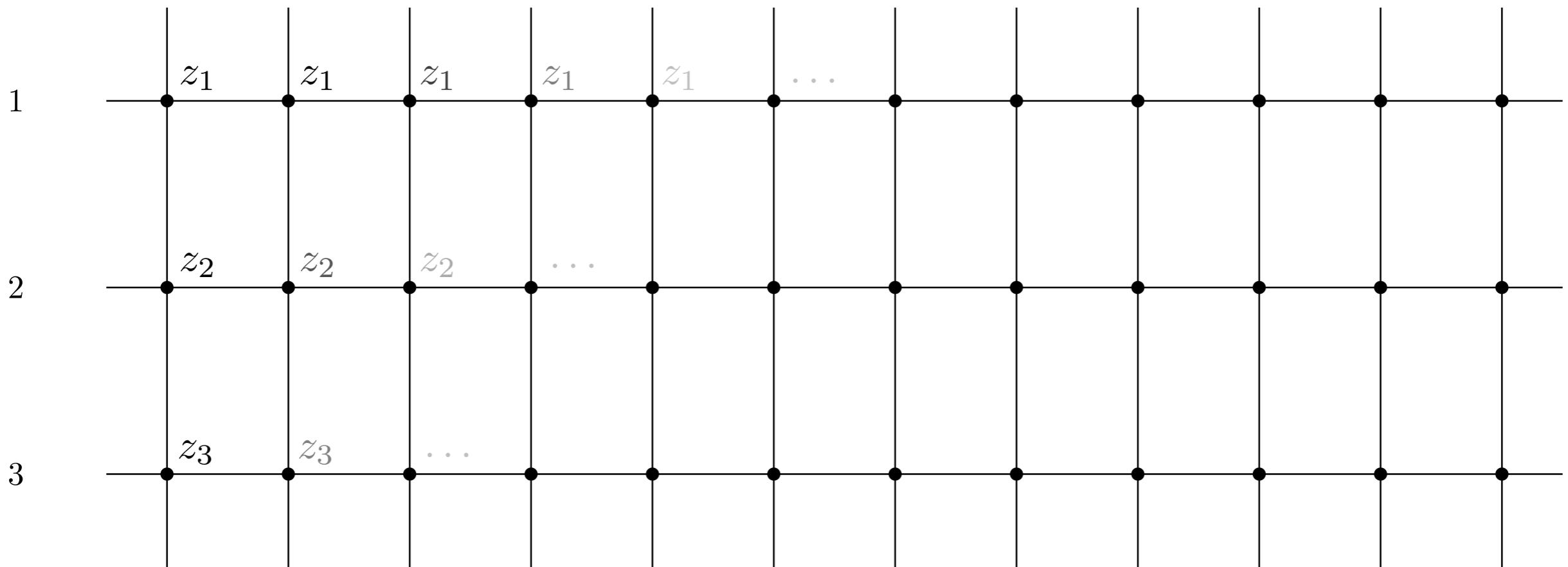
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Row parameter $z_i \in \mathbb{C}^\times$



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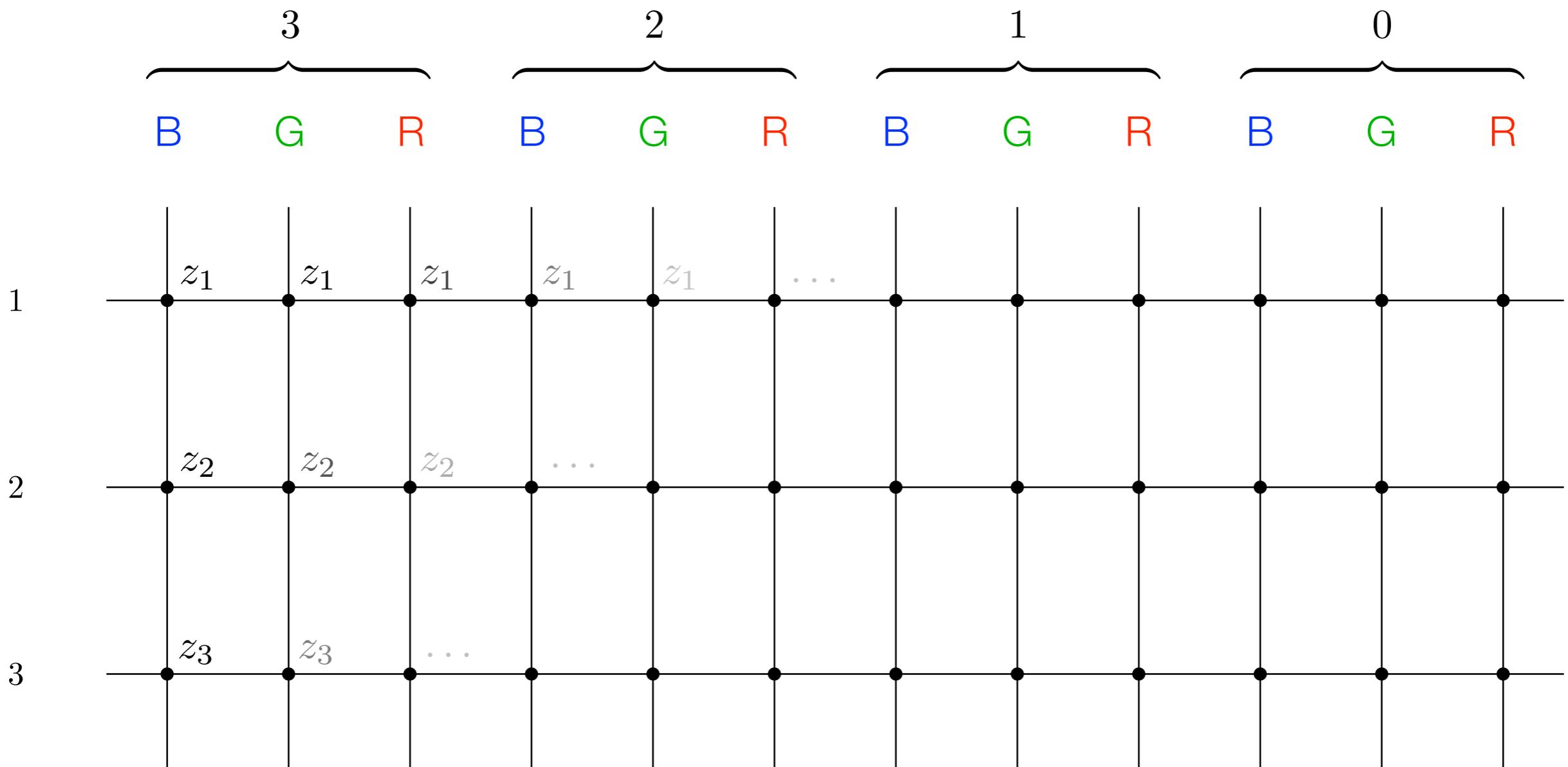
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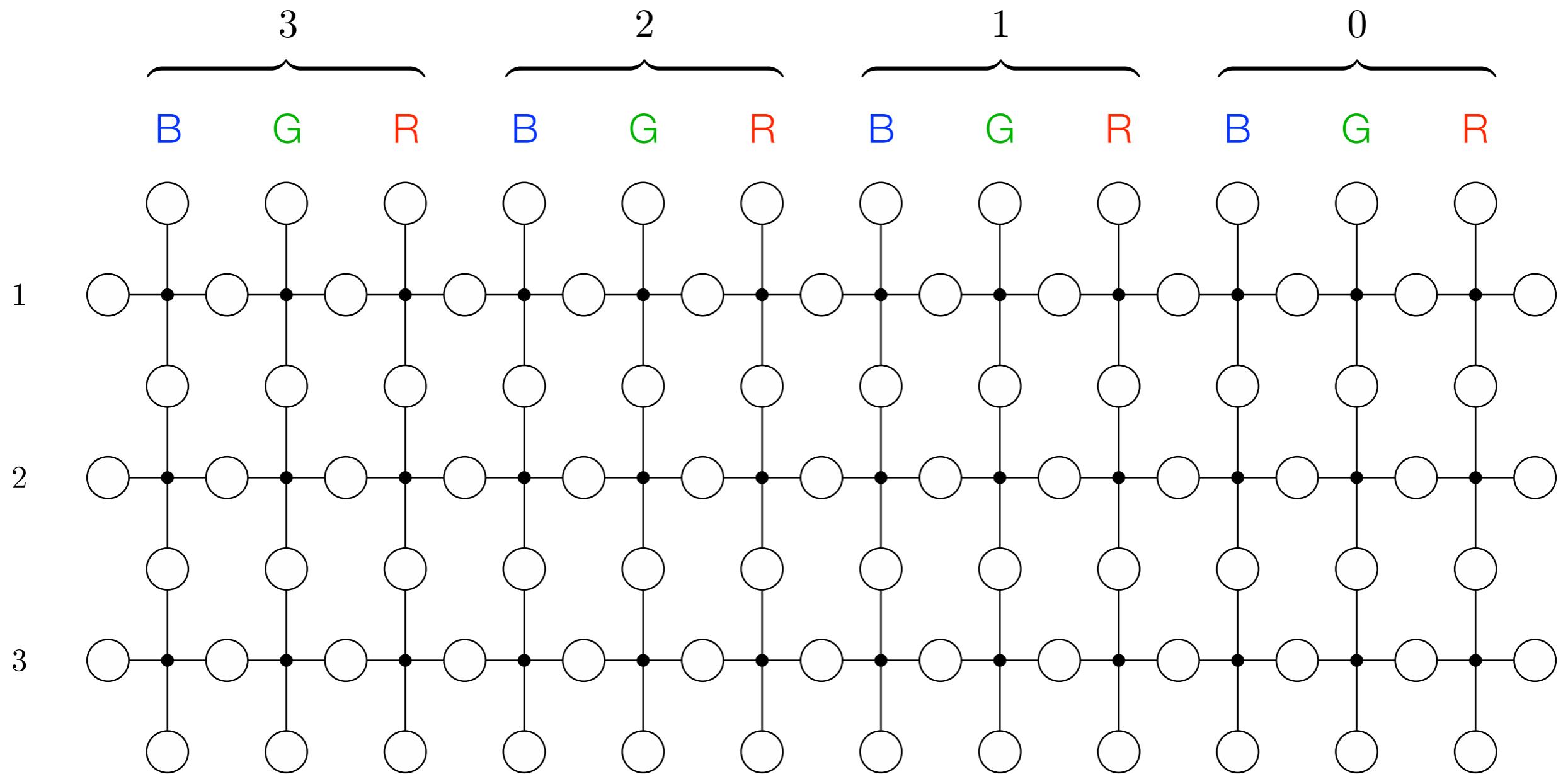
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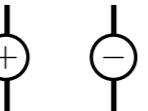
Row parameter $z_i \in \mathbb{C}^\times$

Number color blocks from right to left

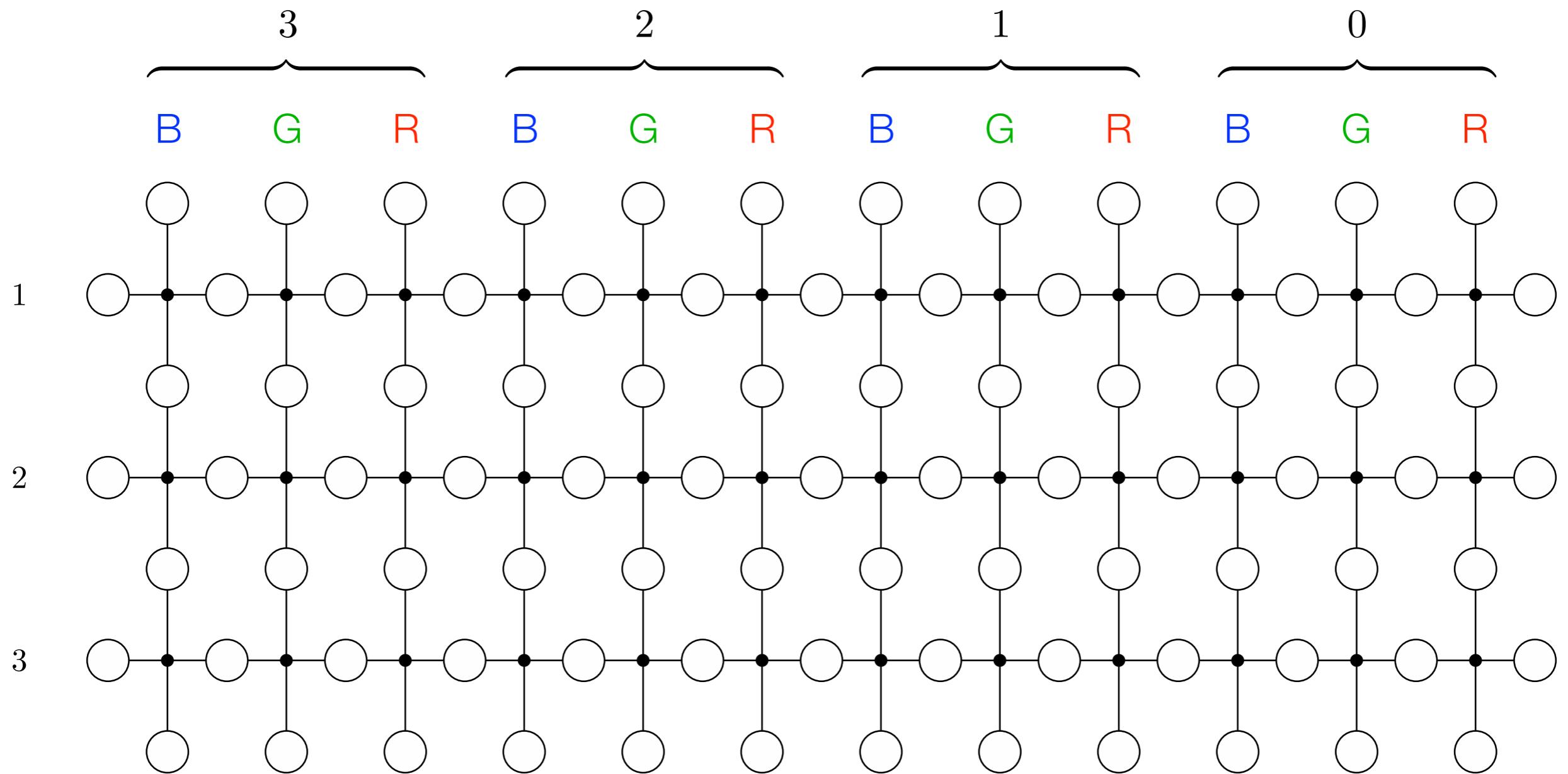


Boundary

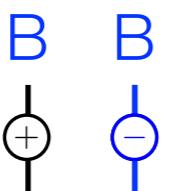


Vertical edges: 

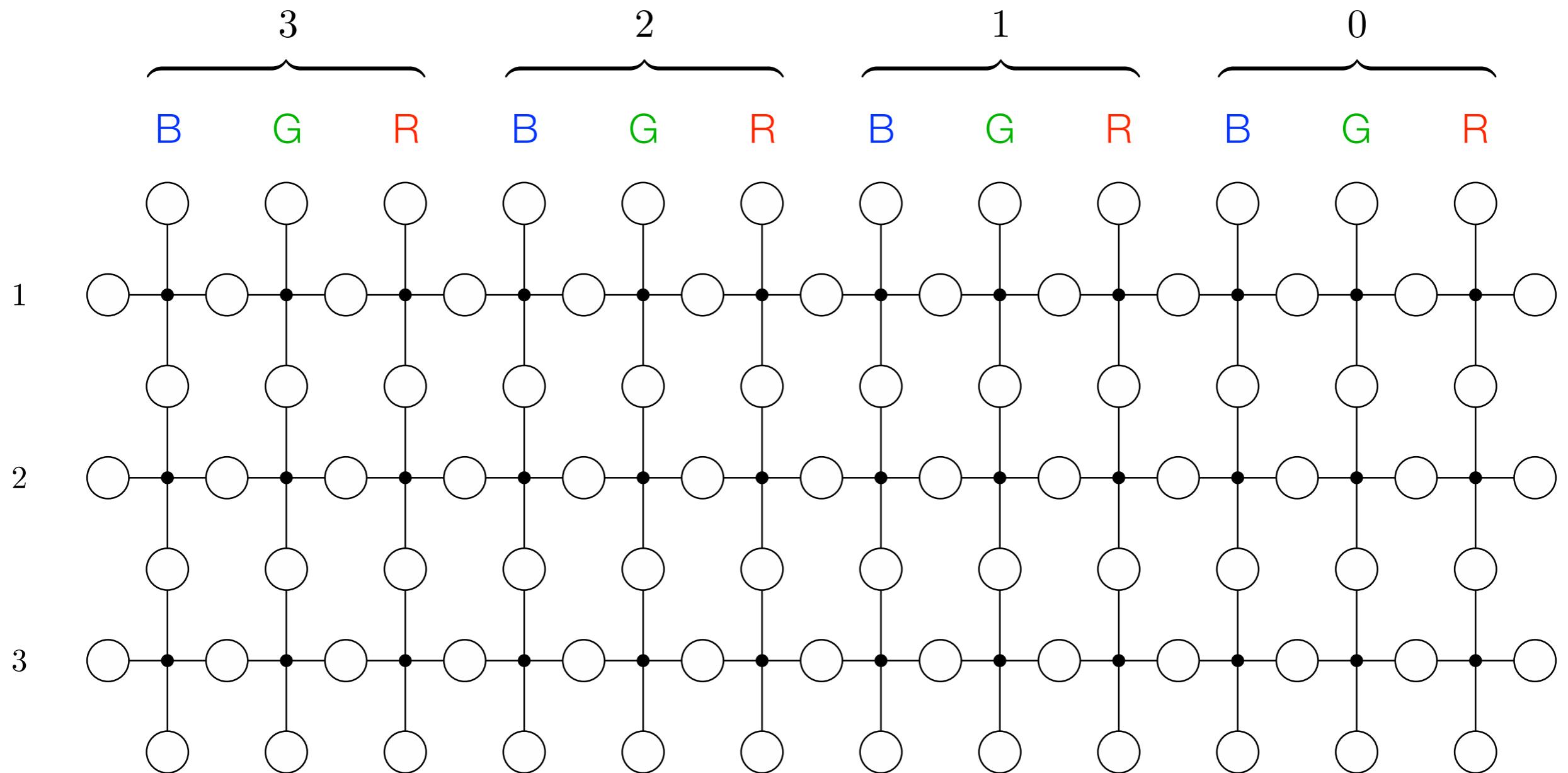
Boundary



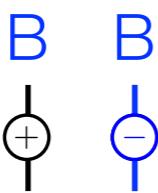
Vertical edges:



Boundary

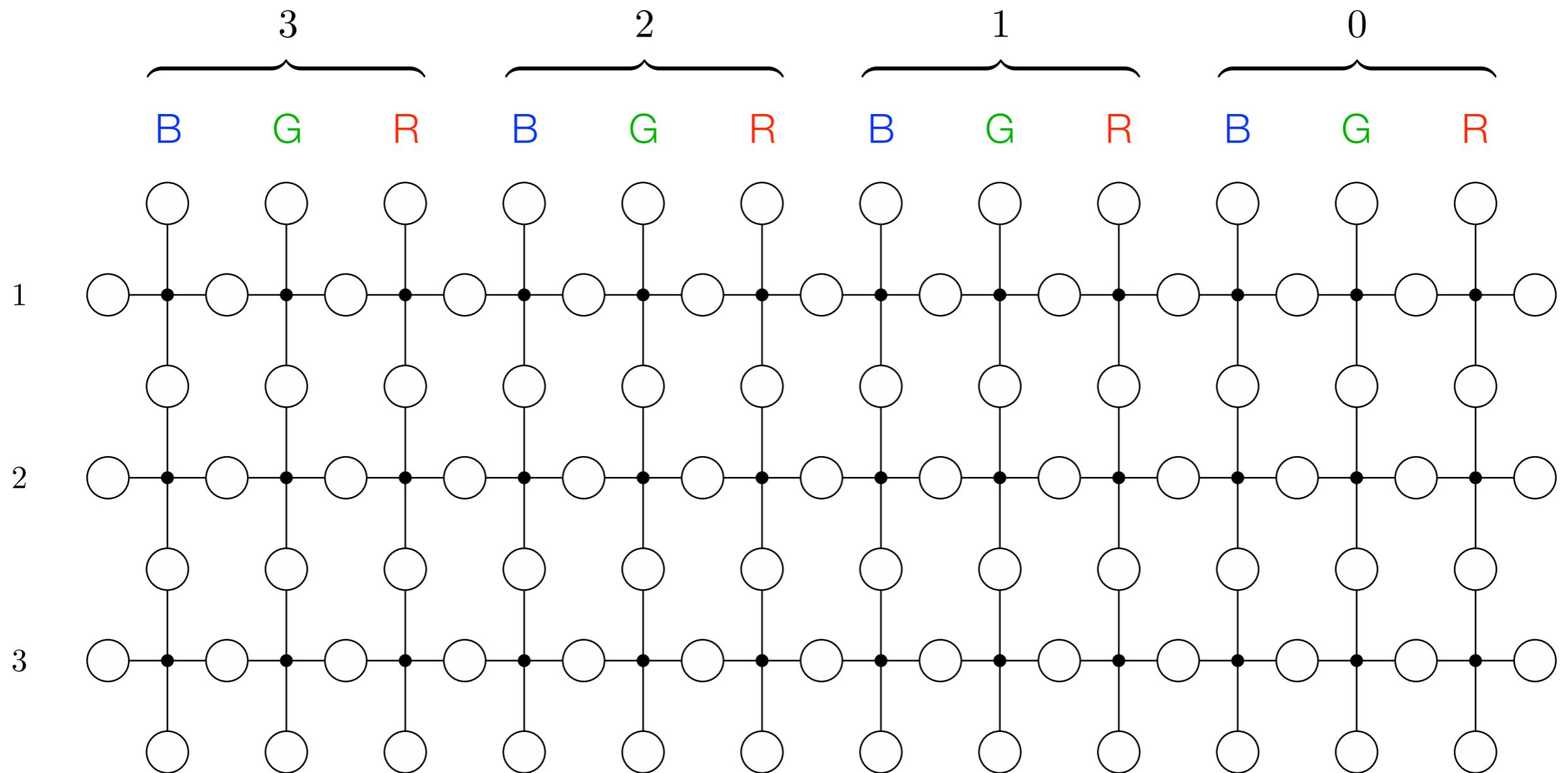
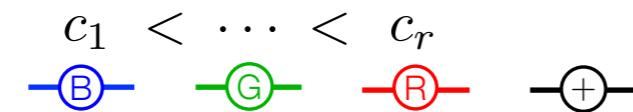


Vertical edges:

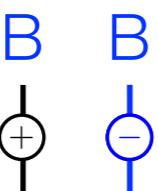


Boundary

Horizontal edges:

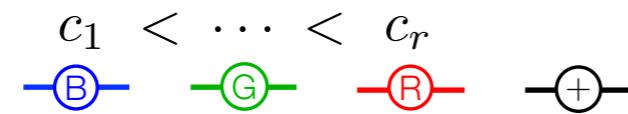


Vertical edges:



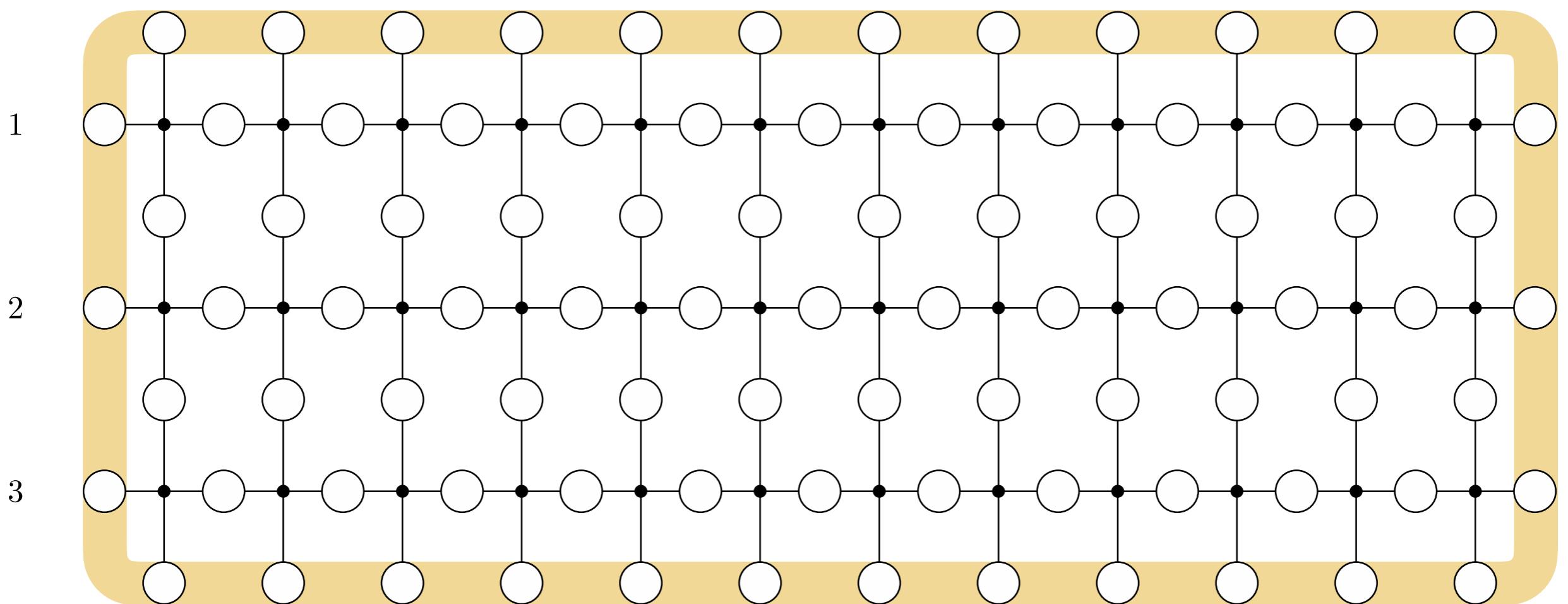
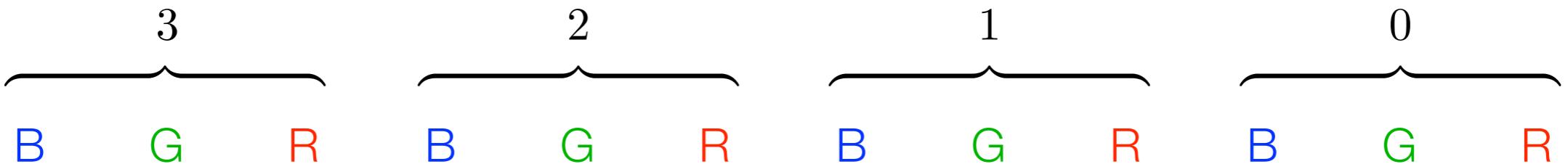
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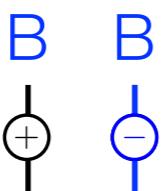


Boundary data: $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}_{\geq 0}^r$

$w \in W = S_r$

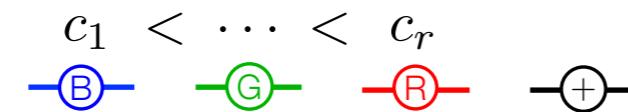


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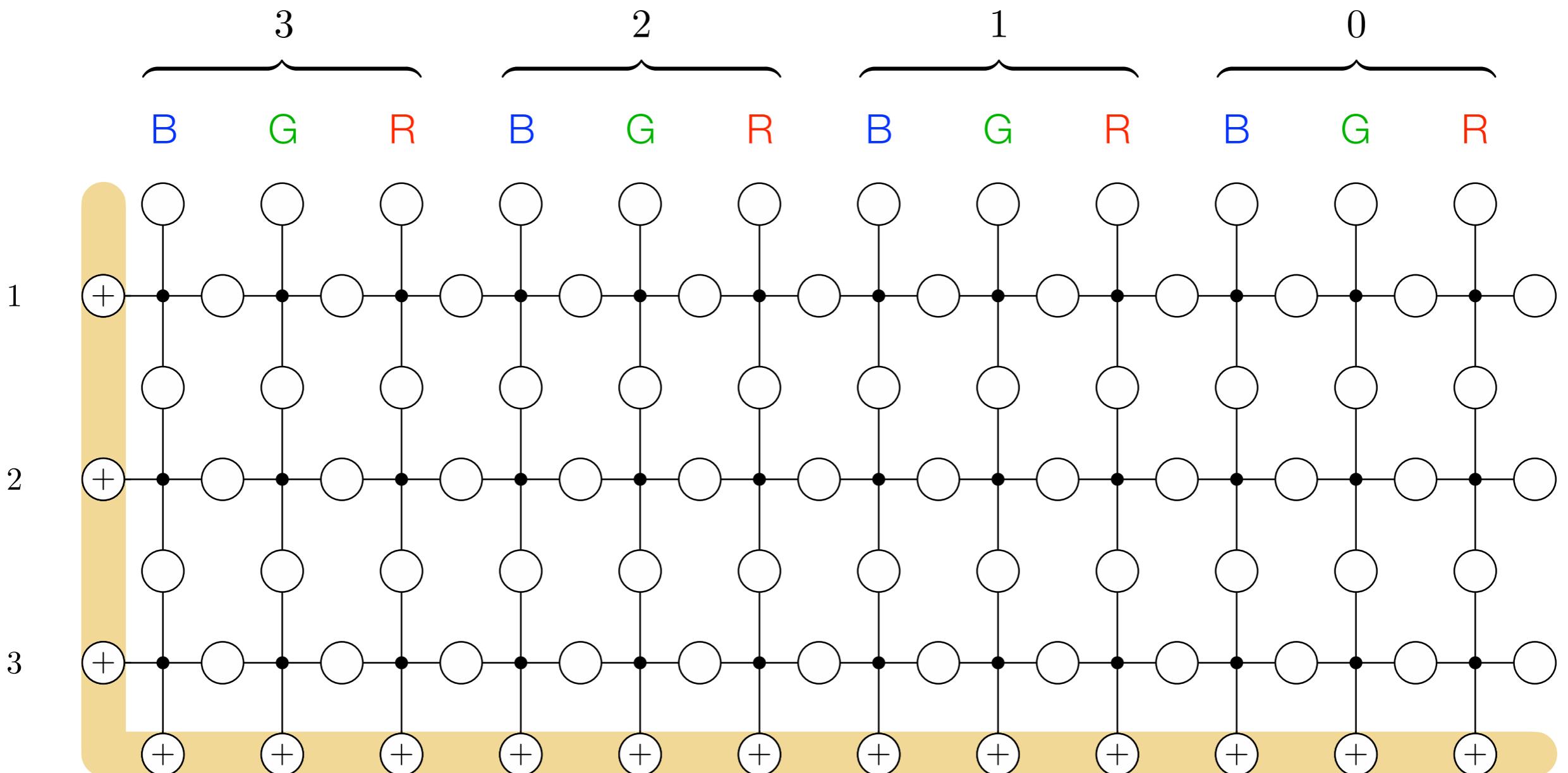
Left and bottom boundary: \oplus

Horizontal edges:

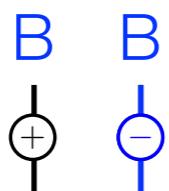


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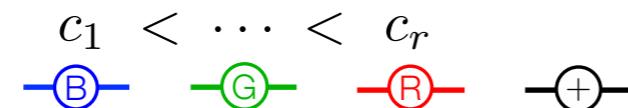


Vertical edges:



Left and bottom boundary: \oplus

Horizontal edges:

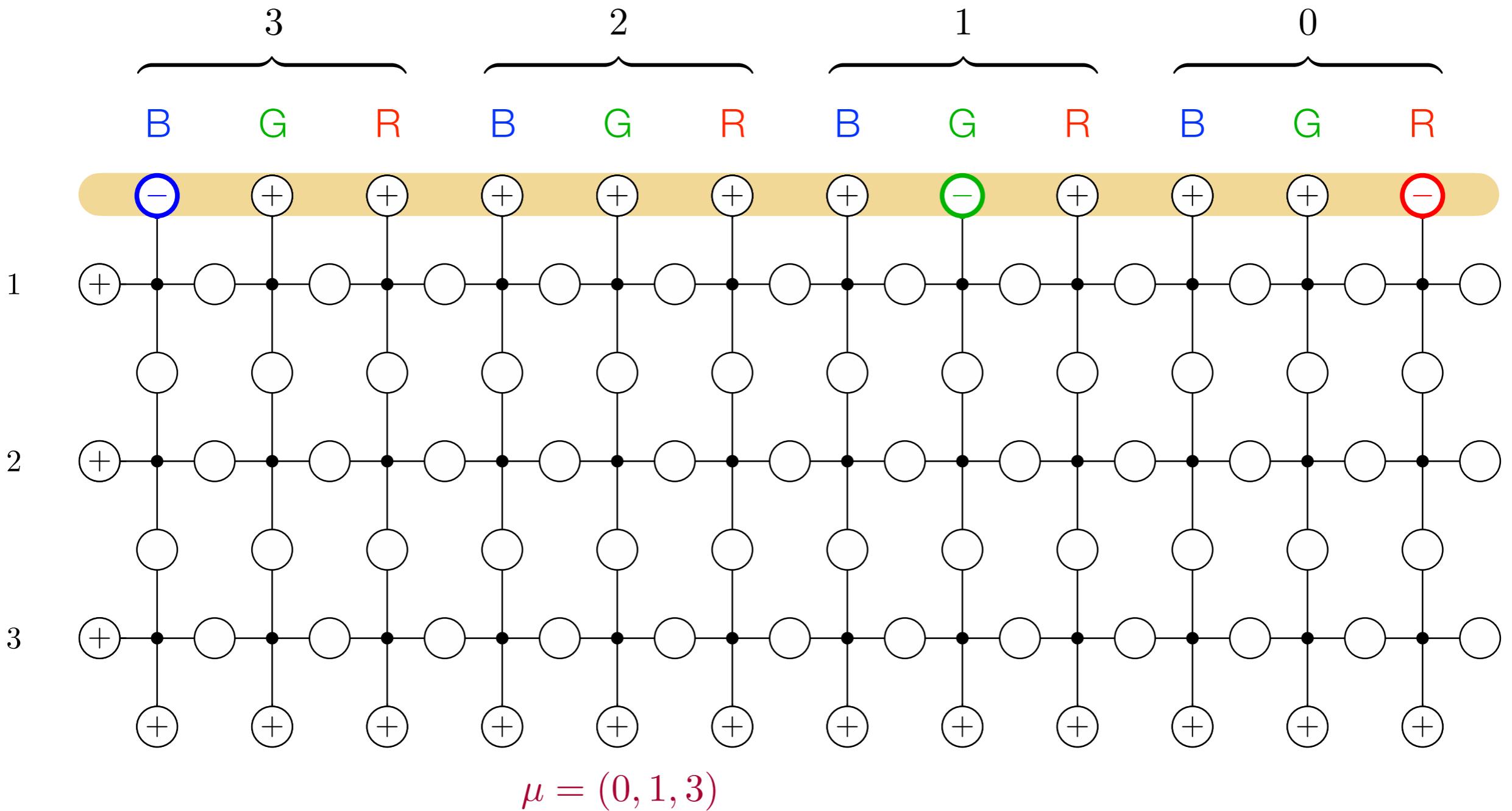


Top boundary:

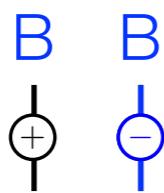
\ominus at block μ_j with color c_{r+1-j}

Boundary data: $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}_{\geq 0}^r$

$$w \in W = S_r$$

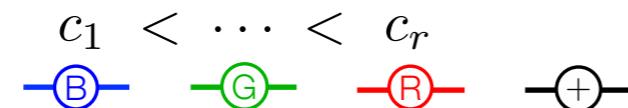


Vertical edges:



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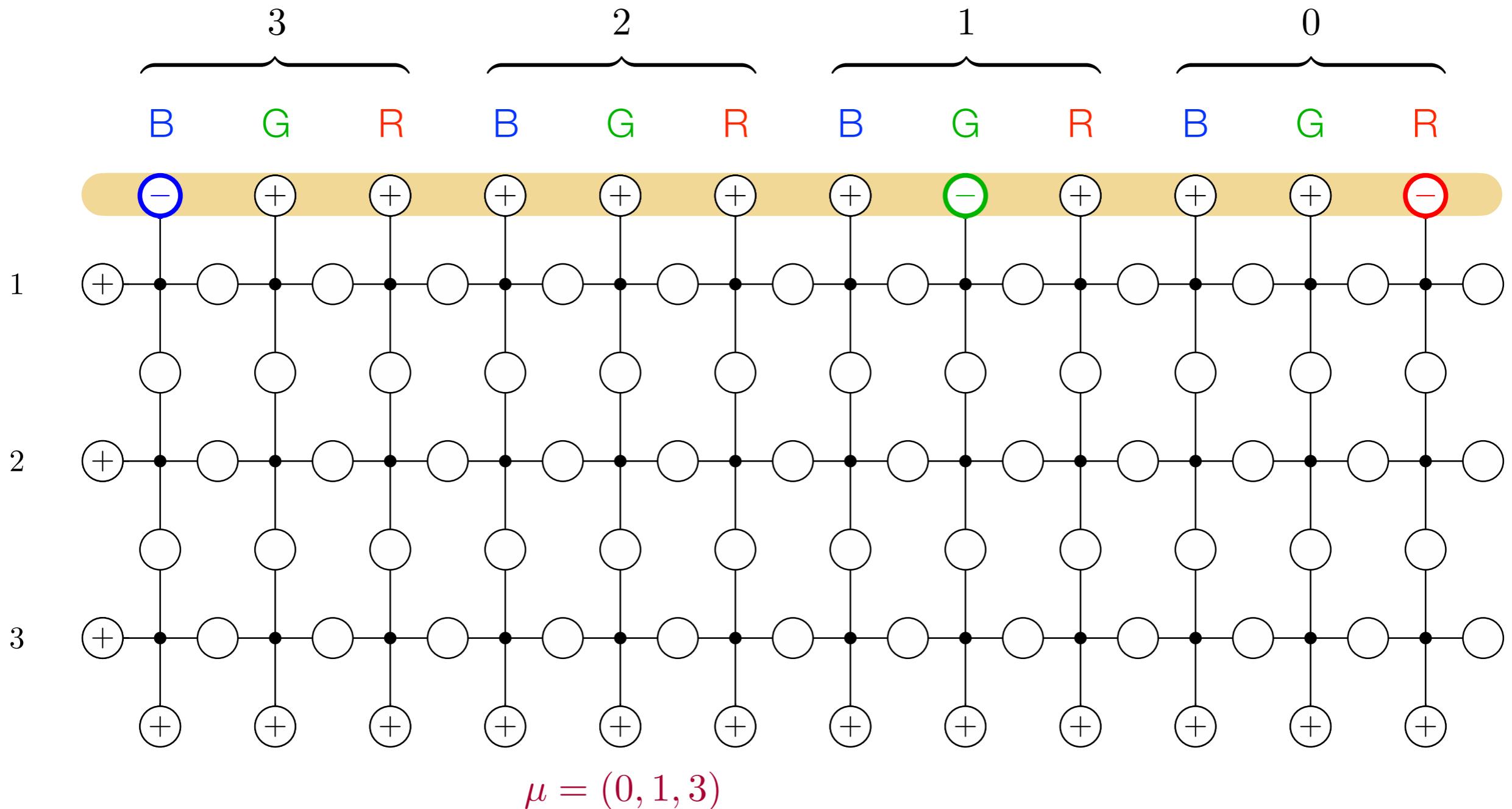
Top boundary:

$$P = (c_r, \dots, c_1)$$

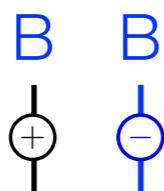
\ominus at block μ_j with color $c_{r+1-j} = (P)_j$

Boundary data: $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}_{\geq 0}^r$

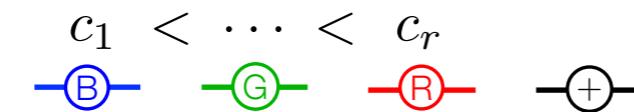
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Vertical edges:



Horizontal edges:



Boundary data: $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}_{\geq 0}^r$

$$w \in W = S_r$$

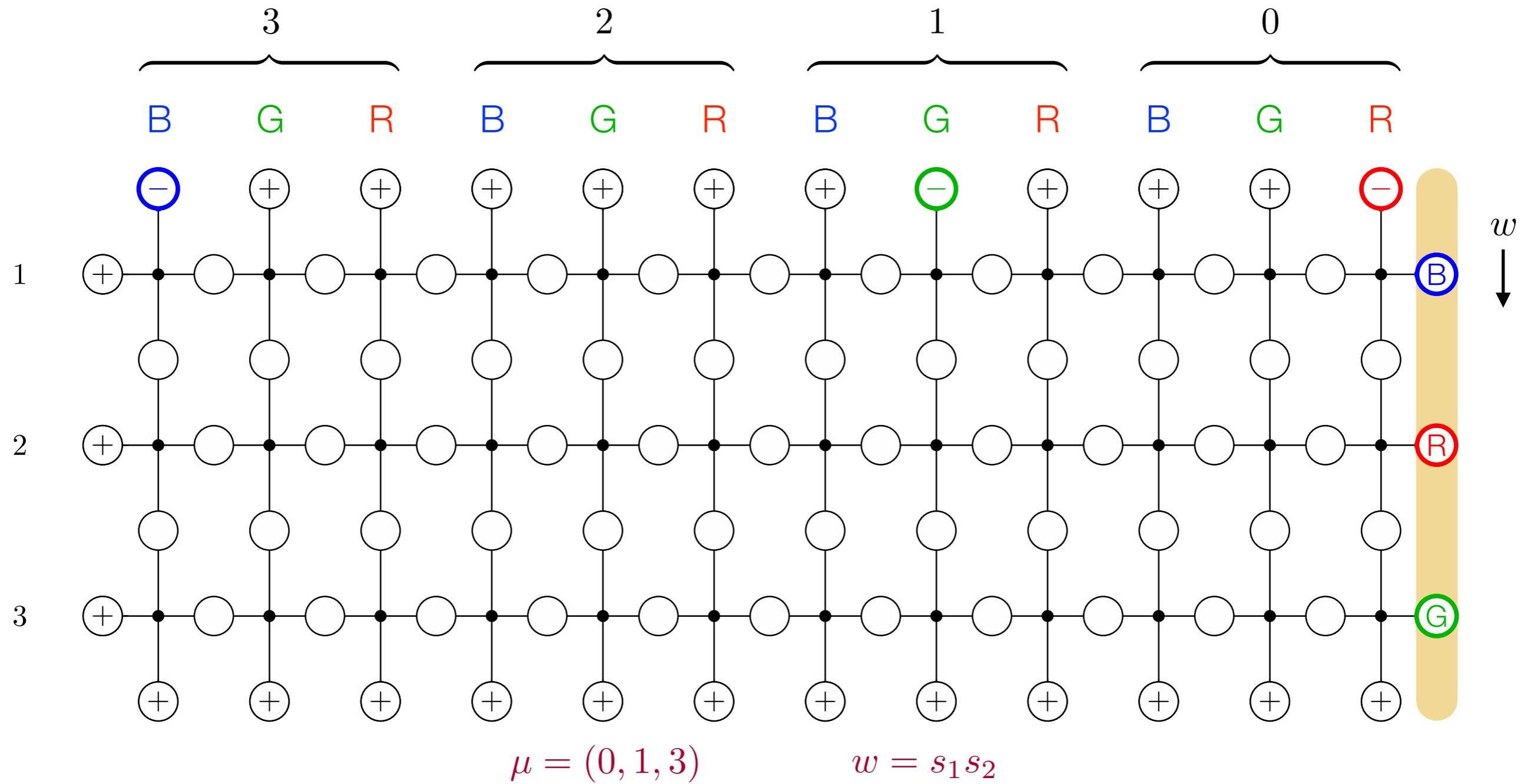
Left and bottom boundary: \oplus

Top boundary:

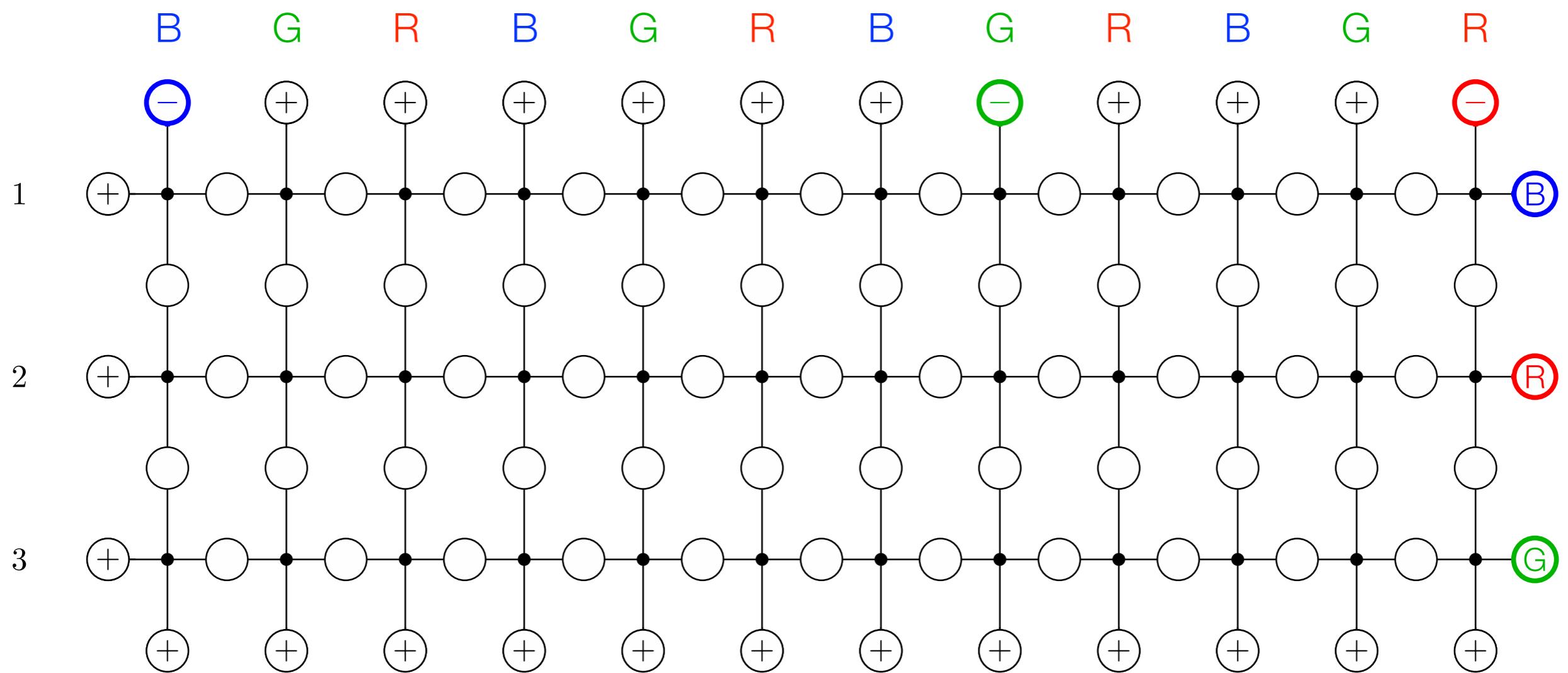
\ominus at block μ_j with color $c_{r+1-j} = (P)_j$

Right boundary:

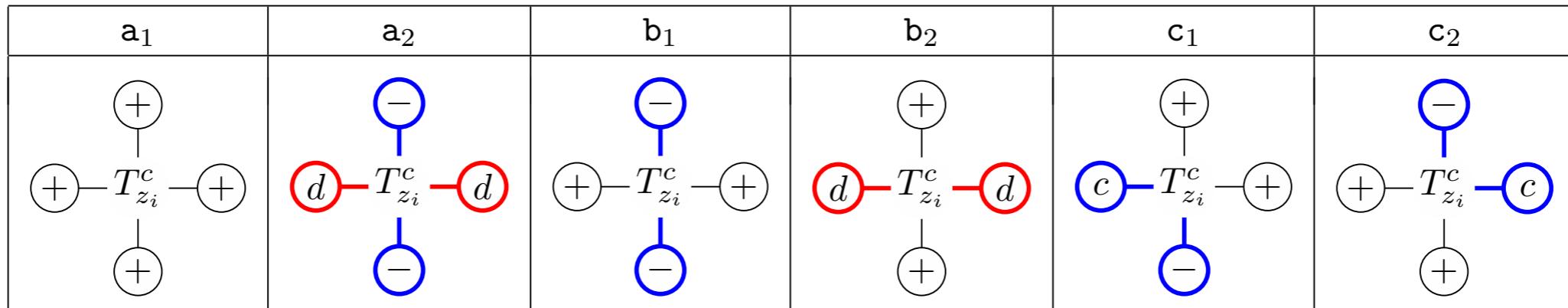
row i with color $c_{r+1-w^{-1}(i)} = (wP)_i$



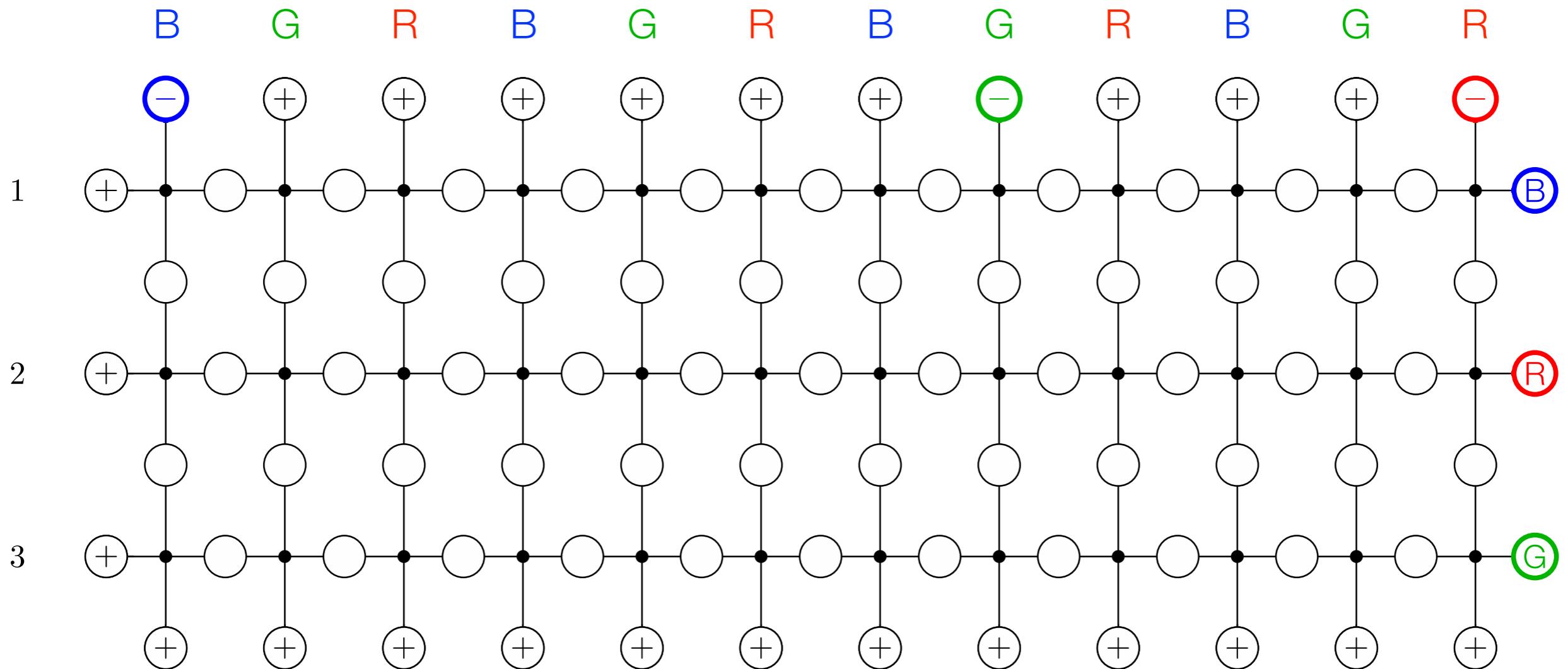
Inner edges from six possible vertex configurations:



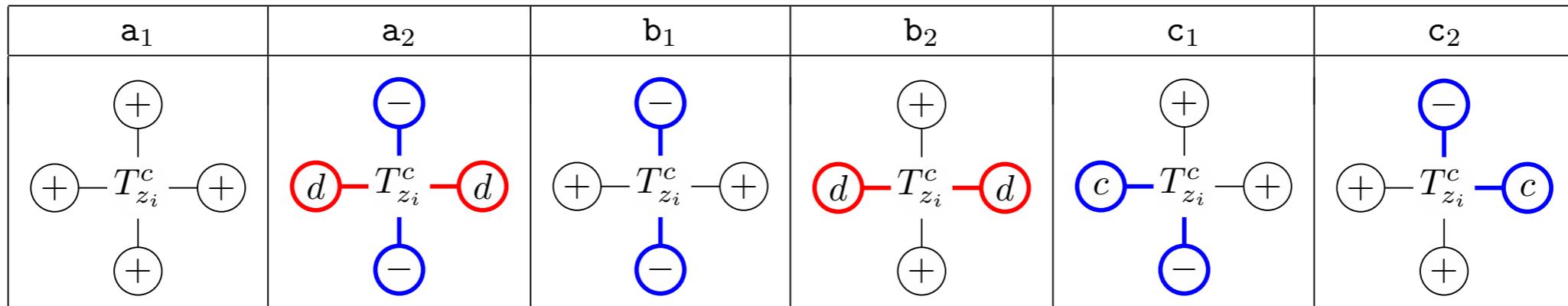
Inner edges from six possible vertex configurations:



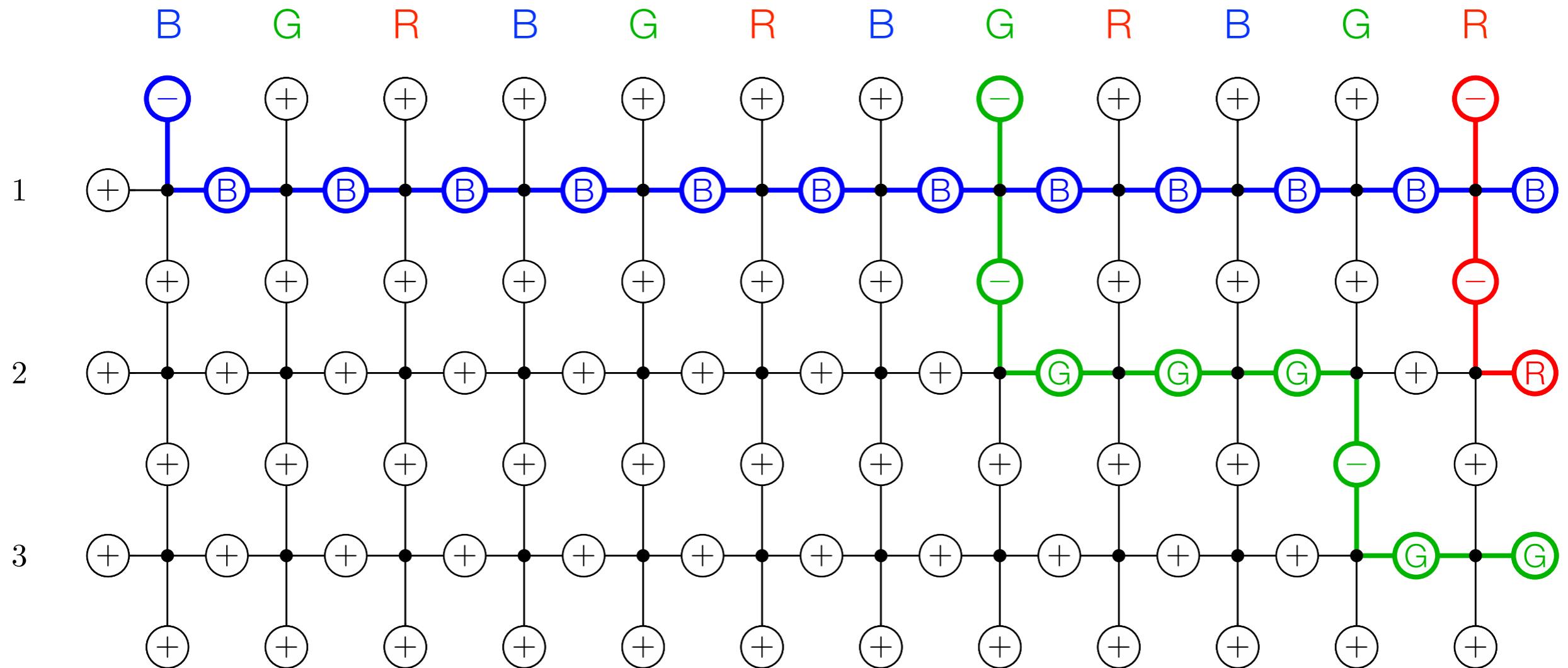
Each vertex is labeled by column color c and parameter z_i for row i denoted $T_{z_i}^c$



Inner edges from six possible vertex configurations:

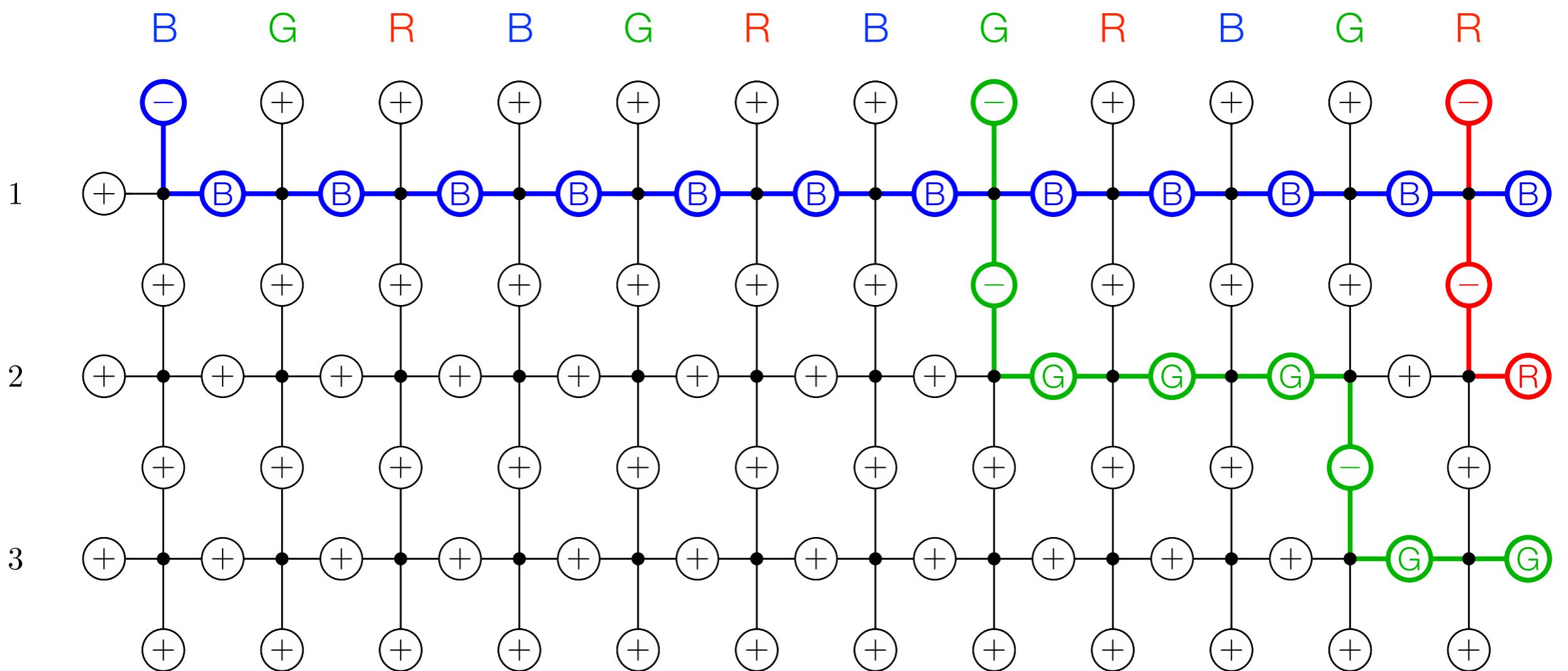


Each vertex is labeled by column color c and parameter z_i for row i denoted $T_{z_i}^c$

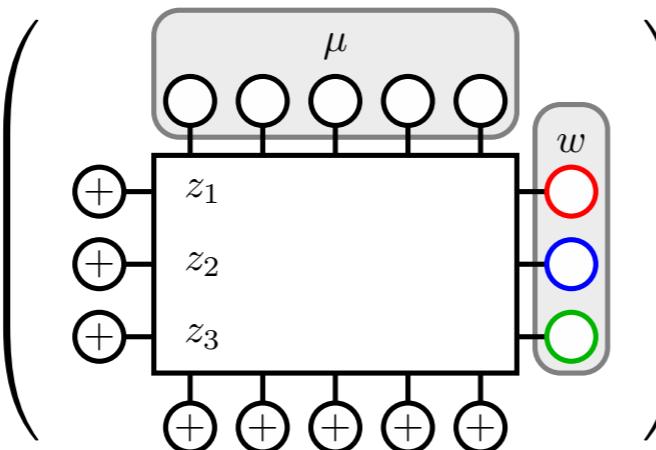


Inner edges from six possible vertex configurations: (parameter $v \in \mathbb{C}$)

	a_1	a_2	b_1	b_2	c_1	c_2
Boltzmann weights	1	$v \text{ if } c > d$ $z_i \text{ if } c = d$ 1 if $c < d$	$-v$	$z_i \text{ if } c = d,$ 1 otherwise	$(1 - v)z_i$	1



Partition function

$$Z_{\mu, w}(\mathbf{z}) := Z \left(\begin{array}{c} \text{Diagram with boundary conditions } \mu \text{ and weights } w \\ \text{with vertical labels } z_1, z_2, z_3 \end{array} \right) = \sum_{\substack{\text{states with} \\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$


Partition function

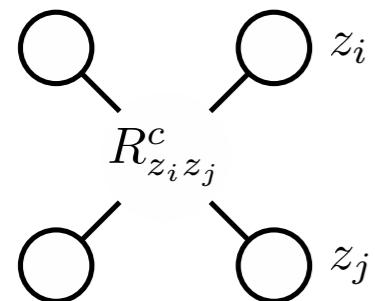
$$Z_{\mu,w}(\mathbf{z}) := Z \left(\begin{array}{c} \text{Diagram with boundary conditions } \mu \text{ and } w \\ \text{with internal states } z_1, z_2, z_3 \end{array} \right) = \sum_{\substack{\text{states with} \\ \text{given boundary}}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges:

Partition function

$$Z_{\mu, w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \\ & \circ & \circ & \circ & \circ & \circ & \mu \\ \hline & \oplus & & & & & \\ z_1 & \oplus & & & & & \\ z_2 & \oplus & & & & & \\ z_3 & \oplus & & & & & \\ & \oplus & \oplus & \oplus & \oplus & \oplus & \\ \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

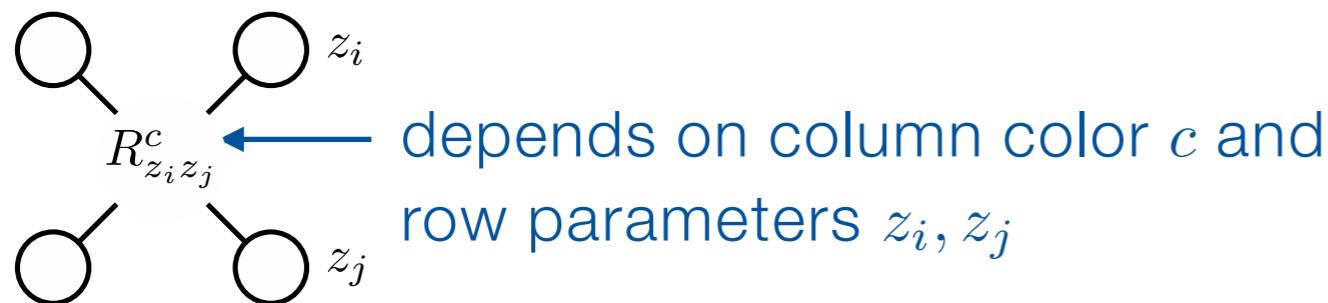
There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges:    



Partition function

$$Z_{\mu, w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \\ & \mu & & & & & \\ \hline & z_1 & z_2 & z_3 & & & w \\ \hline & + & + & + & + & + & \\ \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: 

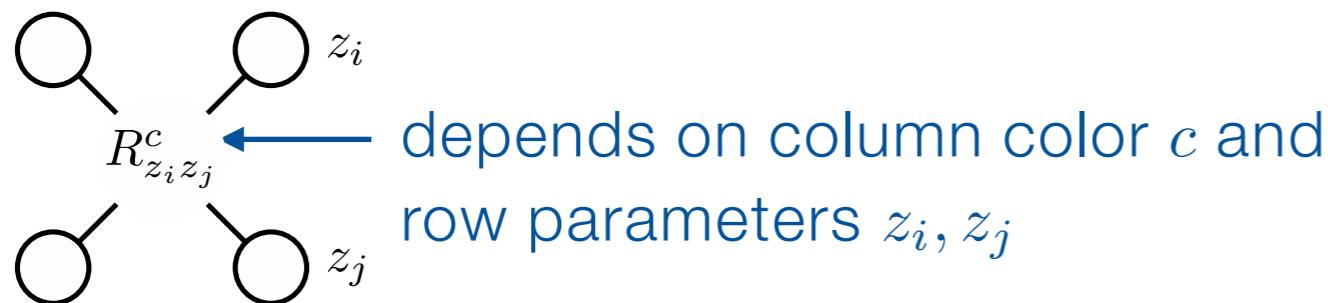


Partition function

$$Z_{\mu, w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \\ & \mu & & & & & \\ \hline & z_1 & z_2 & z_3 & & & w \\ \hline & + & + & + & + & + & \\ \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges:  (table of weights in paper)

[arXiv:1906.04140]

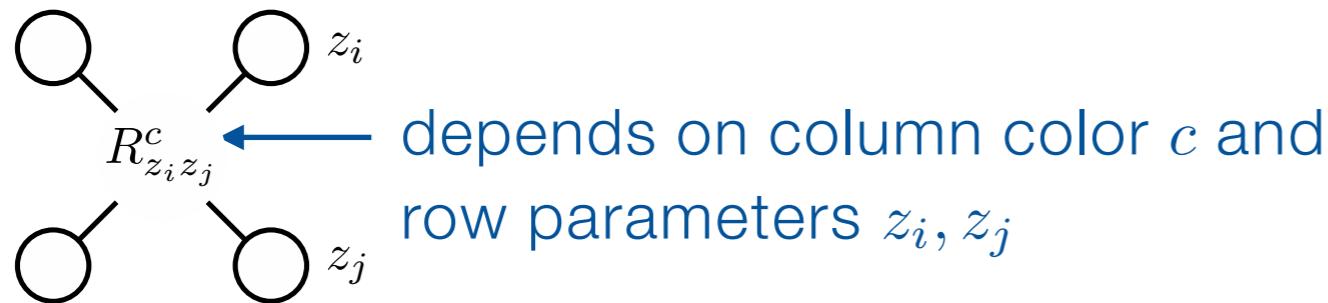


Partition function

$$Z_{\mu,w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \mu \\ & \circ & \circ & \circ & \circ & \circ & \\ \hline & z_1 & & & & & w \\ & + & & & & & \\ & z_2 & & & & & \\ & + & & & & & \\ & z_3 & & & & & \\ & + & & & & & \\ \hline & + & + & + & + & + & \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: (table of weights in paper)

[arXiv:1906.04140]



$$Z \left(\begin{array}{cc} z_j & b \\ z_i & a \end{array} \begin{array}{c} * \\ R_{z_i z_j}^c \end{array} \begin{array}{cc} c \\ * \end{array} \begin{array}{c} T_{z_i}^c \\ * \end{array} \begin{array}{cc} d \\ e \end{array} \begin{array}{c} z_i \\ z_j \end{array} \right) = Z \left(\begin{array}{cc} z_j & b \\ z_i & a \end{array} \begin{array}{c} * \\ T_{z_j}^c \end{array} \begin{array}{c} c \\ * \end{array} \begin{array}{c} * \\ R_{z_i z_j}^{c'} \end{array} \begin{array}{cc} d \\ e \end{array} \begin{array}{c} z_i \\ z_j \end{array} \right)$$

Partition function

$$Z_{\mu,w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \\ & \mu & & & & & \\ \hline & z_1 & z_2 & z_3 & & & w \\ \oplus & & & & & & \\ \oplus & & & & & & \\ \oplus & & & & & & \\ \hline & + & + & + & + & + & \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: (table of weights in paper)

[arXiv:1906.04140]



$$Z \left(\begin{array}{cc} z_j & c \\ * & T_{z_i}^c \\ \hline z_i & a \end{array} \right) = Z \left(\begin{array}{cc} z_j & c \\ b & * \\ \hline z_i & a \end{array} \right) + Z \left(\begin{array}{cc} z_j & c \\ * & R_{z_i z_j}^{c'} \\ \hline z_i & f \end{array} \right)$$

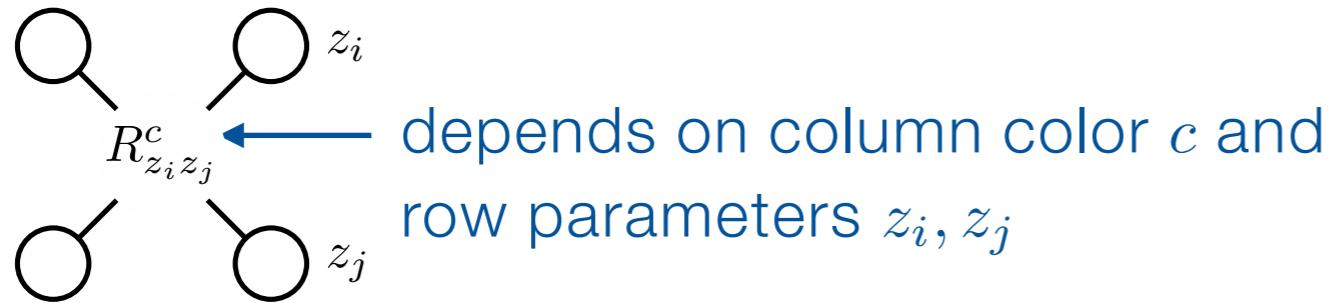
Next column color (wraps)

Partition function

$$Z_{\mu,w}(\mathbf{z}) := Z \left(\begin{array}{c|ccccc|c} & & & & & & \mu \\ & \circ & \circ & \circ & \circ & \circ & \\ \hline & z_1 & & & & & w \\ & + & & & & & \\ & z_2 & & & & & \\ & + & & & & & \\ & z_3 & & & & & \\ & + & & & & & \\ \hline & + & + & + & + & + & \end{array} \right) = \sum_{\text{states with given boundary}} \prod_{\text{vertices}} \text{weights}$$

There exists an (auxiliary) Yang–Baxter equation for this system with R-vertices that only involves horizontal edges: (table of weights in paper)

[arXiv:1906.04140]



$$Z \left(\begin{array}{ccccc} z_j & b & * & c & d \\ & R_{z_i z_j}^c & T_{z_i}^c & z_i \\ z_i & a & * & T_{z_j}^c & e \\ & & & & z_j \\ & & & f & \end{array} \right) = Z \left(\begin{array}{ccccc} z_j & b & T_{z_j}^c & * & d \\ & & & R_{z_i z_j}^{c'} & z_i \\ z_i & a & T_{z_i}^c & * & e \\ & & & f & z_j \\ & & & & \end{array} \right)$$

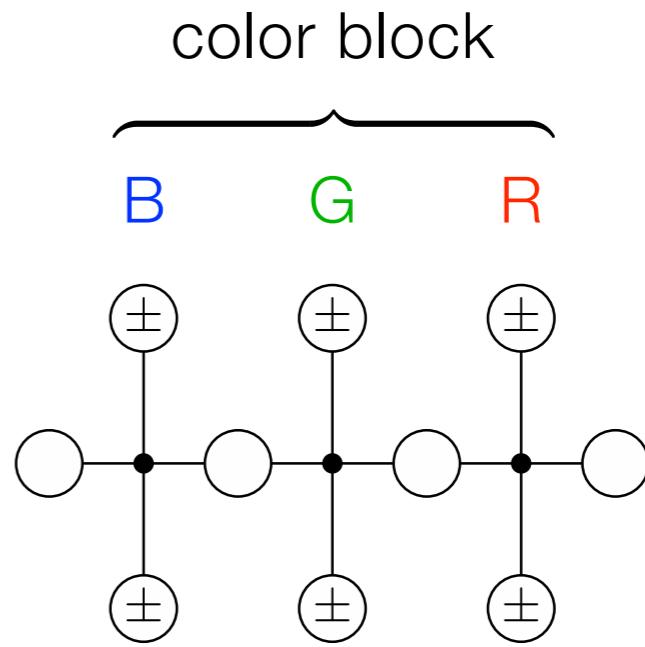
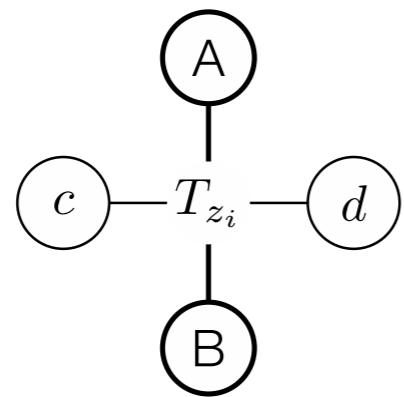
Next column color (wraps)

unfused

Fusion

$$A, B \in \mathcal{P}(\{B, G, R\})$$

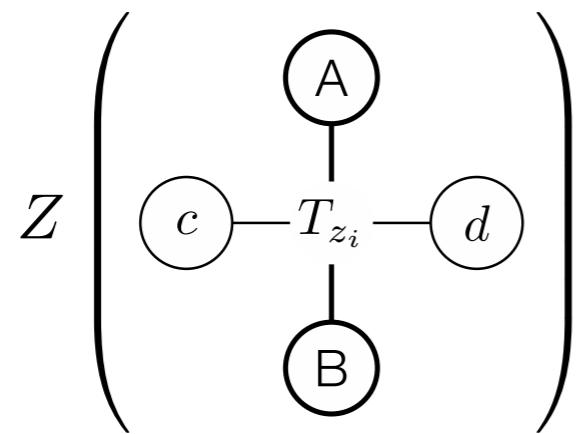
$$c, d \in \{B, G, R, +\}$$



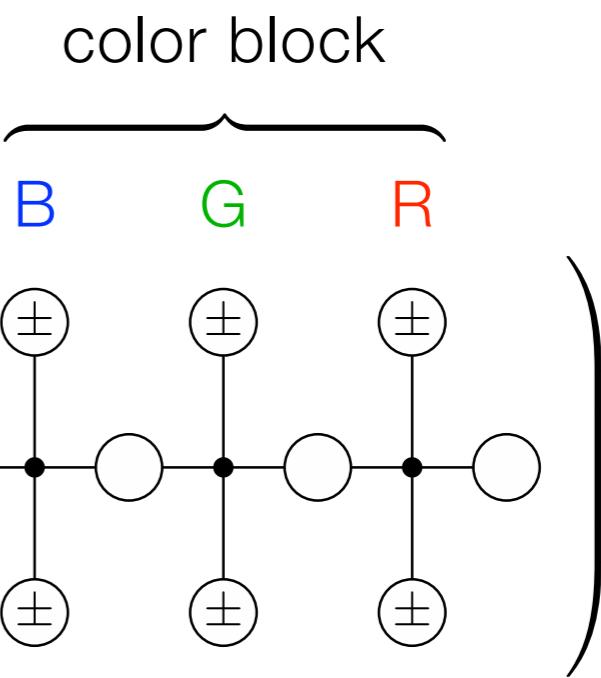
Fusion

$$A, B \in \mathcal{P}(\{B, G, R\})$$

$$c, d \in \{B, G, R, +\}$$



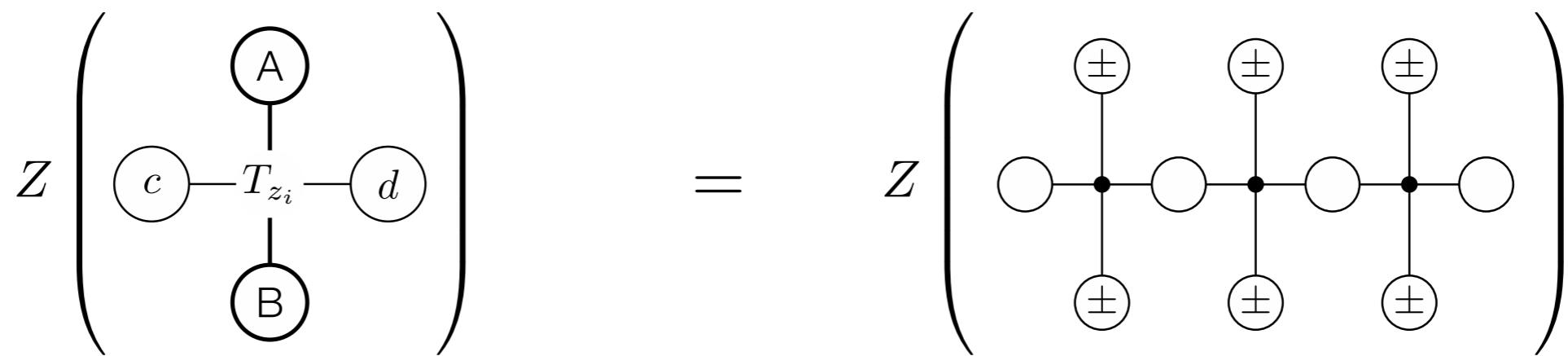
=



Fusion

$$A, B \in \mathcal{P}(\{B, G, R\})$$

$$c, d \in \{B, G, R, +\}$$



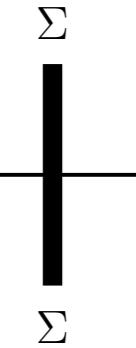
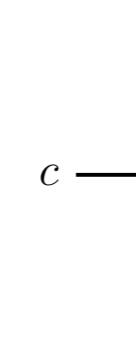
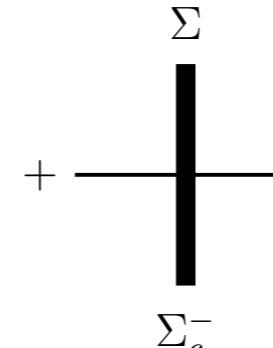
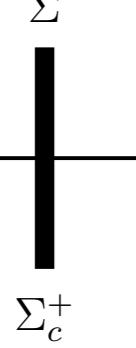
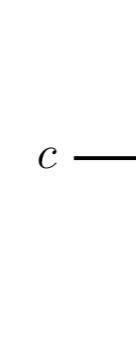
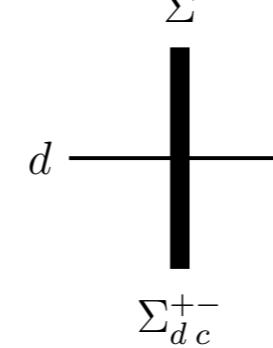
Example $r = 2$

$c > d$	$c < d$
 $(1 - v)z_i \quad \text{if } c > d,$ $(-v)(1 - v)z_i \quad \text{if } c < d.$	 left: 1 right: $(1 - v)z_i$

Figure 8 in arXiv:1906.04140

Fusion

Fused Boltzmann weights: $(r \text{ colors})$

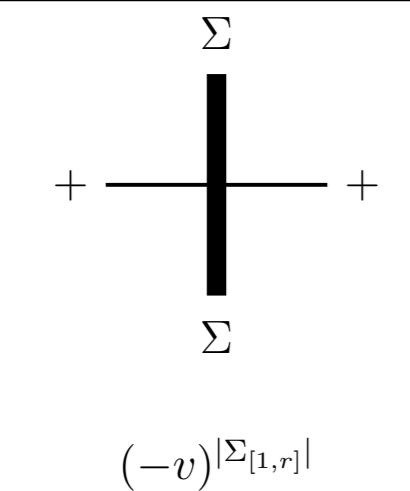
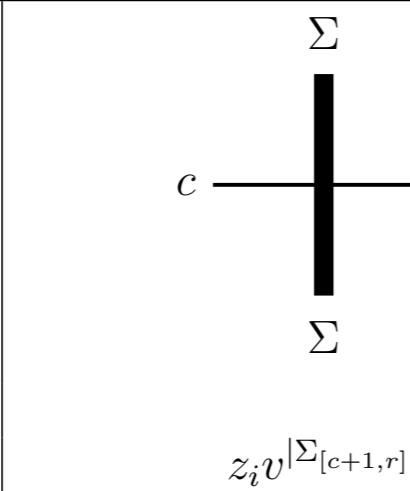
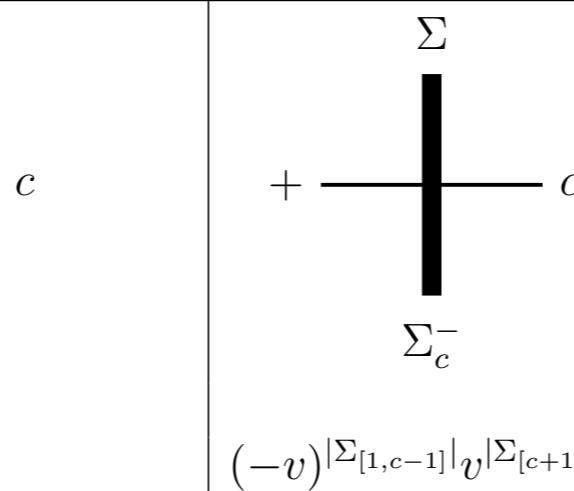
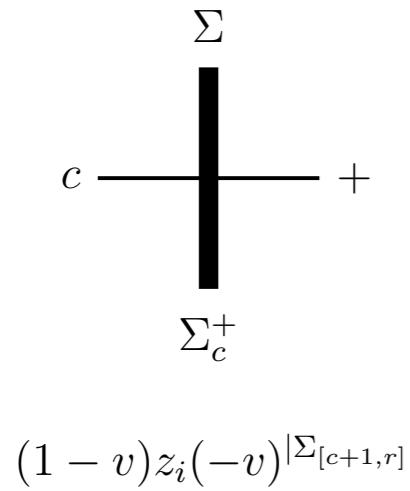
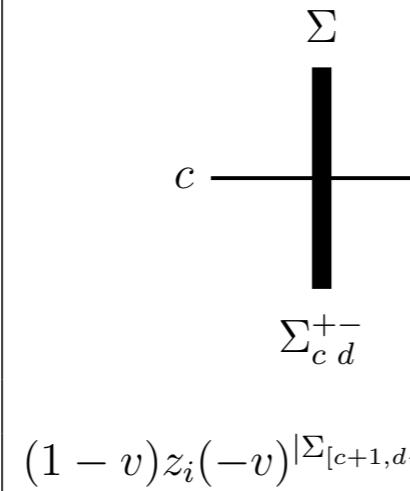
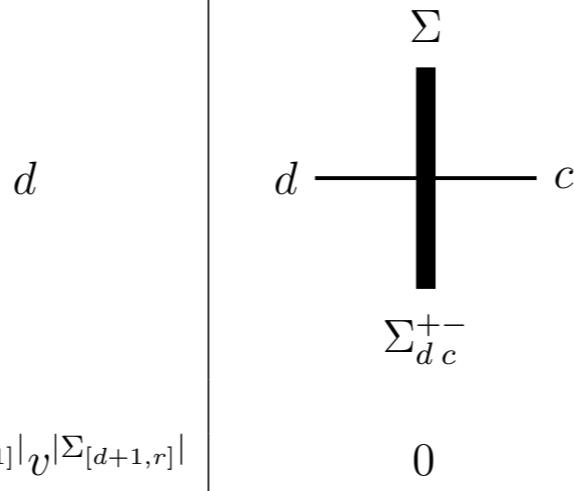
		
$(-v)^{ \Sigma_{[1,r]} }$	$z_i v^{ \Sigma_{[c+1,r]} }$	$(-v)^{ \Sigma_{[1,c-1]} } v^{ \Sigma_{[c+1,r]} }$
		
$(1-v) z_i (-v)^{ \Sigma_{[c+1,r]} }$	$(1-v) z_i (-v)^{ \Sigma_{[c+1,d-1]} } v^{ \Sigma_{[d+1,r]} }$	0

$$\Sigma \in \mathcal{P}(\{c_1, \dots, c_r\})$$

Figure 12 in arXiv:1906.04140

Fusion

Fused Boltzmann weights: $(r \text{ colors})$

$$\Sigma \in \mathcal{P}(\{c_1, \dots, c_r\})$$

$$\Sigma_a^+ = \Sigma \cup \{a\} \text{ if } a \notin \Sigma$$

$$\Sigma_a^- = \Sigma \setminus \{a\} \text{ if } a \in \Sigma$$

$$\Sigma_{a,b}^{+-} = \Sigma \cup \{a\} \setminus \{b\} \text{ if } a \in \Sigma, b \notin \Sigma$$

$$\Sigma_{[a,b]} = \{c \in \Sigma \mid a \leq c \leq b\}$$

Figure 12 in arXiv:1906.04140

Fusion

The fused lattice model has an ordinary Yang–Baxter equation with R-matrix independent of column which comes from a Drinfeld twist of $U_q(\widehat{\mathfrak{gl}}(r|1))$.

Fusion

The fused lattice model has an ordinary Yang–Baxter equation with R-matrix independent of column which comes from a Drinfeld twist of $U_q(\widehat{\mathfrak{gl}}(r|1))$.

$$Z \left(\begin{array}{c} z_j \text{---} b \\ | \quad | \\ R_{z_i z_j} \text{---} * \\ | \quad | \\ z_i \text{---} a \end{array} \text{---} \begin{array}{c} * \text{---} T_{z_i} \text{---} d \\ | \\ * \text{---} T_{z_j} \text{---} e \\ | \\ f \end{array} \text{---} \begin{array}{c} c \\ | \\ z_i \end{array} \right) = Z \left(\begin{array}{c} z_j \text{---} b \\ | \quad | \\ T_{z_j} \text{---} * \\ | \quad | \\ z_i \text{---} a \end{array} \text{---} \begin{array}{c} * \text{---} R_{z_i z_j} \text{---} d \\ | \\ * \text{---} T_{z_i} \text{---} e \\ | \\ f \end{array} \text{---} \begin{array}{c} c \\ | \\ z_i \\ | \\ d \\ | \\ z_j \\ | \\ e \\ | \\ z_j \end{array} \right)$$

Fusion

The fused lattice model has an ordinary Yang–Baxter equation with R-matrix independent of column which comes from a Drinfeld twist of $U_q(\widehat{\mathfrak{gl}}(r|1))$.

$$Z \left(\begin{array}{c} z_j \\ \circlearrowleft \\ b \\ R_{z_i z_j} \\ \circlearrowleft \\ z_i \\ a \end{array} \begin{array}{c} * \\ \circlearrowleft \\ c \\ T_{z_i} \\ - \\ d \\ z_i \end{array} \begin{array}{c} \\ \circlearrowleft \\ * \\ \circlearrowleft \\ e \\ z_j \end{array} \begin{array}{c} \\ \circlearrowleft \\ f \\ T_{z_j} \\ - \\ * \\ \circlearrowleft \\ z_j \end{array} \end{array} \right) = Z \left(\begin{array}{c} z_j \\ \circlearrowleft \\ b \\ T_{z_j} \\ - \\ * \\ \circlearrowleft \\ z_i \\ a \end{array} \begin{array}{c} * \\ \circlearrowleft \\ c \\ R_{z_i z_j} \\ \circlearrowleft \\ d \\ z_i \end{array} \begin{array}{c} \\ \circlearrowleft \\ * \\ \circlearrowleft \\ e \\ z_j \end{array} \begin{array}{c} \\ \circlearrowleft \\ f \\ T_{z_i} \\ - \\ * \\ \circlearrowleft \\ z_j \end{array} \end{array} \right)$$

$R_{z_i, z_j} = R_{z_i, z_j}^{c_1}$

fused

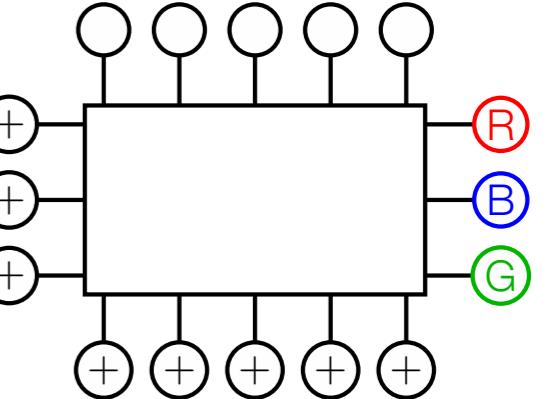
Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \\ \text{---} \\ z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ * \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \\ \text{---} \\ z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right)$$

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{---} b \\ | \\ R_{z_i z_j} \text{---} * \\ | \\ z_i \text{---} a \end{array} \begin{array}{c} * \\ \text{---} T_{z_i} \\ | \\ * \\ \text{---} T_{z_j} \\ | \\ f \end{array} \begin{array}{c} c \\ \text{---} d \\ | \\ z_i \end{array} \right) = Z \left(\begin{array}{c} z_j \text{---} b \\ | \\ T_{z_j} \text{---} * \\ | \\ z_i \text{---} a \end{array} \begin{array}{c} * \\ \text{---} R_{z_i z_j} \\ | \\ * \\ \text{---} T_{z_i} \\ | \\ f \end{array} \begin{array}{c} c \\ \text{---} d \\ | \\ z_i \\ \text{---} e \\ | \\ z_j \end{array} \right)$$

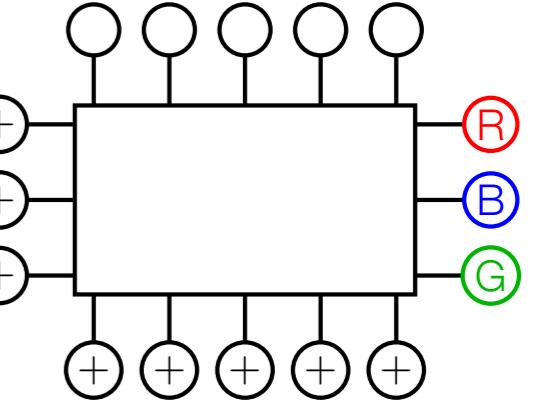
$Z_{\mu, w}(\mathbf{z}) :=$



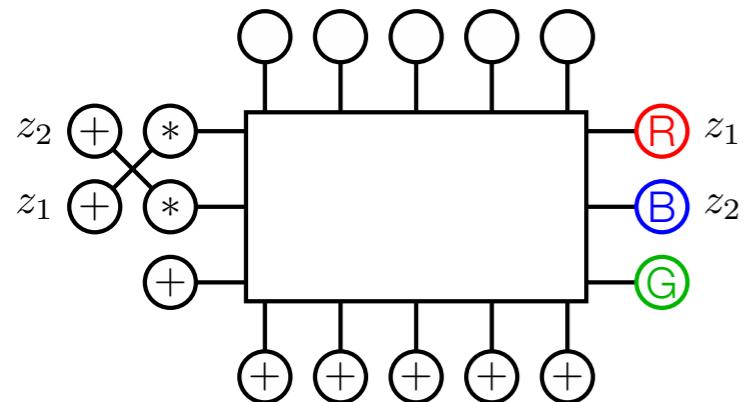
Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right)$$

$$Z_{\mu, w}(\mathbf{z}) :=$$



Train argument



Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ e \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right)$$

$Z_{\mu, w}(\mathbf{z}) :=$

Train argument

sum over internal states

Recursion relations

$$Z \left(\begin{array}{c} z_j \\ \textcircled{b} \\ R_{z_i z_j} \\ z_i \\ \textcircled{a} \\ * \\ T_{z_j} \\ e \\ z_j \\ f \end{array} \right) = Z \left(\begin{array}{c} c \\ z_j \\ \textcircled{b} \\ T_{z_j} \\ * \\ z_i \\ \textcircled{a} \\ T_{z_i} \\ * \\ f \\ d \\ z_i \\ e \\ z_j \end{array} \right)$$

$Z_{\mu, w}(\mathbf{z}) :=$

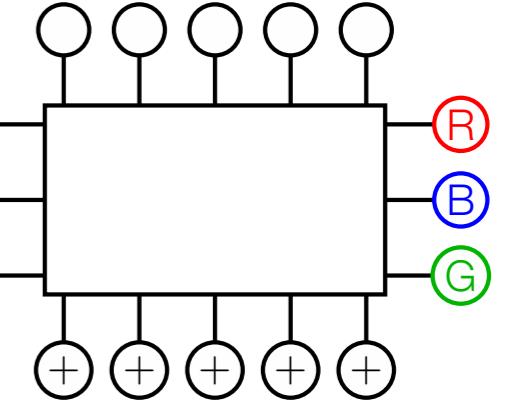
Train argument

sum over internal states

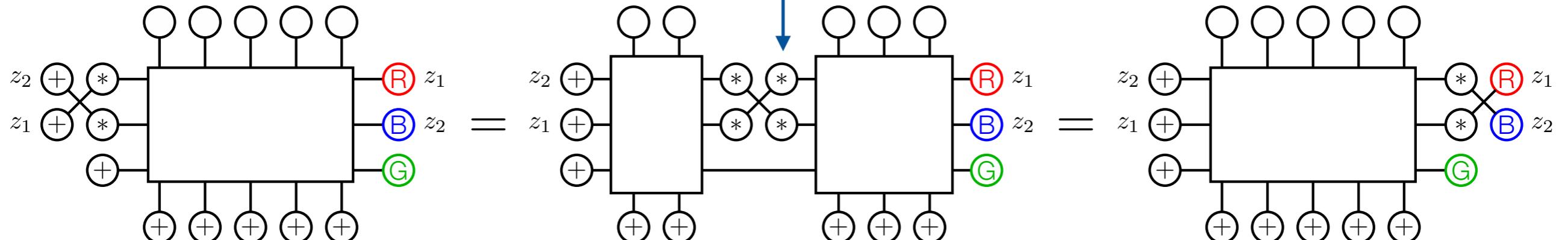
Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ e \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right)$$

$$Z_{\mu, w}(\mathbf{z}) :=$$



Train argument

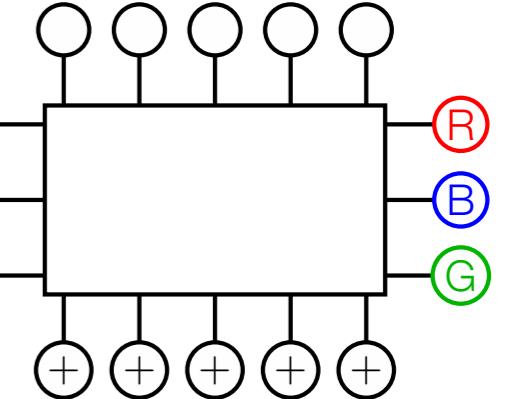


Insert admissible R-vertices and compare

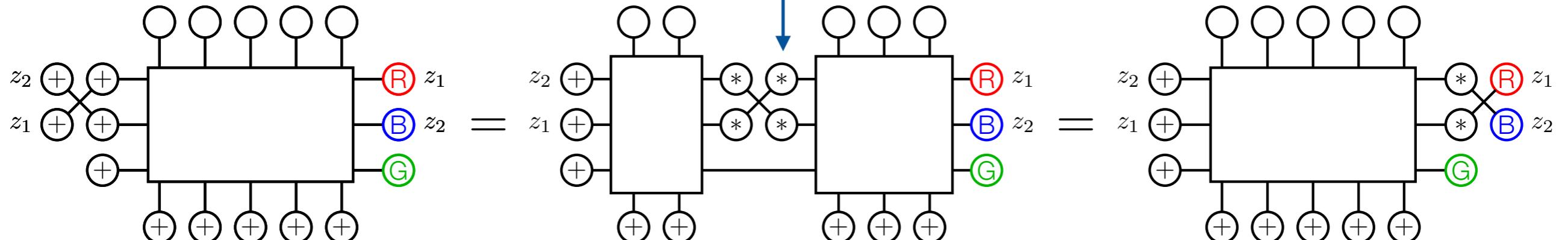
Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ e \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \end{array} \right)$$

$$Z_{\mu, w}(\mathbf{z}) :=$$



Train argument

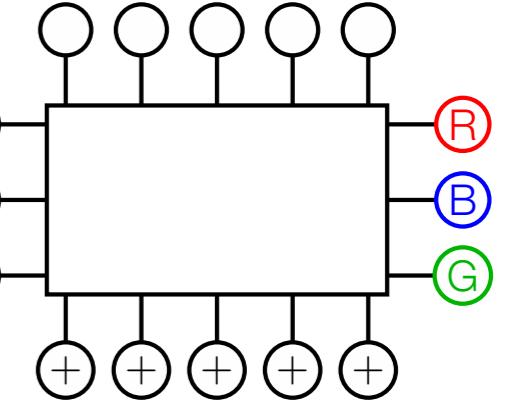


Insert admissible R-vertices and compare

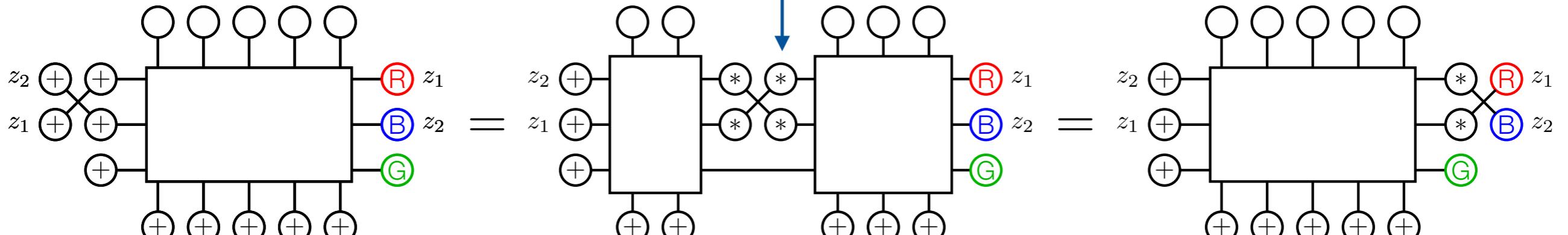
Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ e \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right)$$

$Z_{\mu, w}(\mathbf{z}) :=$



Train argument



Insert admissible R-vertices and compare

The odd and even parts of the $U_q(\widehat{\mathfrak{gl}}(r|1))$ only mix in the middle steps

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ e \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right)$$
$$Z_{\mu, w}(\mathbf{z}) :=$$

A diagram showing a block labeled $Z_{\mu, w}(\mathbf{z})$ with 5 input nodes and 5 output nodes. The inputs are labeled with '+' symbols. The outputs are labeled with color-coded circles: R (red), B (blue), G (green).

Train argument

The diagram illustrates a recursion relation for a neural network block. It shows three configurations of inputs and outputs:

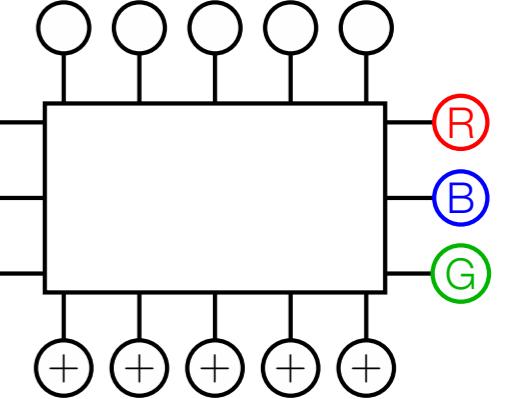
- Inputs z_2 and z_1 , outputs R (red), B (blue), and G (green).
- Inputs z_2 and z_1 , outputs R (red), R (red), B (blue), B (blue), and G (green).
- Inputs z_2 and z_1 , outputs B (blue), R (red), R (red), B (blue), B (blue), and G (green).

The middle configuration is equal to the sum of the other two.

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ * \\ \text{---} \\ d \\ \text{---} \\ z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ * \\ \text{---} \\ d \\ \text{---} \\ z_i \\ \text{---} \\ e \\ \text{---} \\ z_j \end{array} \right)$$

$Z_{\mu, w}(\mathbf{z}) :=$



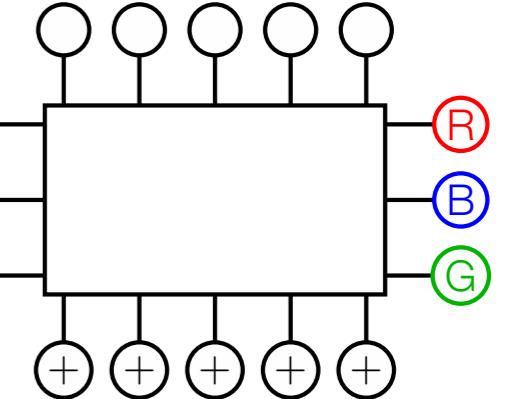
Train argument

$$z_2 \quad z_1 \quad \begin{array}{c} z_1 \\ \text{---} \\ + \end{array} \quad \begin{array}{c} z_2 \\ \text{---} \\ + \end{array} \quad Z_{\mu, w}(\mathbf{z}) \quad = \quad z_2 \quad z_1 \quad \begin{array}{c} z_1 \\ \text{---} \\ + \end{array} \quad \begin{array}{c} z_2 \\ \text{---} \\ + \end{array} \quad \begin{array}{c} R \\ \text{---} \\ R \\ \text{---} \\ B \\ \text{---} \\ B \\ \text{---} \\ G \end{array} \quad + \quad z_2 \quad z_1 \quad \begin{array}{c} z_1 \\ \text{---} \\ + \end{array} \quad \begin{array}{c} z_2 \\ \text{---} \\ + \end{array} \quad \begin{array}{c} B \\ \text{---} \\ R \\ \text{---} \\ R \\ \text{---} \\ B \\ \text{---} \\ G \end{array}$$

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ z_i \text{ } a \end{array} \begin{array}{c} * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \\ \text{---} \\ T_{z_j} \\ \text{---} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ * \\ \text{---} \\ e \text{ } z_j \\ \text{---} \\ f \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ } b \\ \text{---} \\ T_{z_j} \\ \text{---} \\ z_i \text{ } a \\ \text{---} \\ * \\ \text{---} \\ f \end{array} \begin{array}{c} * \\ \text{---} \\ R_{z_i z_j} \\ \text{---} \\ * \\ \text{---} \\ T_{z_i} \\ \text{---} \\ * \end{array} \begin{array}{c} c \\ \text{---} \\ d \text{ } z_i \\ \text{---} \\ e \text{ } z_j \\ \text{---} \\ f \end{array} \right)$$

$Z_{\mu, w}(\mathbf{z}) :=$



Train argument

$$z_2 \text{ } + \\ z_1 \text{ } + \\ \text{---} \\ Z_{\mu, w}(\mathbf{z}) \\ \text{---} \\ + \\ z_2 \text{ } + \\ z_1 \text{ } + \\ \text{---} \\ Z_{\mu, w}(s_1 \mathbf{z}) \\ \text{---} \\ + \\ z_2 \text{ } + \\ z_1 \text{ } + \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---}$$

$\text{R } z_1$ $\text{B } z_2$ G

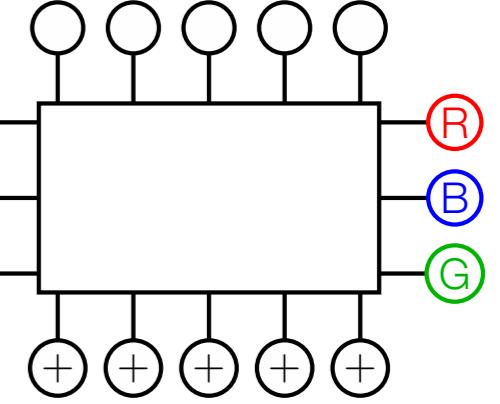
$\text{R } z_1$ $\text{B } z_2$ G

$\text{B } z_1$ $\text{R } z_2$ G

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{---} b \\ | \\ R_{z_i z_j} \text{---} * \\ | \\ z_i \text{---} a \end{array} \begin{array}{c} c \\ | \\ * \\ | \\ T_{z_i} \text{---} d \\ | \\ z_i \end{array} \begin{array}{c} d \\ | \\ z_i \end{array} \right) = Z \left(\begin{array}{c} z_j \text{---} b \\ | \\ T_{z_j} \text{---} * \\ | \\ z_i \text{---} a \end{array} \begin{array}{c} c \\ | \\ * \\ | \\ R_{z_i z_j} \text{---} e \\ | \\ z_j \end{array} \begin{array}{c} e \\ | \\ z_j \end{array} \right)$$

$$Z_{\mu, w}(\mathbf{z}) :=$$



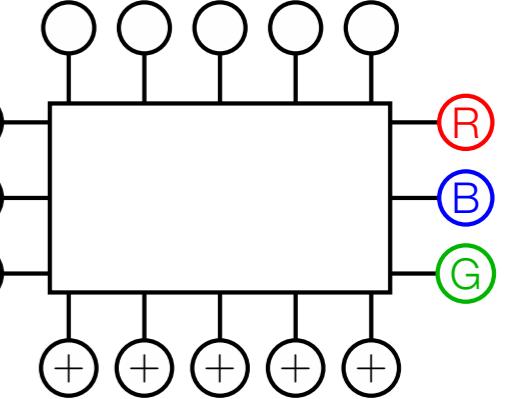
Train argument

$$z_2 \text{---} + \text{---} Z_{\mu, w}(\mathbf{z}) \text{---} \begin{matrix} z_1 \\ R \\ B \\ G \end{matrix} = z_2 \text{---} + \text{---} Z_{\mu, w}(s_1 \mathbf{z}) \text{---} \begin{matrix} z_1 \\ R \\ B \\ G \end{matrix} + z_2 \text{---} + \text{---} Z_{\mu, s_1 w}(s_1 \mathbf{z}) \text{---} \begin{matrix} z_1 \\ R \\ B \\ G \end{matrix}$$

Recursion relations

$$Z \left(\begin{array}{c} z_j \text{ (b)} \\ R_{z_i z_j} \\ z_i \text{ (a)} \end{array} \begin{array}{c} * \\ T_{z_i} \\ * \\ * \\ T_{z_j} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \\ z_i \\ e \\ z_j \end{array} \right) = Z \left(\begin{array}{c} z_j \text{ (b)} \\ T_{z_j} \\ z_i \text{ (a)} \end{array} \begin{array}{c} * \\ R_{z_i z_j} \\ * \\ * \\ T_{z_i} \\ f \end{array} \begin{array}{c} c \\ \text{---} \\ d \\ z_i \\ e \\ z_j \end{array} \right)$$

$$Z_{\mu, w}(\mathbf{z}) :=$$



Train argument

$$\begin{array}{c} z_2 \\ z_1 \end{array} \begin{array}{c} + \\ + \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} Z_{\mu, w}(\mathbf{z}) \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} R \\ B \\ G \end{array} = \begin{array}{c} z_2 \\ z_1 \end{array} \begin{array}{c} + \\ + \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} Z_{\mu, w}(s_1 \mathbf{z}) \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} R \\ B \\ G \end{array} + \begin{array}{c} z_2 \\ z_1 \end{array} \begin{array}{c} + \\ + \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} Z_{\mu, s_1 w}(s_1 \mathbf{z}) \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} R \\ B \\ G \end{array}$$

Solving for $Z_{\mu, s_1 w}$ gives a recursion relation in w .

Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\text{ } \textcircled{G} \text{ polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\text{(finite geometric series)} \end{aligned}$$

Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$s_i f(\mathbf{z}) = f(s_i \mathbf{z}) \quad \mathbf{z}^\lambda = \prod_i z_i^{\lambda_i} \quad \mathbf{z}^{\alpha_i} = z_i/z_{i+1}$
 $T_i \curvearrowright \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r)$
(finite geometric series)

Base case

Recursion relations

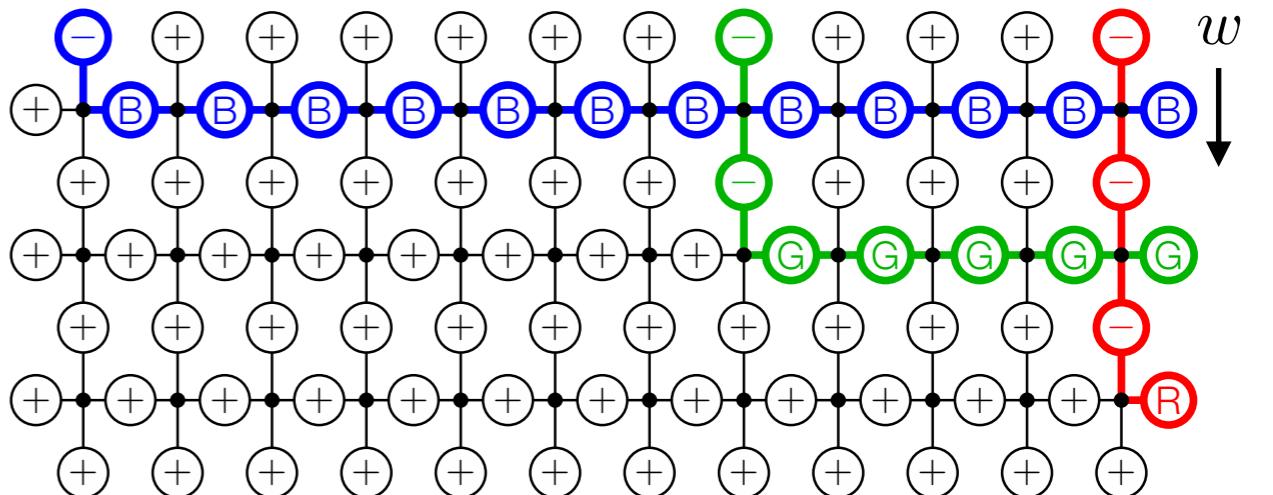
$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case



Recursion relations

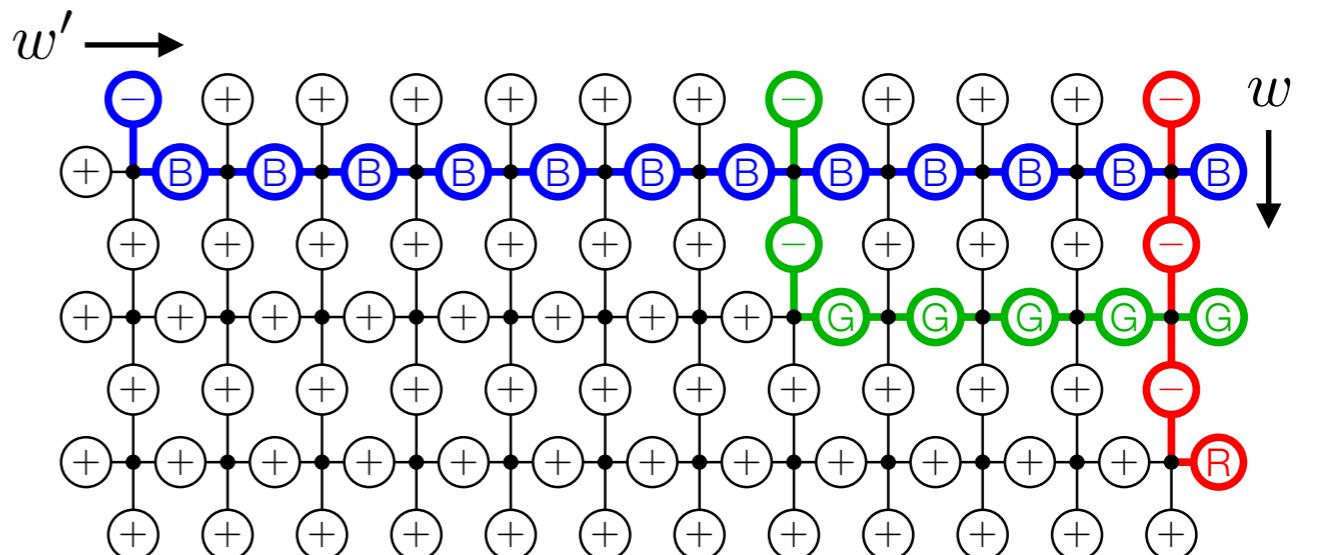
$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case



Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

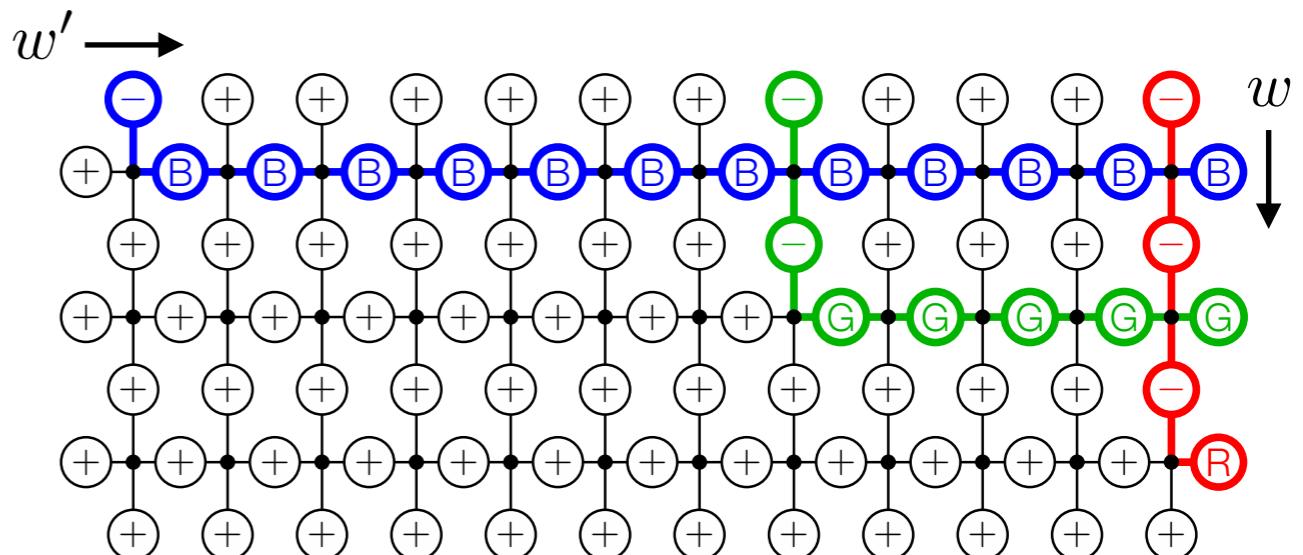
Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case

Block number μ_j with color $(P)_j$



Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

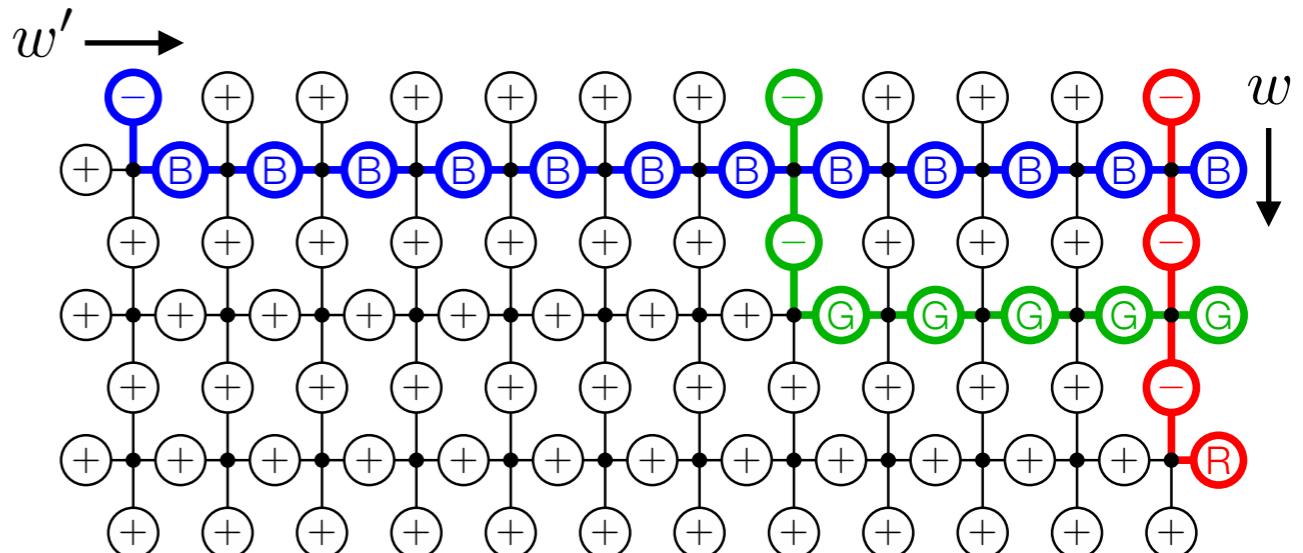
$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case

Block number $(w' \mu)_j$ with color $(w' P)_j$

Positions and colors left-to-right



Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

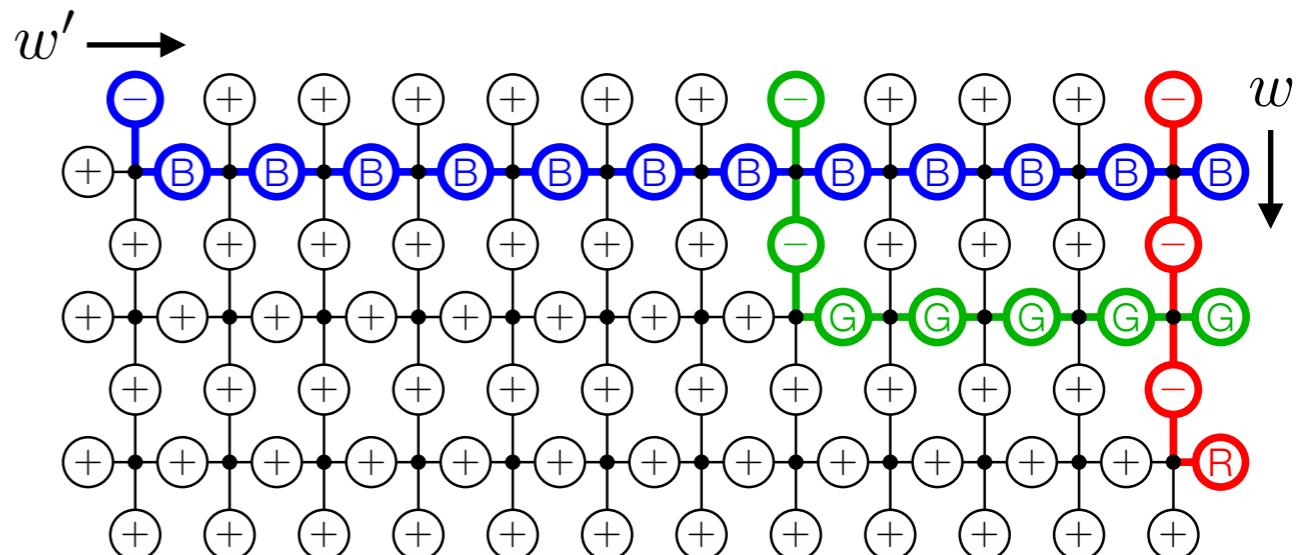
$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case $w' = w$

Block number $(w' \mu)_j$ with color $(w' P)_j$

Positions and colors left-to-right



Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

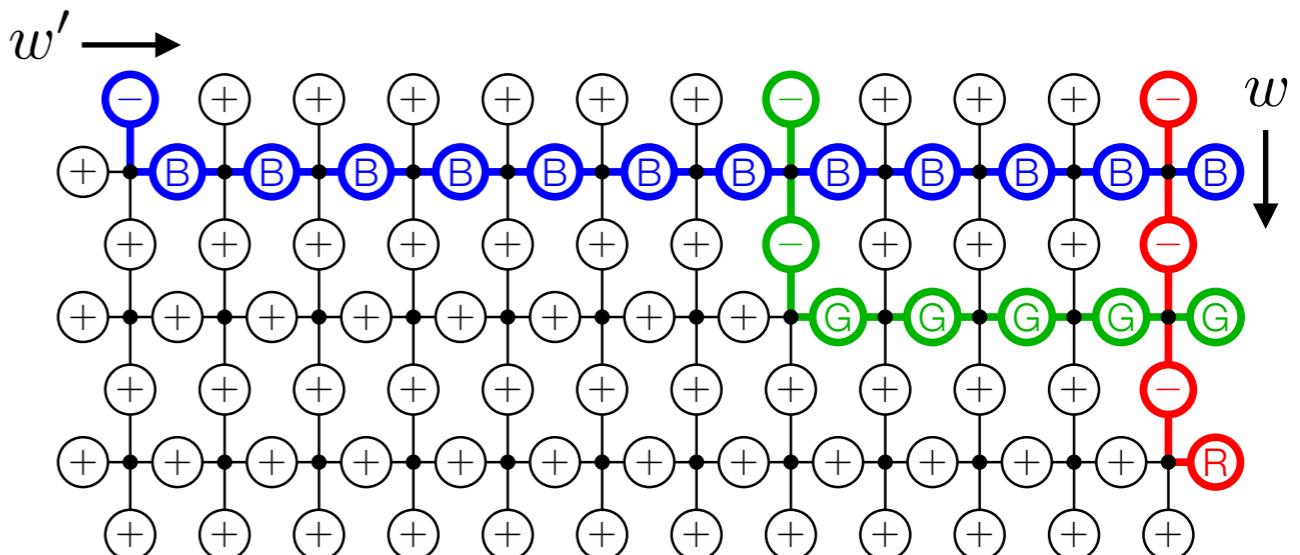
$$\begin{aligned} s_i f(\mathbf{z}) &= f(s_i \mathbf{z}) & \mathbf{z}^\lambda &= \prod_i z_i^{\lambda_i} & \mathbf{z}^{\alpha_i} &= z_i/z_{i+1} \\ T_i &\in \text{polynomials in } \mathbf{z} = (z_1, \dots, z_r) \\ &\quad (\text{finite geometric series}) \end{aligned}$$

Base case $w' = w$

Block number $(w' \mu)_j$ with color $(w' P)_j$

Positions and colors left-to-right

$$Z_{\mu, w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w' \mu} \quad \text{unique state}$$



Recursion relations

$$Z_{\mu, s_i w}(\mathbf{z}) := \begin{cases} T_i Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) > \ell(w) \\ T_i^{-1} Z_{\mu, w}(\mathbf{z}) & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

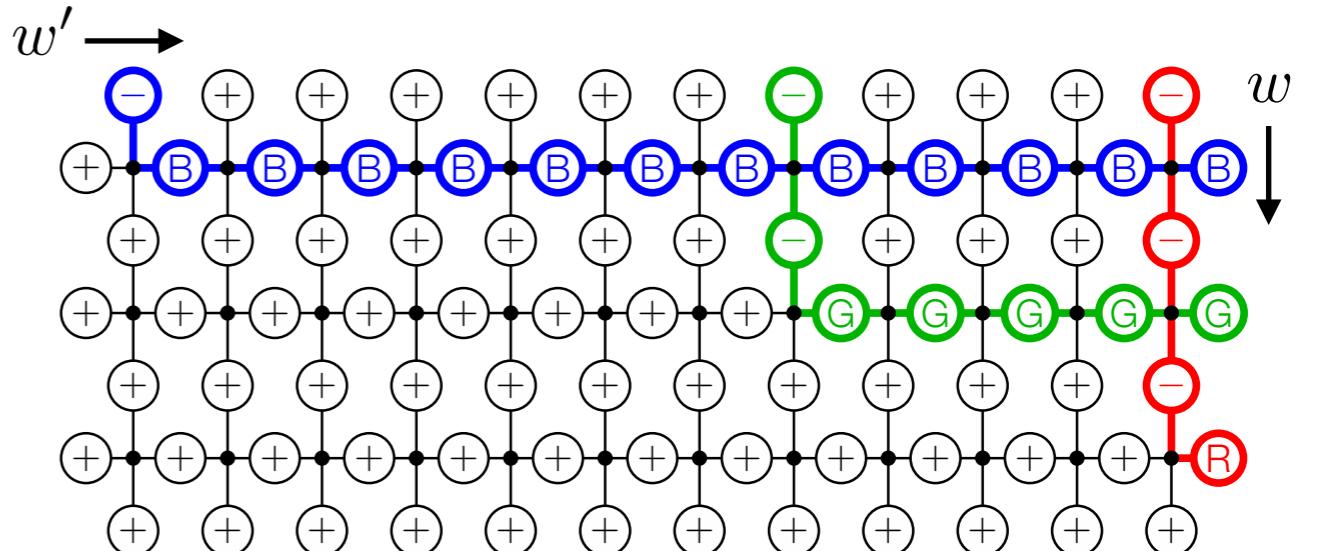
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Base case $w' = w$

Block number $(w' \mu)_j$ with color $(w' P)_j$

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Corollary: [arXiv:1906.04140]

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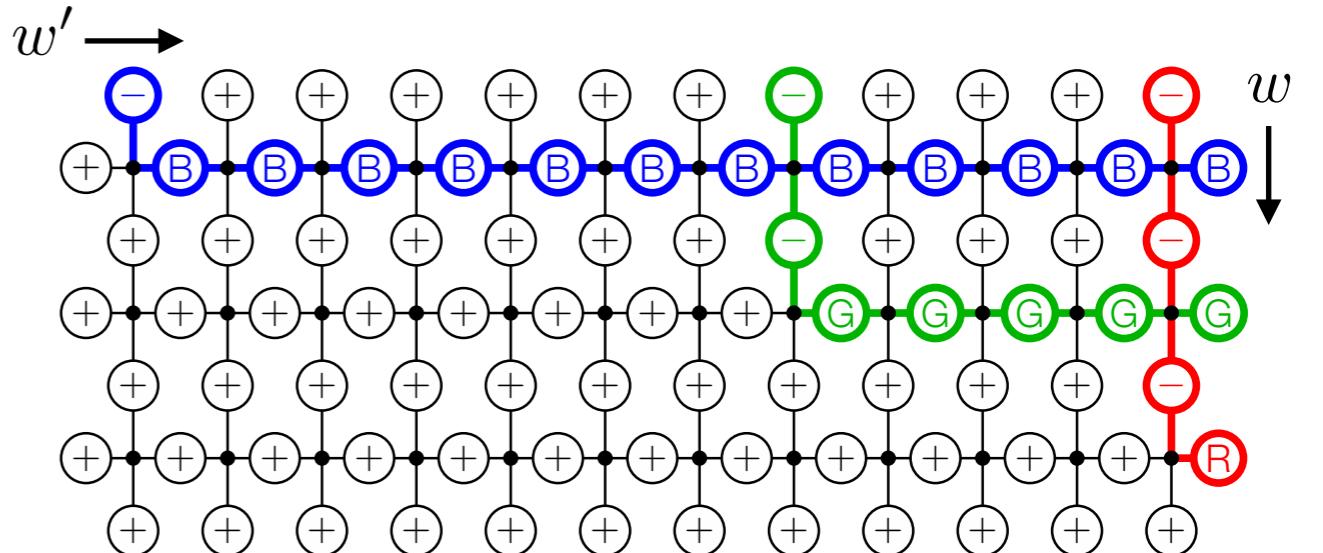
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$$\begin{aligned} w' &\rightarrow s_{i_1} w' \rightarrow \cdots \rightarrow s_{i_k} \cdots s_{i_1} w' = w \\ e_{i_m} &= \begin{cases} 1 & \text{ascent} \\ -1 & \text{descent} \end{cases} \text{ at step } m \end{aligned}$$

Representation theory

Setup

Details in [arXiv:1906.04140]

Setup

F non-archimedean local field, \mathfrak{o} ring of integers

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\mathbb{Q}_p

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Let $\mathbf{G} = \mathrm{GL}_r$, $G = \mathbf{G}(F)$. Standard maximal split torus \mathbf{T} .

Principal series representation

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Modular quasicharacter 

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$$\Phi_w^{\mathbf{z}}(bw'k) = \begin{cases} \delta^{1/2} \tau_{\mathbf{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases} \quad b \in B, w' \in W, k \in J$$

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Base case

$\phi_w(\mathbf{z}; g)$ is determined by its values on $g = \varpi^{-\lambda} w'$ with $\lambda \in \mathbb{Z}^r$ and $w' \in W = S_r$ such that

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$$\xrightarrow[w'\mu = \lambda + \rho]{} \mu \in \mathbb{Z}_{\geq 0}^r$$

$\rho = (r-1, \dots, 1, 0)$

Bijection between data determining the values for Iwahori Whittaker functions and the boundary data for the lattice model.

Base case $w' = w$

$$\phi_{w'}(\mathbf{z}; \varpi^{-\lambda} w') = v^{\ell(w')} \mathbf{z}^\lambda$$

Base case

$\phi_w(\mathbf{z}; g)$ is determined by its values on $g = \varpi^{-\lambda} w'$ with $\lambda \in \mathbb{Z}_{\geq 0}^r$ and $w' \in W = S_r$ such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 & \text{if } (w')^{-1} \alpha_i \in \Delta^+ \\ -1 & \text{if } (w')^{-1} \alpha_i \in \Delta^- \end{cases}$$

positive roots
↓
dominant weight

w' -almost dominant weight λ

$$\begin{array}{c} w'\mu = \lambda + \rho \\ \longleftrightarrow \\ \rho = (r-1, \dots, 1, 0) \end{array} \quad \mu \in \mathbb{Z}_{\geq 0}^r$$

Bijection between data determining the values for Iwahori Whittaker functions and the boundary data for the lattice model.

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Compare: $Z_{\mu, w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu} = v^{\ell(w')} \mathbf{z}^{\lambda + \rho}$

Recursion relations

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Intertwining integral $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z}) \rightarrow I(w\mathbf{z})$

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Intertwining integral $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z}) \rightarrow I(w\mathbf{z})$ $\mathcal{A}_w^{\mathbf{z}} \Phi(g) = \int_{N \cap wN_- w^{-1}} \Phi(w^{-1}ng) dn$

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$$\Omega_{\mathbf{z}} \qquad \mathcal{A}_{s_i}^{\mathbf{z}} \qquad \Phi_w^{\mathbf{z}}$$

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[Casselman 80,
Brubaker–Bump–Licata 15]

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Leads to recursion relations

$$\mathbf{z}^\rho \phi_{s_i w}(\mathbf{z}; g) = \begin{cases} T_i \ \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) > \ell(w), \\ T_i^{-1} \ \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

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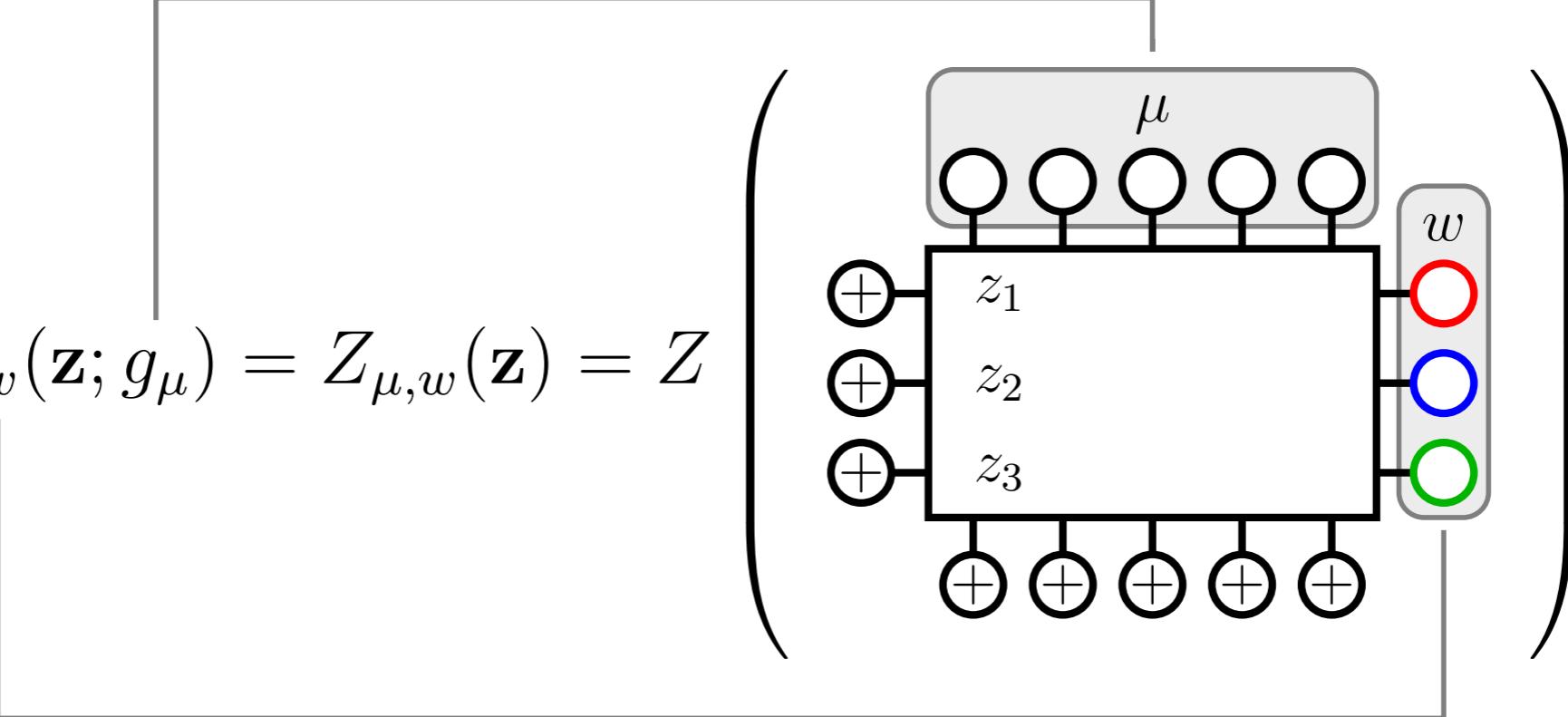
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Same Demazure operators as before

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$

Main theorem

$$\mathbf{z}^\rho \phi_w(\mathbf{z}; g_\mu) = Z_{\mu, w}(\mathbf{z}) = Z$$


$$g_\mu = \varpi^{-\lambda} w' \quad w' \mu = \lambda + \rho$$

Non-distinct colors

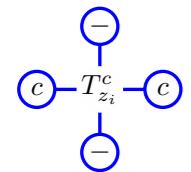
a_1	a_2	b_1	b_2	c_1	c_2
1	$v \quad \text{if } c > d$ $z_i \quad \text{if } c = d$ $1 \quad \text{if } c < d$	$-v$	$z_i \quad \text{if } c = d,$ $1 \quad \text{otherwise}$	$(1 - v)z_i$	1

Non-distinct colors

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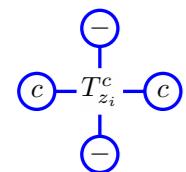
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is necessary for our system to satisfy the Yang–Baxter equations, but because of our **boundary conditions** it does not appear in $Z_{\mu,w}(\mathbf{z})$

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If we use only one color we recover a system equivalent to the [Tokuyama ice model](#) which computes [spherical Whittaker functions](#) $\phi^\circ = \sum_{w \in W} \phi_w$

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Non-distinct colors

Non-trivial feature:

Non-distinct colors

Non-trivial feature: (fused systems)

$$Z \left(\begin{array}{c} \textcircled{+} \textcircled{-} \textcircled{-} \textcircled{+} \textcircled{-} \\ z_1 z_2 z_3 \\ \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \end{array} \right) + Z \left(\begin{array}{c} \textcircled{+} \textcircled{-} \textcircled{-} \textcircled{+} \textcircled{-} \\ z_1 z_2 z_3 \\ \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \end{array} \right) = Z \left(\begin{array}{c} \textcircled{+} \textcircled{-} \textcircled{-} \textcircled{+} \textcircled{-} \\ z_1 z_2 z_3 \\ \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \textcircled{+} \end{array} \right)$$

The diagram illustrates a non-trivial feature of fused systems. It shows three terms being summed. Each term consists of a vertical stack of nodes. The top row contains five nodes: the first four are labeled with $\textcircled{+}$, $\textcircled{-}$, $\textcircled{-}$, and $\textcircled{+}$ respectively, and the fifth node is labeled with a color-coded circle (Red, Blue, or Green). The middle row contains three nodes labeled z_1 , z_2 , and z_3 . The bottom row contains five nodes, all of which are $\textcircled{+}$. The three terms differ in the color of the fifth node: the first has a Red circle, the second has a Blue circle, and the third has a Green circle. The sum of these three terms results in a fourth term where the fifth node is Red, indicating a non-commutative or non-associative property of the fusion operation.

Non-distinct colors

Non-trivial feature: (fused systems)

$$Z \left(\begin{array}{c} \text{+} \\ z_1 \\ z_2 \\ z_3 \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \end{array} \right) + Z \left(\begin{array}{c} \text{+} \\ z_1 \\ z_2 \\ z_3 \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \end{array} \right) = Z \left(\begin{array}{c} \text{+} \\ z_1 \\ z_2 \\ z_3 \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \\ \text{+} \end{array} \right)$$

The diagram illustrates a cancellation of states in fused systems. It shows three terms being summed. Each term consists of a vertical stack of nodes connected by horizontal edges. The top row contains four nodes with signs (+, -, +, -) and four colored outputs (R, B, G, R). The middle row contains three nodes with signs (+, +, +) and three colored outputs (G, B, R). The bottom row contains five nodes with signs (+, +, +, +, +) and five black output nodes. The first two terms cancel each other out, resulting in the third term.

In particular, we have a cancellation of states where red and green vertical edges overlap.

Non-distinct colors

Non-trivial feature: (fused systems)

$$Z \left(\begin{array}{c} \text{Diagram 1: } z_1, z_2, z_3 \\ \text{with vertical edges } +, -, \text{ and labels R, B, G} \end{array} \right) + Z \left(\begin{array}{c} \text{Diagram 2: } z_1, z_2, z_3 \\ \text{with vertical edges } +, -, \text{ and labels G, B, R} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 3: } z_1, z_2, z_3 \\ \text{with vertical edges } +, -, \text{ and labels R, B, R} \end{array} \right)$$

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These models compute parahoric Whittaker functions $\sum_{\tilde{w} \in W_P} \phi_{w\tilde{w}}$

Non-distinct colors

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block-triangular

$$\left(\begin{array}{cccc} o & p & p & p \\ o & o & p & p \\ o & o & o & p \\ o & o & o & o \end{array} \right) \quad \left(\begin{array}{cccc} o & o & p & p \\ o & o & p & p \\ o & o & o & o \\ o & o & o & o \end{array} \right) \quad \left(\begin{array}{ccccc} o & o & o & o \\ o & o & o & o \\ o & o & o & o \\ o & o & o & o \end{array} \right)$$

Iwahori \subseteq parahoric \subseteq spherical

Non-distinct colors

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Same model
different boundary conditions

Iwahori \subseteq parahoric \subseteq spherical

Special polynomials

TABLE 2. Relations between different Whittaker functions and associated special polynomials.

Whittaker function	Special polynomial
Spherical Whittaker function $\sum_{w \in W} \phi_w(\mathbf{z}; \varpi^{-\lambda})$	Schur polynomial $= \prod_{\alpha \in \Delta^+} (1 - v \mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z})$
Li's Whittaker function $\sum_{w \in W} (-v)^{-\ell(w)} \phi_w(\mathbf{z}; \varpi^{-\lambda})$	Hall-Littlewood polynomial $= \mathbf{z}^{-\rho} P_{\lambda+\rho}(\mathbf{z}, v^{-1})$
Iwahori Whittaker function $\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda})$	Non-symmetric Macdonald polynomial $= (-v)^{\ell(w)} \mathbf{z}^{-\rho} w_0 E_{w_0 w(\lambda+\rho)}(\mathbf{z}; \infty, v)$
Parahoric Whittaker function $\psi_1^\mathbf{J}(\mathbf{z}; \varpi^{-\lambda})$	Macdonald polynomial with prescribed symmetry $= \mathbf{z}^{-\rho} S_{\lambda+\rho}^{(\emptyset, \mathbf{J})}(\mathbf{z}; 0, v^{-1}) a_{\lambda+\rho}^{(\emptyset, \mathbf{J})}$

Quantum groups

$\mathrm{GL}_r(F)$

spherical

$U_q(\widehat{\mathfrak{gl}}(1|1))$

[Brubaker–Bump–Friedberg 09]

Iwahori

$U_q(\widehat{\mathfrak{gl}}(r|1))$

[Brubaker–Buciumas–Bump–HG
arXiv:1906.04140]

Quantum groups

$\mathrm{GL}_r(F)$

metaplectic
 n -cover of $\mathrm{GL}_r(F)$

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[Brubaker–Bump–Friedberg 09]

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[Brubaker–Buciumas–Bump 19]

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Metaplectic cover

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

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The irreducible representations of \tilde{T} are n^r -dimensional.

See for example [Savin 04, McNamara 16]

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Basis of Whittaker functionals enumerated by

$$\theta \in \Lambda/n\Lambda \simeq (\mathbb{Z}/n\mathbb{Z})^r$$

 Weight lattice

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| W |

Metaplectic version

$$c_1 < \underbrace{\cdots < c_r}_{x \in \mathbb{Z}/n\mathbb{Z}}$$

Horizontal edges: $\textcolor{blue}{B}, \textcolor{green}{G}, \textcolor{red}{R}, 0, \dots, n - 1$

Metaplectic version

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Vertical edges: $+, -$



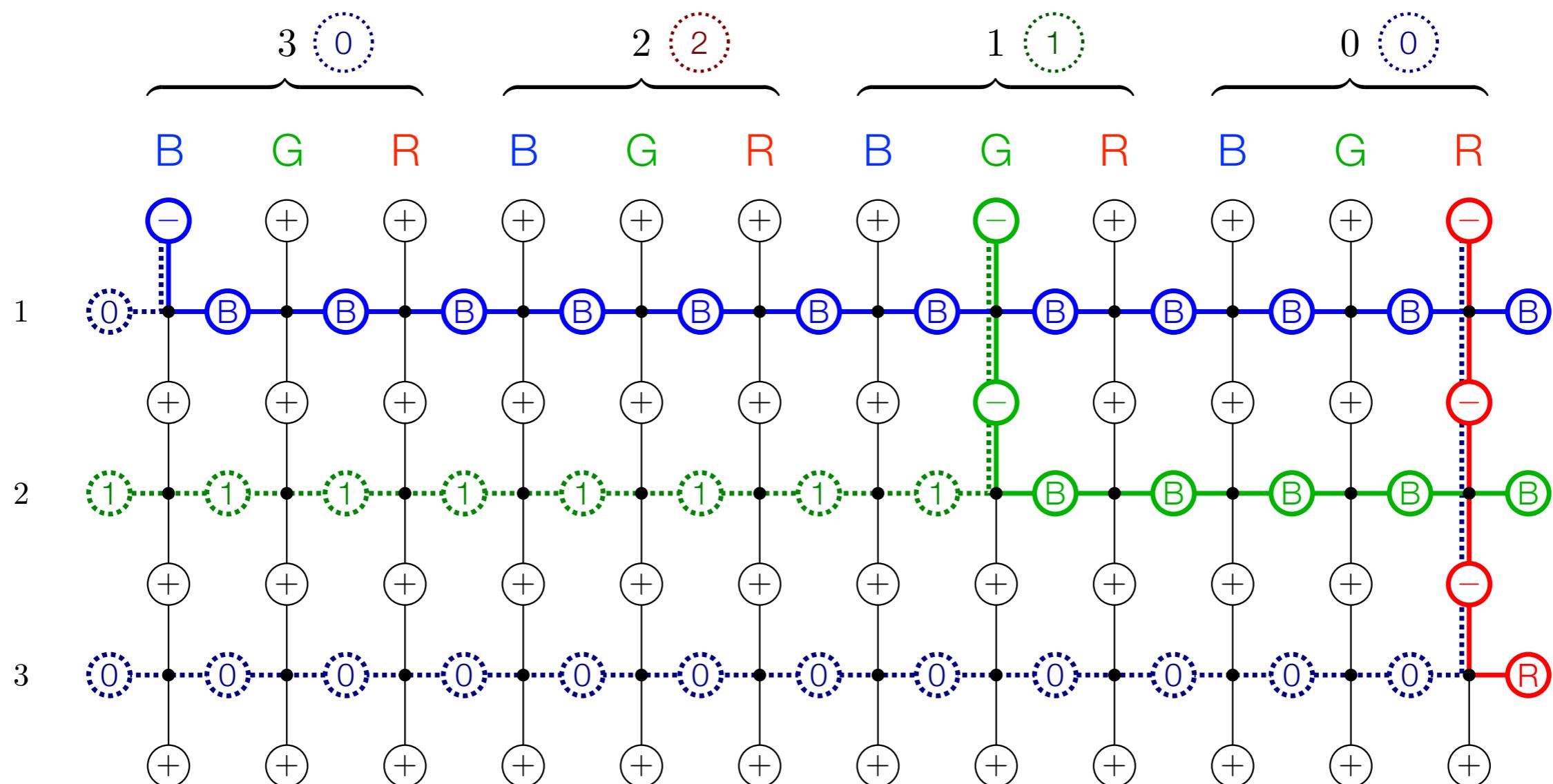
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R-matrix: $U_q(\widehat{\mathfrak{gl}}(n|r)).$

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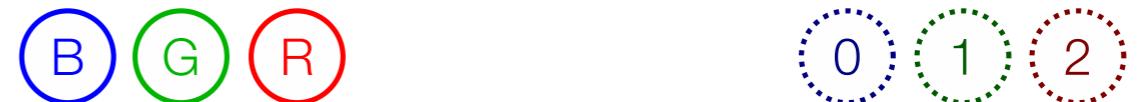
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[Kazhdan–Patterson 84,
McNamara 16 (more generally)]

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Omega_{\mathbf{z}}^{\theta} & \mathcal{A}_{s_i}^{\mathbf{z}} & \Phi_{\mathbf{w}}^{\mathbf{z}} \end{array}$$

Similar to [Lemma 6.3,
Patnaik–Puskás 17]

Metaplectic version

$$c_1 < \dots < c_r \quad x \in \mathbb{Z}/n\mathbb{Z}$$

Horizontal edges: $\textcolor{blue}{B}, \textcolor{green}{G}, \textcolor{red}{R}, 0, \dots, n-1$

color and it's superpartner "scolor"

Vertical edges: $+, -$

R-matrix: $U_q(\widehat{\mathfrak{gl}}(n|r)).$

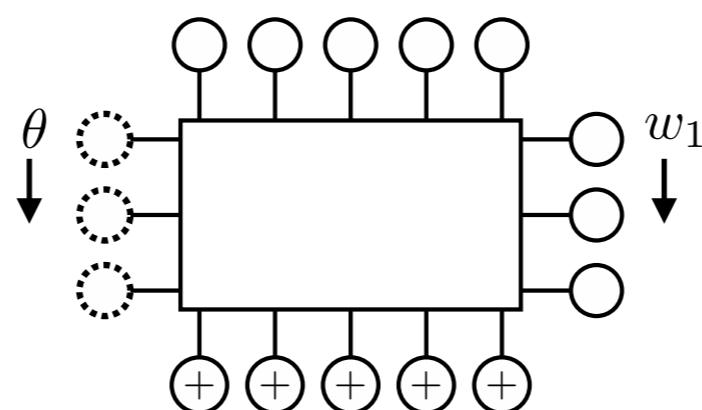


[Kazhdan–Patterson 84,
McNamara 16 (more generally)]

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$\Omega_{\mathbf{z}}^\theta \quad \quad \quad \mathcal{A}_{s_i}^{\mathbf{z}} \quad \quad \quad \Phi_{w_i}^{\mathbf{z}}$$

Similar to [Lemma 6.3,
Patnaik–Puskás 17]



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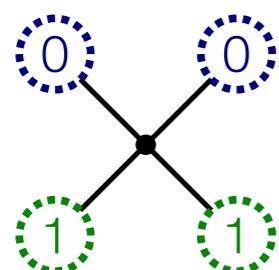
R-matrix: $U_q(\widehat{\mathfrak{gl}}(n|r)).$



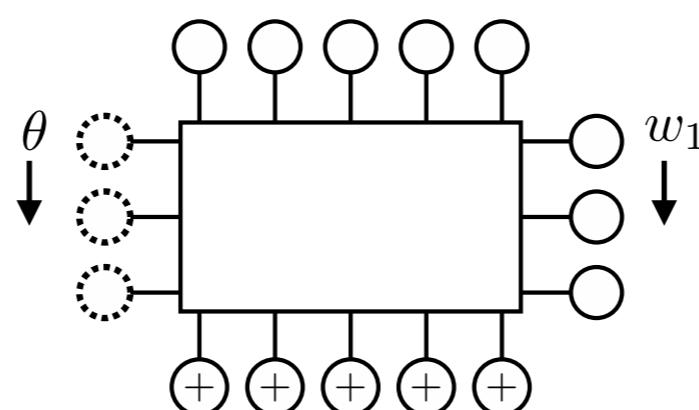
[Kazhdan–Patterson 84,
McNamara 16 (more generally)]

$$\downarrow \Omega_{\mathbf{z}}^\theta \qquad \qquad \mathcal{A}_{s_i}^{\mathbf{z}} \qquad \qquad \downarrow \Phi_{\mathbf{w}}^{\mathbf{z}}$$

Similar to [Lemma 6.3,
Patnaik–Puskás 17]



$U_q(\widehat{\mathfrak{gl}}(n))$



Metaplectic version

$$c_1 < \cdots < c_r \quad x \in \mathbb{Z}/n\mathbb{Z}$$

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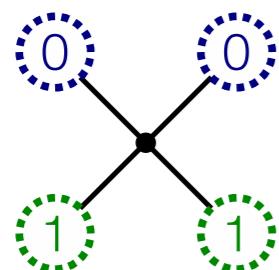
R-matrix: $U_q(\widehat{\mathfrak{gl}}(n|r)).$



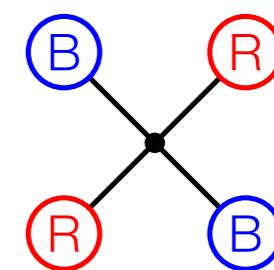
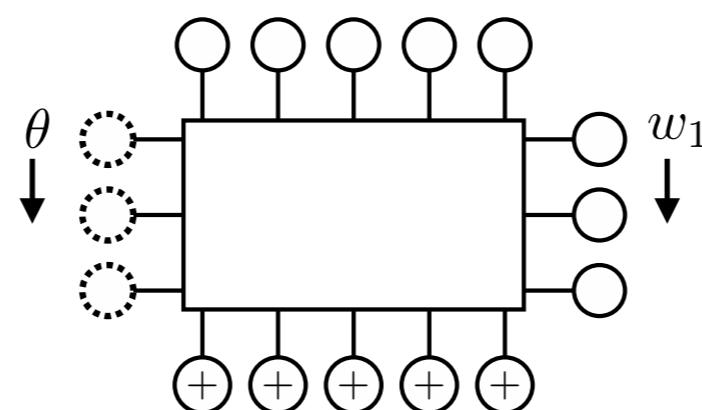
[Kazhdan–Patterson 84,
McNamara 16 (more generally)]

$$\downarrow \Omega_{\mathbf{z}}^\theta \qquad \qquad \downarrow \mathcal{A}_{s_i}^{\mathbf{z}} \qquad \qquad \downarrow \Phi_{\mathbf{w}}^{\mathbf{z}}$$

Similar to [Lemma 6.3,
Patnaik–Puskás 17]



$U_q(\widehat{\mathfrak{gl}}(n))$



$U_q(\widehat{\mathfrak{gl}}(r))$

Thank you!

Slides will be made available at
<https://hgustafsson.se>

