Automorphic forms and their Fourier coefficients

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Slides available at https://hgustafsson.se

Papers

Results based on a series of papers with:

Dmitry Gourevitch (Weizmann Institute, Israel), Axel Kleinschmidt (Albert Einstein Institute, Potsdam), Daniel Persson (Chalmers, Gothenburg) and Siddhartha Sahi (Rutgers, New Jersey)

- A reduction principle for Fourier coefficients of automorphic forms
  Mathematische Zeitschrift volume 300, pages 2679–2717 (2022)

- Fourier coefficients of minimal and next-to-minimal automorphic representations of simply-laced groups
  Canadian Journal of Mathematics, 74(1), 122-169 (2020)

- Eulerianity of Fourier coefficients of automorphic forms
  AMS : Representation Theory 25 (2021) 481-507

Review of automorphic forms and applications in string theory is based on the book:

- Eisenstein series and automorphic representations
  Philipp Fleig, Henrik P. A. Gustafsson, Axel Kleinschmidt, Daniel Persson
  Cambridge University Press, Cambridge Studies in Advanced Mathematics (2018)
  ISBN 9781107189928

Links to preprints available at https://hgustafsson.se

Outline

- From modular forms to automorphic forms
- Eisenstein series
- Fourier coefficients
- Adelic lift
- Automorphic representations
- Results for computing Fourier coefficients
- Applications to string theory

Why study Fourier coefficients of modular/automorphic forms?

- Contain arithmetic information:
  - The number of integer solutions to $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$ is given by the n-th Fourier coefficient of a modular form [Jacobi 1829]
  - Counting rational points of elliptic curves by Fourier coefficient of cusp forms (modularity theorem) [Wiles, Taylor, Diamond, Conrad, Breuil 95–01]
  - Dimensions of representations of finite sporadic groups in a phenomenon called Moonshine [Conway–Norton 79, Borcherds 92]
- Langlands program:
  - Galois representations $\longleftrightarrow$ automorphic representations with equality of L-functions which are related to Fourier coefficients of automorphic forms.
- Physics:
  - Count the number of quantum states of instantons and black holes.

Modular forms

Function $f : \mathcal{H} \to \mathbb{C}$ on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$$\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathcal{H} : \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \qquad \gamma(z) = \frac{az+b}{cz+d}$$

Satisfying:

- $f(\gamma(z)) = (cz+d)^k f(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ for some weight $k \in \mathbb{N}$
- Holomorphic $\quad \overline{\partial} f = 0$
- Polynomial growth

Typical example: holomorphic Eisenstein series

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

From modular forms to automorphic forms

Modular forms $\rightsquigarrow$ Automorphic forms

SL_2 $\qquad\qquad$ G $\qquad$ E_8 E_7 E_6 $\mathrm{SO}_{n,n}$ SL_n GL_n

$$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$$

$$z = g(i) \longleftrightarrow g\mathrm{SO}_2(\mathbb{R})$$

$\mathrm{SO}_2(\mathbb{R}) = \mathrm{Stab}(i)$

$\gamma(z) = \gamma g(i)$

Representatives $g = \begin{pmatrix} \sqrt{y} & x \\ 0 & 1/\sqrt{y} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$ $\qquad g(i) = x + iy$

maximal compact subgroup

$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \longrightarrow G(\mathbb{R})/K_{\mathbb{R}}$$

More generally, a function $\varphi : G(\mathbb{R}) \to \mathbb{C}$

$$f(z) \longrightarrow \varphi(g)$$

From modular forms to automorphic forms

Modular transformation factor with a weight is difficult to generalize

$$f(\gamma(z)) = (cz+d)^k f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})$$

Instead we will require automorphic invariance

$$\varphi(\gamma g) = \varphi(g) \quad \text{for all } \gamma \in G(\mathbb{Z})$$

This is not such a big restriction as it seems if we work with $G(\mathbb{R})$ instead of $G(\mathbb{R})/K_{\mathbb{R}}$.

Modular form $f : \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to \mathbb{C}$ can be lifted to $\mathrm{SL}_2(\mathbb{Z})$-invariant function $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ that transforms under $\mathrm{SO}_2(\mathbb{R})$

$$\varphi(g) := (ci+d)^{-k} f(g(i)) \qquad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$$

Then $\varphi(\gamma g) = \varphi(g) \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$\varphi\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ik\theta}\varphi(g) \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$$

Automorphic forms

Smooth function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ satisfying:

- Automorphic invariant: $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Z})$.
- Annihilated by (some) polynomials in G-invariant differential operators.

  E.g. eigenfunction to Casimir operator or Laplacian. Compare $\overline{\partial} f = 0$
- K-finiteness: $\mathrm{span}\{g \mapsto \varphi(gk) : k \in K\}$ is finite dimensional.

  Often right-invariant under K (called spherical) Maximal compact subgroup.
- Polynomial growth

SL_2 Eisenstein series

Typical example of automorphic form on $\mathcal{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$

Non-holomorphic Eisenstein series

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2 \right\}$$

$$E_s(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{(\mathrm{Im}\, z)^s}{|cz+d|^{2s}} = \sum_{\gamma\in B(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} (\mathrm{Im}\, \gamma(z))^s$$

$s \in \mathbb{C}$

Manifestly $\mathrm{SL}_2(\mathbb{Z})$-invariant

Compare with the holomorphic Eisenstein series, a modular form of weight k

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

$$\bar{\partial} G_k = 0$$

$$\big(\Delta - s(s-1)\big) E_s = 0 \qquad \Delta = 4y(\partial_x^2 + \partial_y^2)$$

E_s invariant under $\mathrm{SL}_2(\mathbb{Z})$ while G_k transforms with weight k.

SL_2 Eisenstein series

Typical example of automorphic form on $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$

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$s \in \mathbb{C}$

Manifestly $\mathrm{SL}_2(\mathbb{Z})$-invariant

Character on B

To be able to generalize to other groups: Let $\chi_s(g) = \mathrm{Im}\big(g(i)\big)^s$.

$G = \mathrm{SL}_2$: $\quad E_s(g) = \displaystyle\sum_{\gamma\in B(\mathbb{Z})\backslash\,\mathrm{SL}_2(\mathbb{Z})} \chi_s(\gamma g)$ $\qquad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} k$ $\qquad z = x + iy$, $\; k \in \mathrm{SO}_2(\mathbb{R})$

$G = \mathrm{SL}_n$: $\quad E_{\vec{s}}(g) = \displaystyle\sum_{\gamma\in B(\mathbb{Z})\backslash\,\mathrm{SL}_n(\mathbb{Z})} \chi_{\vec{s}}(\gamma g)$ $\qquad$ B upper triangular

$\vec{s} \in \mathbb{C}^r$

Fourier coefficients (SL_2)

For $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have that $\gamma(z) = z + 1$.

$\mathrm{SL}_2(\mathbb{Z})$-invariance $\implies E_s(z+1) = E_s(z)$

Periodic in $x = \mathrm{Re}(z)$

Fourier series

$$E_s(x+iy) = \sum_{m\in\mathbb{Z}} a_m(y) e^{2\pi i m x}$$

Note: not requiring holomorphicity which gives a series in $q = e^{2\pi i z}$

$$a_m(y) = \int_{\mathbb{Z}\backslash\mathbb{R}} E_s(x'+iy) e^{-2\pi i m x'} dx'$$

Fourier coefficients (SL_2)

For $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have that $\gamma(z) = z + 1$.

$\mathrm{SL}_2(\mathbb{Z})$-invariance $\implies E_s(z+1) = E_s(z)$ Periodic in $x = \mathrm{Re}(z)$

Fourier series

$$E_s(x+iy) = \sum_{m\in\mathbb{Z}} a_m(y) e^{2\pi i m x} = \sum_{m\in\mathbb{Z}} \int_{\mathbb{Z}\backslash\mathbb{R}} E_s(x'+x+iy) e^{-2\pi i m x'} dx'$$

$$a_m(y) = \int_{\mathbb{Z}\backslash\mathbb{R}} E_s(x'+iy) e^{-2\pi i m x'} dx' \qquad E_s(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{(\mathrm{Im}\, z)^s}{|cz+d|^{2s}}$$

Eigenequation $\Delta - s(s-1))E_s = 0$ and growth condition imply

$$E_s(x+iy) = C_0 y^s + C_0' y^{1-s} + y^{1/2} \sum_{m\neq 0} C_m K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

Key arithmetic information we want to use in applications is hidden in the constants.

Fourier coefficients (H3)

Consider a function φ on the Heisenberg group

$$\mathbb{H}_3(\mathbb{R}) = N(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

What does it mean to be periodic?

$$\varphi\Big(\underbrace{\overbrace{\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}}^{\gamma \in N(\mathbb{Z})} \overbrace{\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}}^{g \in N(\mathbb{R})}}_{\begin{pmatrix} 1 & x+a & z+c+ay \\ & 1 & y+b \\ & & 1 \end{pmatrix}}\Big) = \varphi\Big(\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}\Big)$$

How to Fourier expand it?

Fourier coefficients (H_3)

$$\varphi\Big(\underbrace{\begin{pmatrix}1&a&c\\&1&b\\&&1\end{pmatrix}\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}}_{\begin{pmatrix}1&x+a&z+c+ay\\&1&y+b\\&&1\end{pmatrix}}\Big) = \varphi\Big(\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}\Big)$$

Simply expanding φ in x, y and z using the Fourier modes $e^{2\pi i(mx+ny+kz)}$ does not work!

Invariant under $\begin{cases} x \to x + a \\ y \to y + b \\ z \to z + c + \cancel{ay} \end{cases}$ while φ is not.

Need to Fourier expand with respect to abelian unipotent subgroups.

Can expand in steps along commutator subgroups, but easier to work backwards.

Fourier coefficients (H3)

Start with the (abelian) center: $Z(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$

$$\varphi(g) = \sum_{k\in\mathbb{Z}} \mathcal{F}_k(g) \quad = \sum_{m,n\in\mathbb{Z}} \mathcal{F}_{m,n}(g) + \sum_{k\neq 0} \mathcal{F}_k(g)$$

$$\mathcal{F}_k(g) = \int_{\mathbb{Z}\backslash\mathbb{R}} \varphi(\begin{pmatrix} 1 & & z' \\ & 1 & \\ & & 1 \end{pmatrix} g) e^{-2\pi i k z'}\, dz'$$

depending on x and y
periodic in x and y for $k=0$

$$\mathcal{F}_0(g) = \sum_{m,n\in\mathbb{Z}} \mathcal{F}_{m,n}(g)$$

$$\mathcal{F}_{m,n}(g) = \int_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi(\begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} g) e^{-2\pi i (mx'+ny')}\, dx' dy' dz'$$

Fourier coefficients (H_3)

$$\mathcal{F}_{m,n}(g) = \int_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} g\right) e^{-2\pi i(mx'+ny')}\, dx'dy'dz'$$

Group theory notation

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(ug)\, \psi_{m,n}(u)^{-1}\, du \;\; = \mathcal{W}_{\psi_{m,n}}$$ Whittaker coefficient

Character

$\psi(uu') = \psi(u)\psi(u')$ $\qquad$ $\psi(u) = 1$ for $u \in N(\mathbb{Z})$ $\qquad$ $N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$

Different unipotent subgroups:

u is unipotent if $(1-u)^N = 0$ for some N

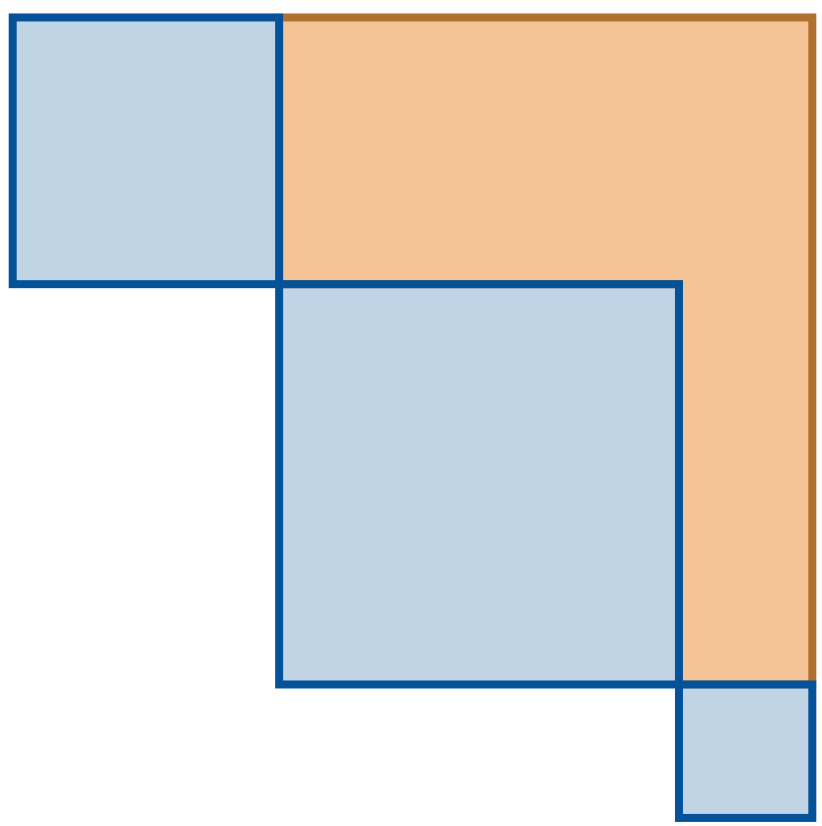
$\log u = -\sum_{k=1}^{N-1} \frac{1}{k}(1-u)^k$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\, \psi(u)^{-1}\, du$$

e.g. $U = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$

Parabolic subgroups

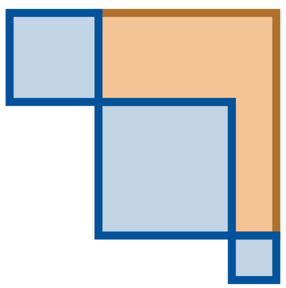
For GL_n and SL_n a standard parabolic subgroup P can be visualized by the following blocks.



Levi Unipotent

$$P = L\,U$$

Parabolic subgroups



$$P = L\,U$$

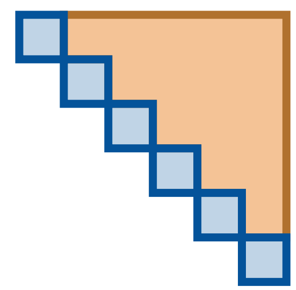
$\mathcal{W}$

Whittaker coefficients

Much is known

Minimal parabolic (Borel) B

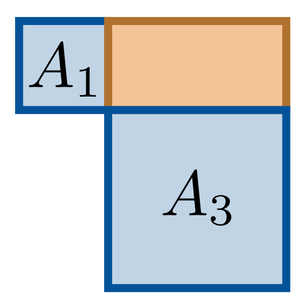
Maximal $U = N$. Small $L =$ torus.



Computation for Eisenstein series reviewed in [Fleig–HG–Kleinschmidt–Persson 18]

Other parabolic P

Smaller U. Larger L.



$\mathcal{F}$

Fourier coefficients difficult to compute directly

Strategy:

including the automorphic form φ iteself

When possible, write the latter $(\mathcal{F})$ in terms of the former $(\mathcal{W})$.

Parabolic subgroups

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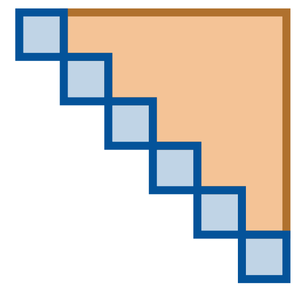

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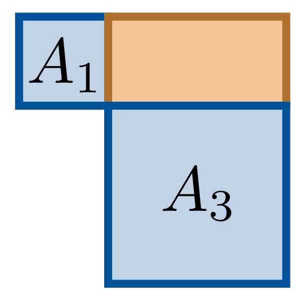

$\mathcal{F}$

Fourier coefficients difficult to compute directly

Strategy:

including the automorphic form φ iteseelf

When possible, write the latter $(\mathcal{F})$ in terms of the former $(\mathcal{W})$.

The other direction is trivial (by integration), but this direction is difficult (requiring successive Fourier expansions)

Strategy example

Parabolic Fourier coefficient in terms of Whittaker coefficients

Let $G = \mathrm{SL}_4$, $U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $\psi^{-1}\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 x_1 + m_2 x_2 + m_3 x_3)}$

$m_1, m_2, m_3 \in \mathbb{Z}$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) e^{2\pi i(x_1 + m_2 x_2 + m_3 x_3)}\, d^3x$$

For now, assume $m_1 = 1$

Goal: write as sums of

$$\mathcal{W}_{m_1,m_4,m_6}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^6} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) e^{2\pi i(m_1 x_1 + m_4 x_4 + m_6 x_6)}\, d^6x$$

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For now, assume $m_1 = 1$

Step 1: Conjugation

By conjugating the integration variable using automorphic invariance one can change the character.

Let $\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_2 & 1 & 0 \\ 0 & -m_3 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z})$. Automorphic invariance gives: $\varphi(g') = \varphi(\gamma_0 g')$.

$$\varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) = \varphi\left(\gamma_0 \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0^{-1} \gamma_0 g\right) = \varphi\left(\begin{pmatrix} 1 & x_1 + m_2 x_2 + m_3 x_3 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right)$$

Strategy example

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) e^{2\pi i (x_1 + m_2 x_2 + m_3 x_3)}\, d^3x$$

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Thus, with a shift in the x_1 integration variable

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right) e^{2\pi i x_1}\, d^3x$$

Can then further expand along next row...

Strategy example

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$$\varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) = \varphi\left(\gamma_0 \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0^{-1} \gamma_0 g\right) = \varphi\left(\begin{pmatrix} 1 & x_1 + m_2 x_2 + m_3 x_3 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right)$$

Thus, with a shift in the x_1 integration variable

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right) e^{2\pi i x_1} \, d^3x$$

Can then further expand along next row...

To do the same with any other $m_1 \neq 0$ would need to conjugate with

$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{m_2}{m_1} & 1 & 0 \\ 0 & -\frac{m_3}{m_1} & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Q})$. But automorphic invariance only for $\mathrm{SL}_4(\mathbb{Z})$.

Tool: Adelic lift

Adeles $\mathbb{A} \supset \mathbb{R}$
(defined in next slide)

Automorphic forms on $G(\mathbb{A})$ —Compute→ $\mathbb{Q}\backslash\mathbb{A}$-Fourier coefficients

Invariant under $G(\mathbb{Q})$

Adelic lift ↑ ↓ Restrict

Automorphic forms on $G(\mathbb{R})$ $\mathbb{Z}\backslash\mathbb{R}$-Fourier coefficients

Invariant under $G(\mathbb{Z})$

Also highlights Eulerian and representation theoretical properties.

For details see [Fleig–HG–Kleinschmidt–Persson 18, §2, §6]

The ring of adeles

$$\mathbb{Q} \xrightarrow[\text{Standard norm } |\cdot|]{\text{Completion of Cauchy sequences}} \mathbb{R} \qquad\qquad \mathbb{Q} \xrightarrow[p\text{-adic norm } |\cdot|_p]{} \mathbb{Q}_p$$

For a prime p and $x \in \mathbb{Q}$ prime factorized as $x = p_1^{k_1} \cdots p_n^{k_n}$ we define the p-adic norm

$$|x|_p = \begin{cases} p_i^{-k_i} & \text{if } p = p_i \text{ for any } i \\ 1 & \text{otherwise} \end{cases}$$

Ring of adeles: $\mathbb{A} = \mathbb{R} \times \prod_{\text{prime } p}' \mathbb{Q}_p$

$\mathbb{Q}$ embeds diagonally in $\mathbb{A}$: $\mathbb{Q} \ni q \mapsto (q; q, q, \ldots) \in \mathbb{A}$.

$\mathbb{Q}$ is discrete in $\mathbb{A}$ and $\mathbb{Q}\backslash\mathbb{A}$ is compact.

Dictionary

Fourier expansion on $\mathbb{Z}\backslash\mathbb{R}$ $\longrightarrow$ Fourier expansion on $\mathbb{Q}\backslash\mathbb{A}$

Additive character on $\mathbb{A}$ trivial on $\mathbb{Q}$

$$f(x) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{Z}\backslash\mathbb{R}} f(x+\xi)e^{2\pi i m\xi}\, d\xi \qquad f(x) = \sum_{m\in\mathbb{Q}} \int_{\mathbb{Q}\backslash\mathbb{A}} f(x+\xi)\mathbf{e}(m\xi)\, d\xi$$

$$G(\mathbb{R}) \longrightarrow G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{\text{prime } p}{}' \, G(\mathbb{Q}_p)$$

$$U(\mathbb{Z})\backslash U(\mathbb{R}) \longrightarrow U(\mathbb{Q})\backslash U(\mathbb{A})$$

$G(\mathbb{Z})$-invariant $\longrightarrow$ $G(\mathbb{Q})$-invariant

For details see [Fleig–HG–Kleinschmidt–Persson 18, §2, §6]

Strategy example (adelic)

Let $G = \mathrm{SL}_4$, $U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $\psi^{-1}\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \mathbf{e}(m_1 x_1 + m_2 x_2 + m_3 x_3)$

$m_1, m_2, m_3 \in \mathbb{Q}$

$m_1 \neq 0$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g \right) \mathbf{e}(m_1 x_1 + m_2 x_2 + m_3 x_3)\, d^3x$$

Using the same steps as before we can show:

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4, m_6 \in \mathbb{Q}} \sum_{\gamma \in \Gamma_4} \mathcal{W}_{\psi_{m_1, m_4, m_6}}[\varphi](\overbrace{\gamma\gamma_0}^{\in \mathrm{SL}_4(\mathbb{Q})} g)$$

Maximal parabolic Fourier coefficient

Translated Whittaker coefficients

Compare with [Piatetski-Shapiro 79, Shalika 74] for cusp form.

Simplifies for small automorphic representations.

Automorphic representations

Let $\mathcal{A}$ denote the space of automorphic forms on $G(\mathbb{A})$.

An automorphic representation is an irreducible component of $\mathcal{A}$ under a specific “$G(\mathbb{A})$-action”.

Can characterize automorphic representations using nilpotent orbits

[Bump 09, Fleig-HG-Kleinschmidt-Persson 18, §5]

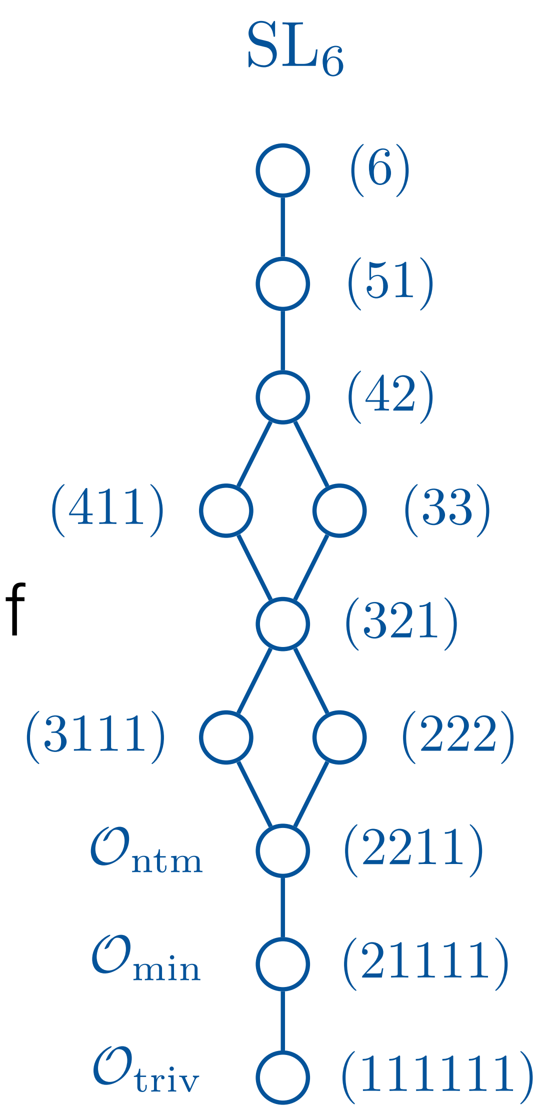
Nilpotent orbits

For $X \in \mathfrak{g}(\mathbb{Q})$ a nilpotent element we define the nilpotent orbit
$\mathcal{O} = \{gXg^{-1} : g \in G(\mathbb{C})\}$

For classical groups (SL_n, SO_n, Sp_n) these orbits are parametrized by partitions of n.

Nilpotent orbits have a partial ordering which, for classical groups, is equivalent to the partial ordering of partitions.

$$(\lambda_1, \ldots, \lambda_n) \leqslant (\mu_1, \ldots, \mu_n) \iff \sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i \text{ for } 1 \leqslant k \leqslant n$$



[Collingwood–McGovern 17]

Nilpotent orbits

The connection between nilpotent orbits and automorphic representations goes via Fourier coefficients

Character ψ on $U(\mathbb{A})$ $\longrightarrow$ Nilpotent element $y \in \mathfrak{g}(\mathbb{Q})$

$\psi_y(u) = \mathbf{e}(\langle y, \log u \rangle)$ $\langle \cdot\,, \cdot \rangle$ Killing form

$$\mathcal{F}_{U,\psi_y}[\varphi](g) = \mathcal{F}_{\gamma U \gamma^{-1}, \psi_{\gamma y \gamma^{-1}}}[\varphi](\gamma g) \qquad \gamma \in G(\mathbb{Q})$$

$$\mathcal{F}_{U,\psi_y}[\varphi] \equiv 0 \iff \mathcal{F}_{\gamma U \gamma^{-1}, \psi_{\gamma y \gamma^{-1}}}[\varphi] \equiv 0$$

Automorphic representations and nilpotent orbits

An automorphic representation π is characterized by a set of nilpotent orbits $\mathrm{WF}(\pi)$ called its wave-front set.

If $\mathcal{O}_y \notin \mathrm{WF}(\pi)$ then $\mathcal{F}_{U,\psi_y}[\varphi] \equiv 0$ for $\varphi \in \pi$ [Gomez–Gourevitch–Sahi 17]

(Similar local statements by Matumoto and Mœglin–Waldspurger)

Minimal automorphic representation:

$\mathrm{WF}(\pi_{\mathrm{min}})$ contains $\mathcal{O}_{\mathrm{min}}$ but no larger orbit.

Next-to-minimal automorphic representation:

$\mathrm{WF}(\pi_{\mathrm{ntm}})$ contains $\mathcal{O}_{\mathrm{ntm}}$ but no larger orbit.

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Small automorphic representations

$\downarrow$

Defining property:

Few non-vanishing Fourier coefficients

Small automorphic representations

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Few non-vanishing Fourier coefficients

SL_4 example: Whittaker coefficients specified by character

$$\psi_{m_1,m_2,m_3}\left(\begin{pmatrix} 1 & x_1 & * & * \\ 0 & 1 & x_2 & * \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \mathbf{e}(m_1x_1 + m_2x_2 + m_3x_3)$$

Maximally degenerate

Minimal representation: only $\mathcal{W}$ with characters $\psi_{m_1,0,0}$, $\psi_{0,m_2,0}$, $\psi_{0,0,m_3}$ survive.

Next-to-minimal representation: also $\psi_{m_1,0,m_3}$ survive.

Realizations

(used for example in string theory applications)

$$E_{\vec{s}}(g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\vec{s}}(\gamma g)$$

$$\vec{s} \in \mathbb{C}^r \longleftrightarrow \lambda = 2s_1\Lambda_1 + \ldots 2s_r\Lambda_r$$

$$\chi_{\vec{s}}(g) = \chi_{\vec{s}}(nak) = a^\lambda$$

For SL_n, $(n > 3)$:

$E_{(s,0,\ldots,0)}(g)$ is in a minimal automorphic representation.

$E_{(0,s,\ldots,0)}(g)$ is in a next-to-minimal automorphic representation.

For E_6, E_7, E_8:

$E_{(3/2,0,\ldots,0)}(g)$ is in a minimal automorphic representation.

$E_{(5/2,0,\ldots,0)}(g)$ is in a next-to-minimal automorphic representation.

[Fleig–HG–Kleinschmidt–Persson 18, Table 6.2]

Reduction principle

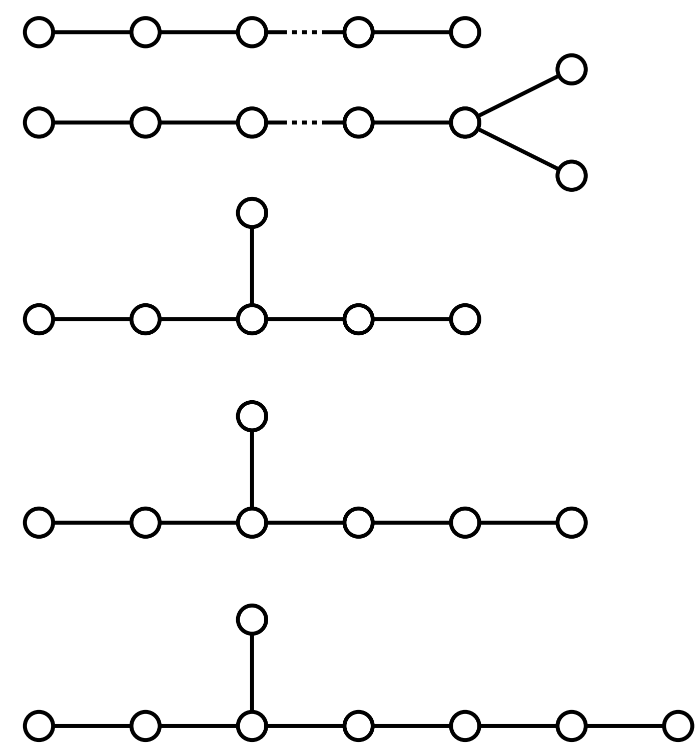
When possible, write difficult Fourier coefficient in terms of Whittaker coefficients.

$\mathcal{F}$ $\mathcal{W}$

Theorem I [Gourevitch–HG–Kleinschmidt–Persson–Sahi 22].

We show that an automorphic form in a minimal or next-to-minimal automorphic representation of a simply-laced group, as well as their Fourier coefficients, can be expressed in terms of Whittaker coefficients and provide an algorithm.

In the general case*, we give the "largest" coefficients that would replace Whittaker coefficients in the above statement: so called Levi-distinguished coefficients.



A precise statement of the algorithm is made using Whittaker pairs which are elements of the Lie algebra describing the Fourier coefficient's unipotent subgroup and character

*Any number field, any central extension of reductive group, any representation.

Explicit formulas

Theorem II [Gourevitch–HG–Kleinschmidt–Persson–Sahi 20].

Formulas for expressing maximal parabolic Fourier coefficients, and φ itself, in terms of Whittaker coefficients for minimal and next-to-minimal representations of simply-laced groups.

Example $G = \mathrm{SO}_{5,5}$:

φ next-to-minimal, U_{α_1} analogous to first row

Character $\psi = \psi_y$ with $y \in \mathfrak{g}^{\times}_{-\alpha_1}(\mathbb{Q})$ in a minimal orbit.

$$\mathcal{F}_{U_{\alpha_1},\psi}[\varphi](g) = \underbrace{\mathcal{W}_{\psi}[\varphi](g)}_{\text{Maximally degenerate}} + \sum_{i=3}^{5} \sum_{\gamma \in \Gamma_i} \sum_{y' \in \mathfrak{g}^{\times}_{-\alpha_i}(\mathbb{Q})} \mathcal{W}_{\psi_{y+y'}}[\varphi](\gamma g)$$

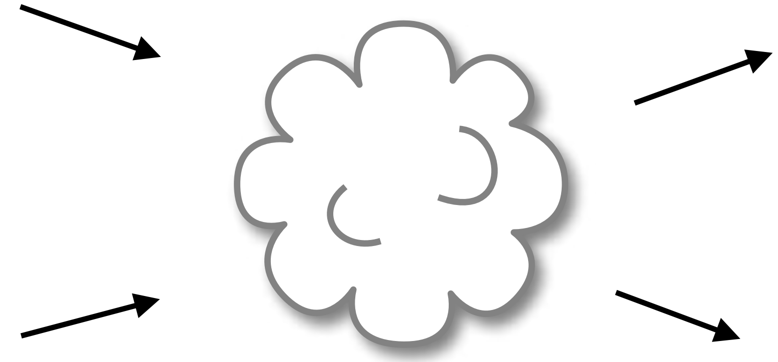
Maximally degenerate

Certain coset representatives in $G(\mathbb{Q})$ specified in paper.

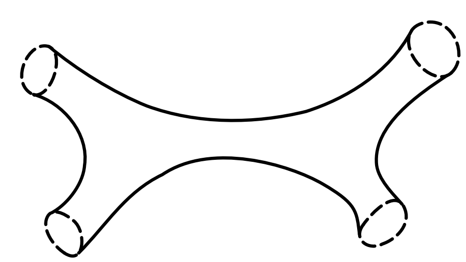
String theory applications

The interaction (scattering) of two gravitons is described by a probability amplitude depending on their incoming and outgoing momenta.

The graviton is a particle that mediates gravity similar to how a photon mediates electromagnetism.



In string theory this is pictured as a string sweeping out a Riemann surface over time



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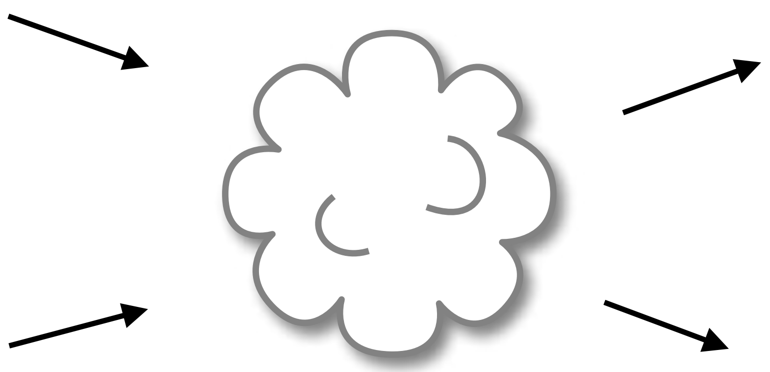

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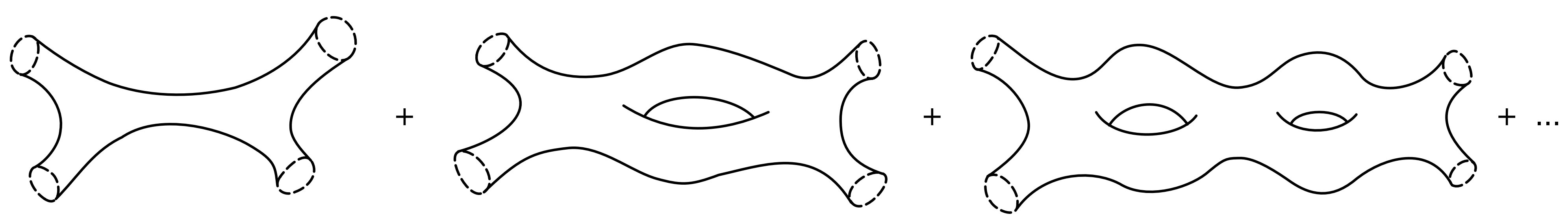

To obtain the full amplitude one has to integrate over all geometries.

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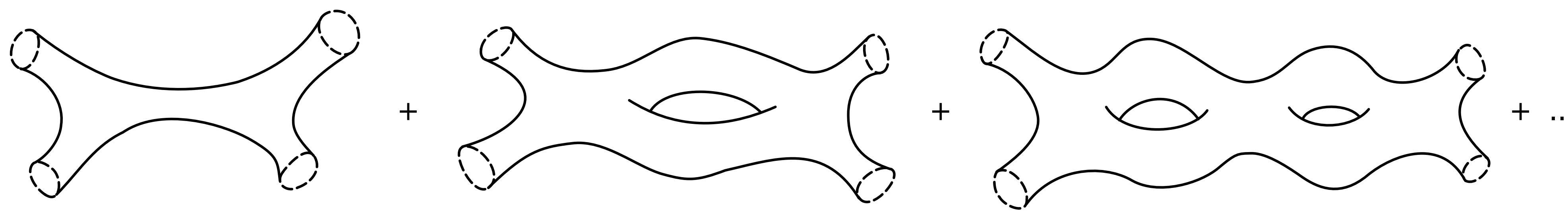
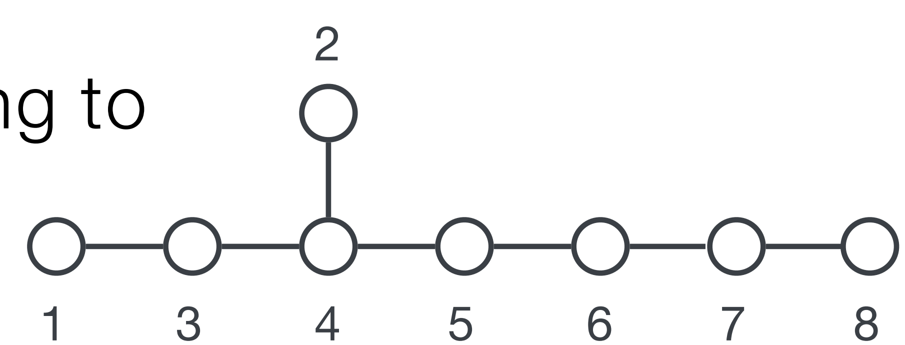


To obtain the full amplitude one has to integrate over all geometries.

In string theory space-time X is a 10-dimensional manifold, but to obtain physics in D dimensions one can for example let $X = \mathbb{R}^D \times T^{10-D}$ where T^d is a d-dimensional torus.

Such a theory is specified by parameters in $G(\mathbb{R})/K_{\mathbb{R}}$ where $G = E_{d+1}$ obtained by restricting to the $d + 1$ first nodes of the Dynkin diagram:



In particular, for $D = 10$, $G = SL_2$

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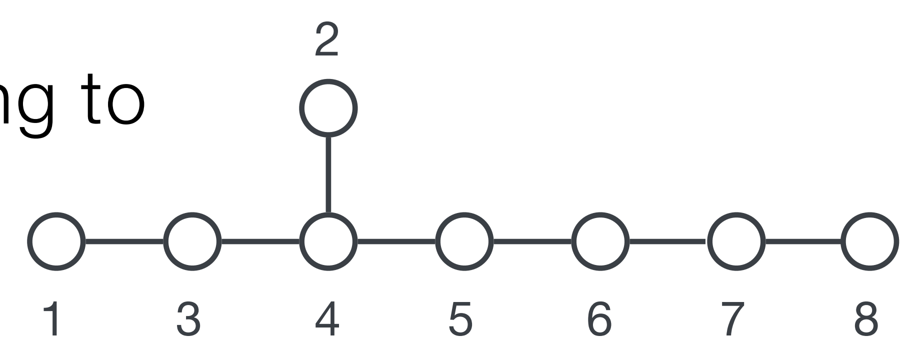

The scattering amplitude can be Taylor expanded with respect to the inverse string tension where higher order terms correspond to quantum corrections and the coefficients are functions $G(\mathbb{R})/K_\mathbb{R} \to \mathbb{C}$.

U-duality $\implies$ $G(\mathbb{Z})$-invariance

automorphic forms

supersymmetry $\implies$ small automorphic representations

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U-duality $\implies$ $G(\mathbb{Z})$-invariance automorphic forms

supersymmetry $\implies$ small automorphic representations

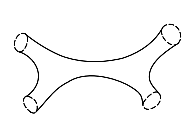
In fact, the first and second order quantum corrections are Eisenstein series in minimal and next-to-minimal representations.

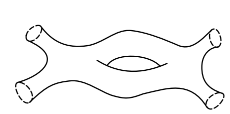
Their Fourier coefficients correspond to different kinds of contributions to the scattering amplitude.

$$E_s(x+iy) = C_0 y^s + C_0' y^{1-s} + y^{1/2} \sum_{m \neq 0} C_m K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$D = 10$

$s = 3/2$





instanton contributions

String theory applications

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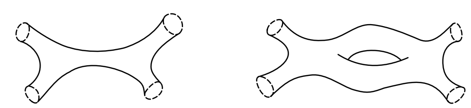

instanton contributions

only these can be computed independently from string perturbation theory

only known approximately from string theory but is here determined by automorphic forms

$$C_m \propto \sum_{d|m} d^2 \qquad m \neq 0$$

sums over the number of quantum states for an instanton of charge *m*

[Green–Gutperle 97]

The arithmetic (p-adic) part of the Fourier coefficients contain information about quantum states for instantons (and black holes).

Thank you!

Slides will be made available at

https://hgustafsson.se