Lattice models and special polynomials in algebr. comb.

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Discrete Seminar – Umeå University

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Slides available at https://hgustafsson.se

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Research areas

Number theory ⟷ Representation theory

Theoretical physics | Quantum algebra | Combinatorics

NT	RT	TP	QA	C	
●	●	●			Automorphic forms and representations
●	●				Langlands program
●	●	●			Whittaker functions for local representations
●	●		●	●	Solvable lattice models
	●		●		Quantum groups
	●			●	(Non)-symmetric special polynomials
	●	●	●		Vertex operator algebras

this talk

https://hgustafsson.se

Papers

Joint work with Ben Brubaker, Valentin Buciumas and Daniel Bump

Vertex operators, solvable lattice models and metaplectic Whittaker functions
Communications in Mathematical Physics 380 (Dec, 2020), 535–579

Colored five-vertex models and Demazure atoms
Journal of Combinatorial Theory, Series A 178 (Feb, 2021)

Colored vertex models and Iwahori Whittaker functions
arXiv:1906.04140

Metaplectic Iwahori Whittaker functions and supersymmetric lattice models
arXiv:2012.15778

Iwahori-metaplectic duality
arXiv:2112.14670

Schur polynomials

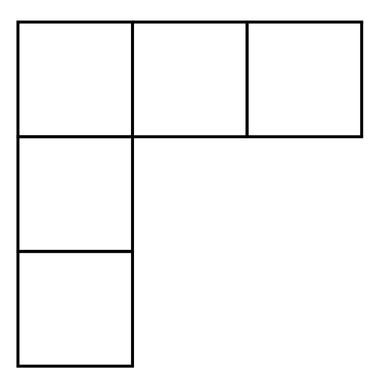
Let λ be a partition of r padded with zeroes to length r. We define the Schur polynomial $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by

$$s_\lambda(\mathbf{z}) = \frac{\det(z_i^{(\lambda+\rho)_j})_{ij}}{\det(z_i^{\rho_j})_{ij}}$$

where $\mathbf{z} = (z_1, \ldots, z_r)$ and $\rho = (r-1, r-2, \ldots, 1, 0)$.

Combinatorial description using Semi-Standard Young Tableaux of shape λ

$\lambda = (3, 1, 1)$



Young diagram

Schur polynomials

Combinatorial description using Semi-Standard Young Tableaux of shape λ

$$\lambda = (3,1,1) \qquad \begin{array}{|c|c|c|} \hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \text{Young diagram}$$

$$\text{Young tableau} \qquad T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{wt}(T) = (2,2,0,0,1)$$

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

Schur polynomials

Combinatorial description using Semi-Standard Young Tableaux of shape λ

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

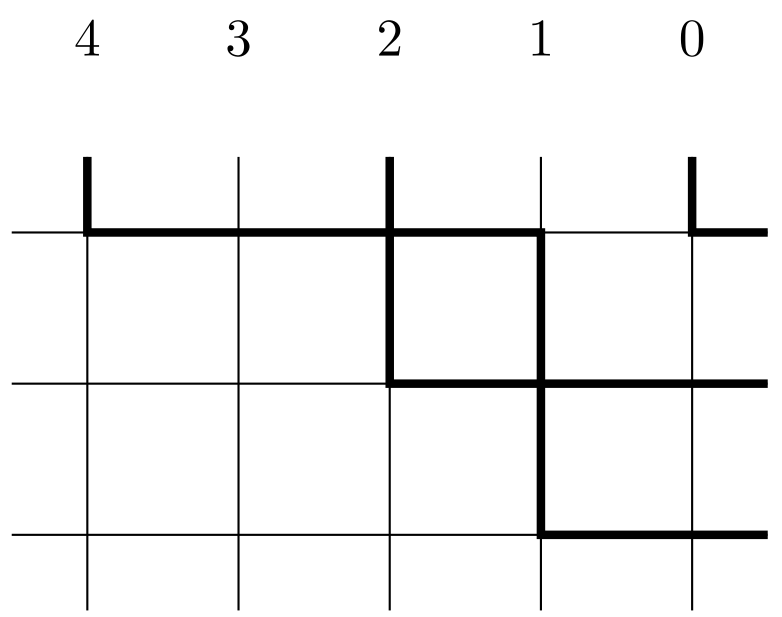
Basis of symmetric polynomials of degree $r = \sum_i \mathrm{wt}(T)_i$

$$s_\lambda(1) = |\mathrm{SSYT}(\lambda)|$$

Lattice paths

SSYT $\longleftrightarrow$ south-east moving lattice paths
(certain)

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$



r rows

Let $\lambda^{(i)}(T)$ be the shape of T after removing labels larger than i

$\lambda^{(3)}(T) = (2,1,0)$ $\quad\lambda^{(2)}(T) = \text{shape}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right) = (1,1)$ $\quad\lambda^{(1)}(T) = \text{shape}\left(\boxed{1}\right) = (1)$

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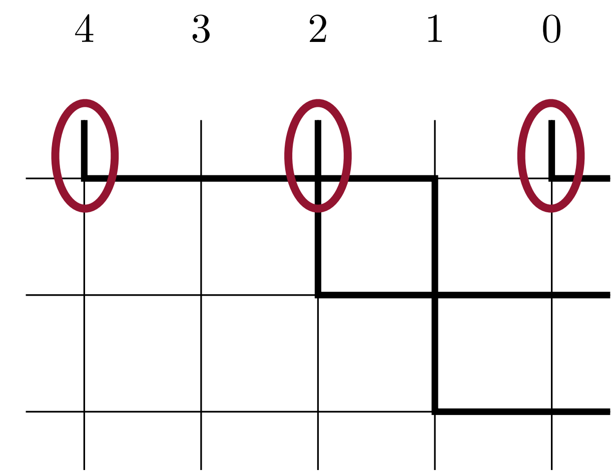
Lattice paths



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We add $\rho^{(r)} = (r-1, r-2, \ldots, 1, 0)$ to each shape to get something called a Gelfand-Tsetlin pattern:

$$\left\{ \begin{array}{c} \lambda^{(3)}(T) + \rho^{(3)} \\ \lambda^{(2)}(T) + \rho^{(2)} \\ \lambda^{(1)}(T) + \rho^{(1)} \end{array} \right\} = \left\{ \begin{array}{ccccc} 4 & & 2 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \right\}$$

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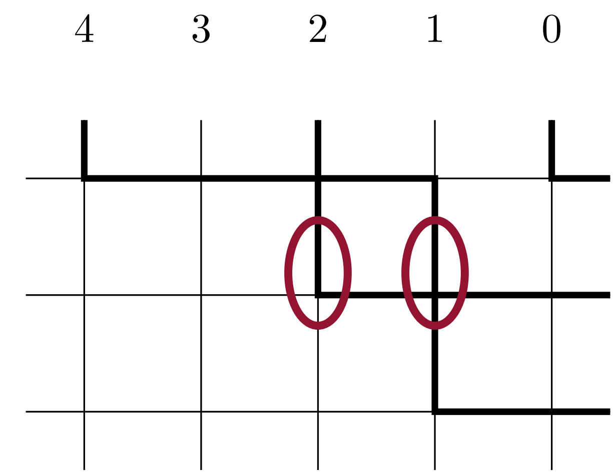
Lattice paths



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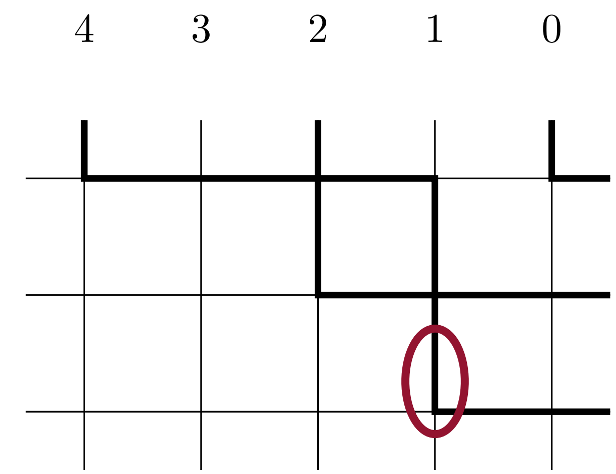
Lattice paths

state $\mathfrak{s}$ $\longrightarrow$



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Lattice paths

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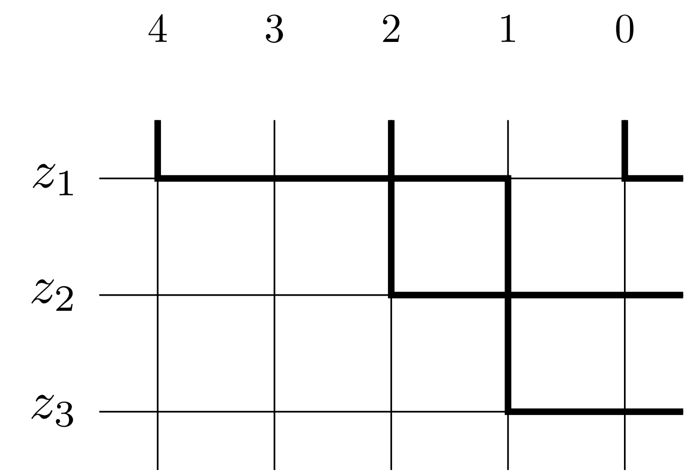
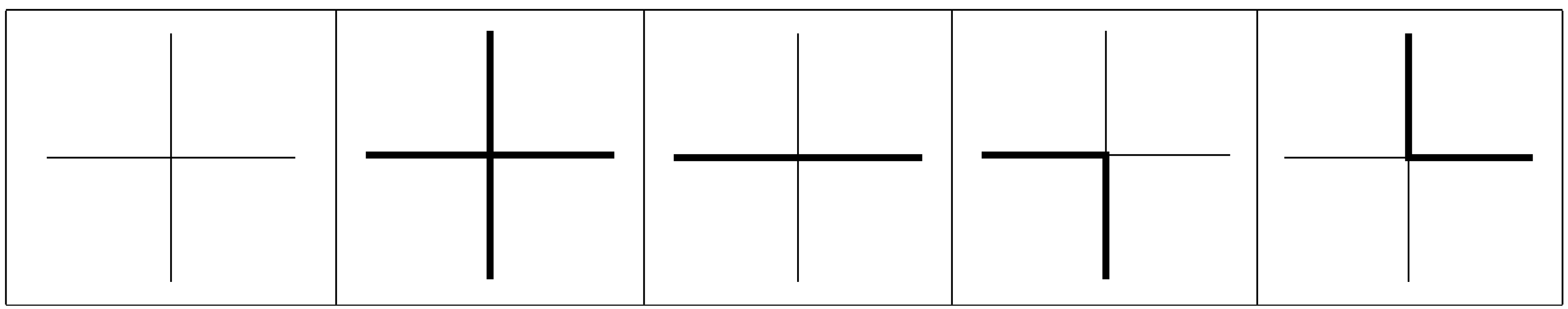


Five different vertex configurations:



$\mathrm{SSYT} \longleftrightarrow$ lattice paths using these vertex configurations

shape λ filled in top boundary edges at columns $\lambda + \rho$

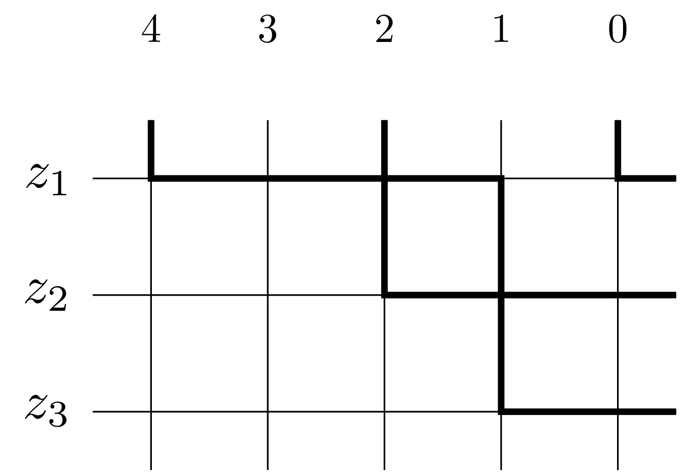
Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$\mathrm{wt}(T)$ counts the number of filled in left-edges in each row

Lattice paths

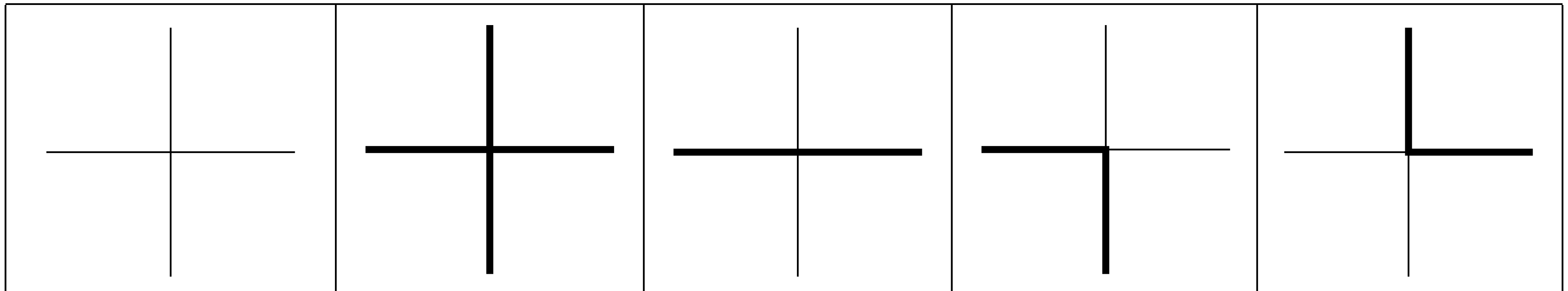
$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

state $\mathfrak{s}$



$\beta(\mathfrak{s}) = z_1^3 z_2^2 z_3$

Five different vertex configurations:



1	z_i	z_i	z_i	1

Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$w_0(z_1, z_2, \dots, z_r) = (z_r, \dots, z_2, z_1)$

Boltzmann weight $\beta(\mathfrak{s}) := \prod_{\text{vertex}}$ vertex weights $= \mathbf{z}^\rho \cdot (w_0 \mathbf{z})^{\mathrm{wt}(T)}$

Partition function $Z(\lambda, \mathbf{z}) := \sum_{\mathfrak{s} \text{ with top } \lambda+\rho} \beta(\mathfrak{s}) = \mathbf{z}^\rho s_\lambda(w_0 \mathbf{z}) = \mathbf{z}^\rho s_\lambda(\mathbf{z})$

Why lattice models?

- Easy to write programs to compute partition functions (symbolically)
- Powerful toolbox statistical mechanics to manipulate lattice models
- New ways to prove identities (e.g. Cauchy identities, functional eq's)
- A bridge for building new connections between widely different mathematical objects

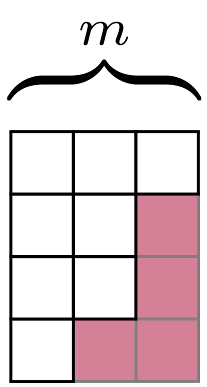
Cauchy identity

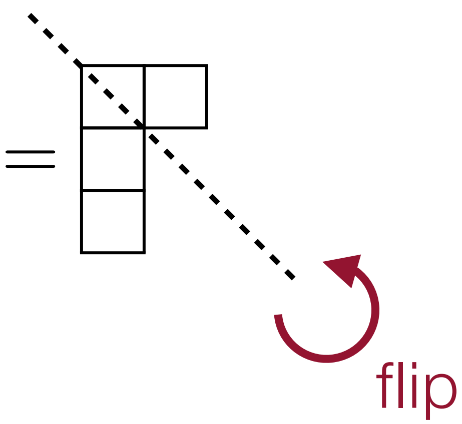
$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\hat{\lambda}'}(\mathbf{y}) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j)$$

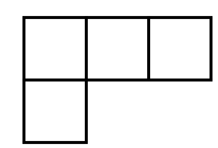
[Macdonald 1992 (0.11')]

$\hat{\lambda}$ = complement of λ

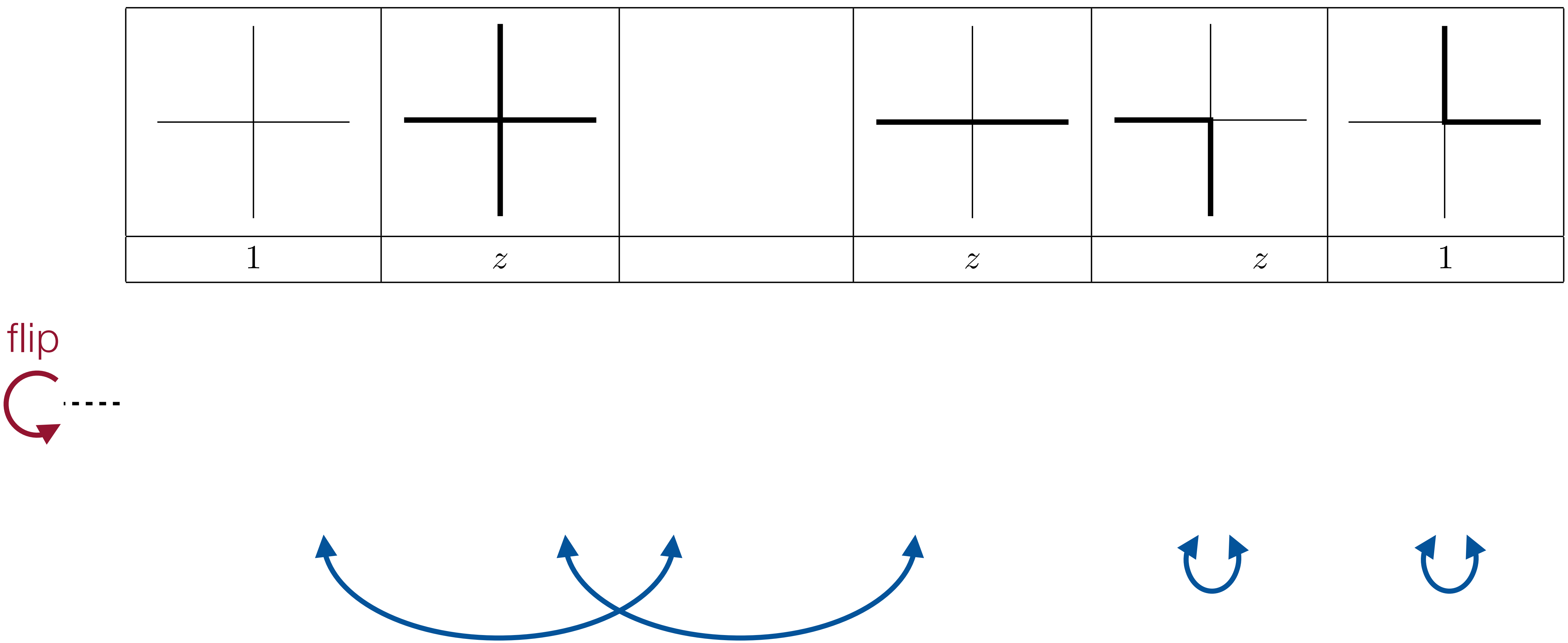
μ' = conjugate of μ

$\lambda =$ 

$\hat{\lambda} =$ 

$\hat{\lambda}' =$ 

Cauchy identity



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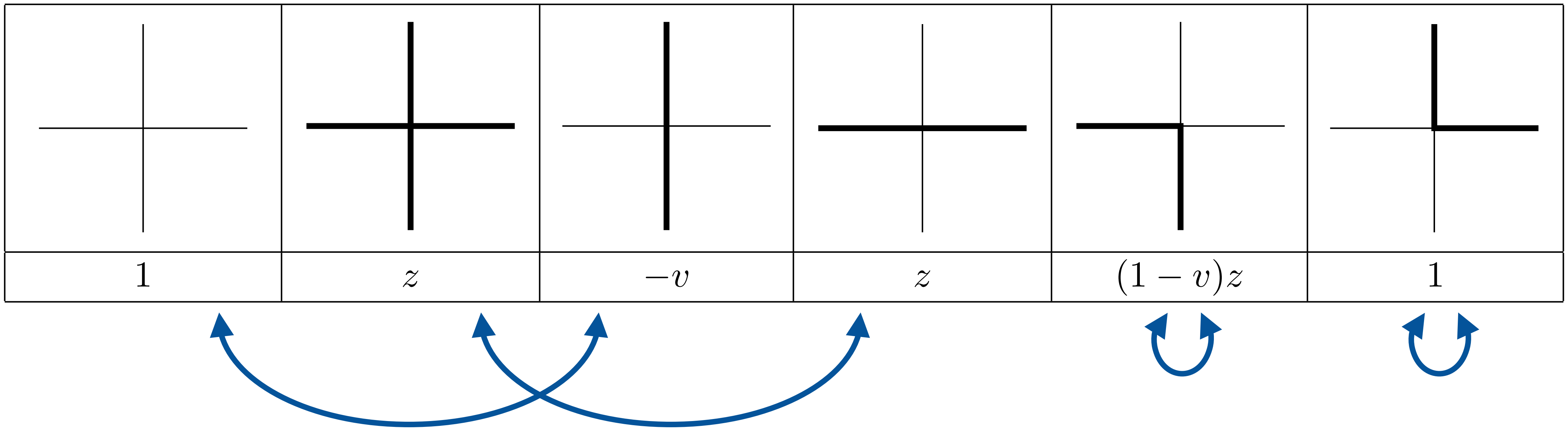


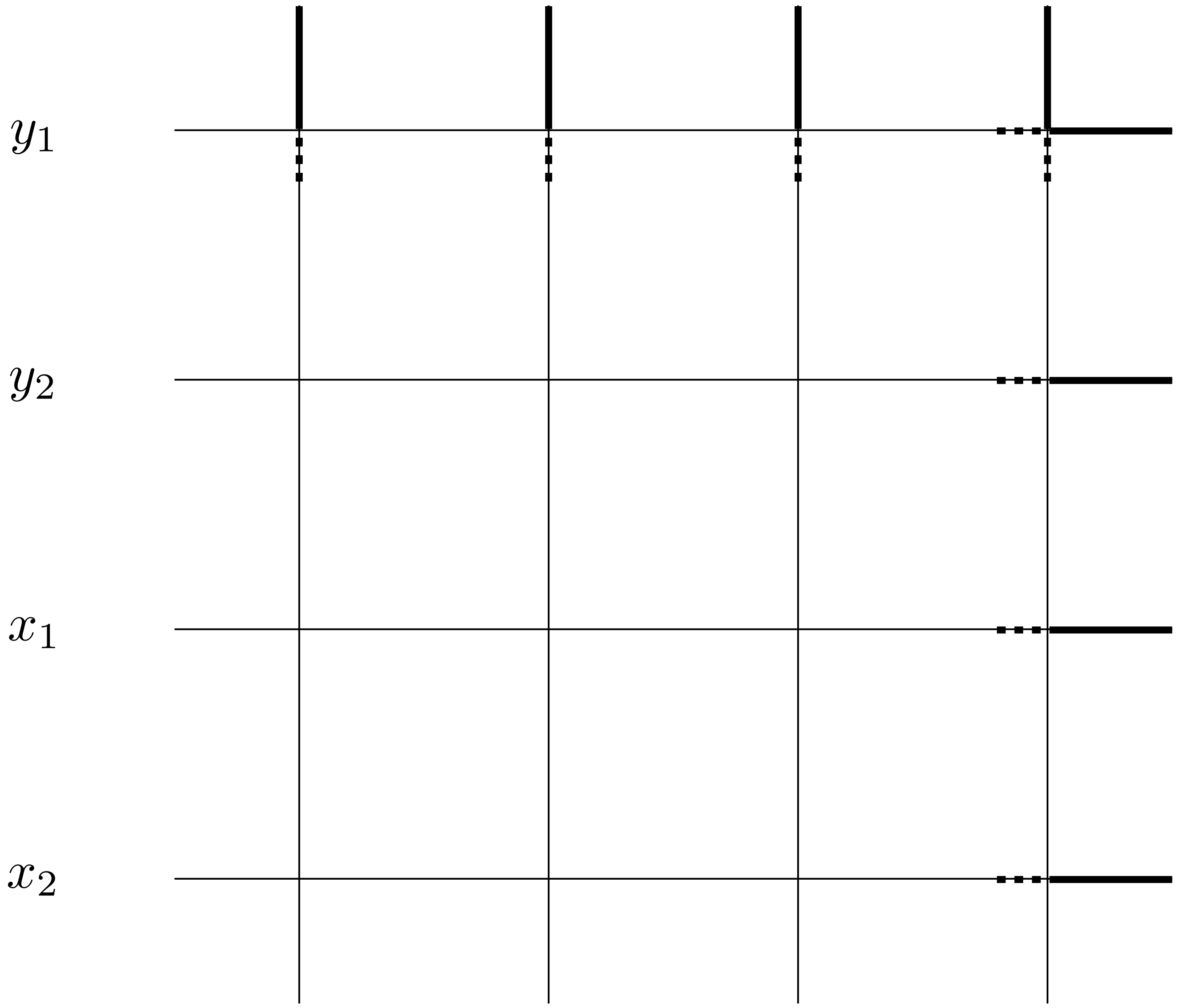
These new weights introduce a slight deformation of the partition function

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^{\rho}\prod_{i<j}\left(1 - v\frac{z_j}{z_i}\right)s_\lambda(\mathbf{z})$$

[Brubaker–Bump–Friedberg 2009]

If $v = -1$ then a flip preserves the Boltzmann weight of the state.

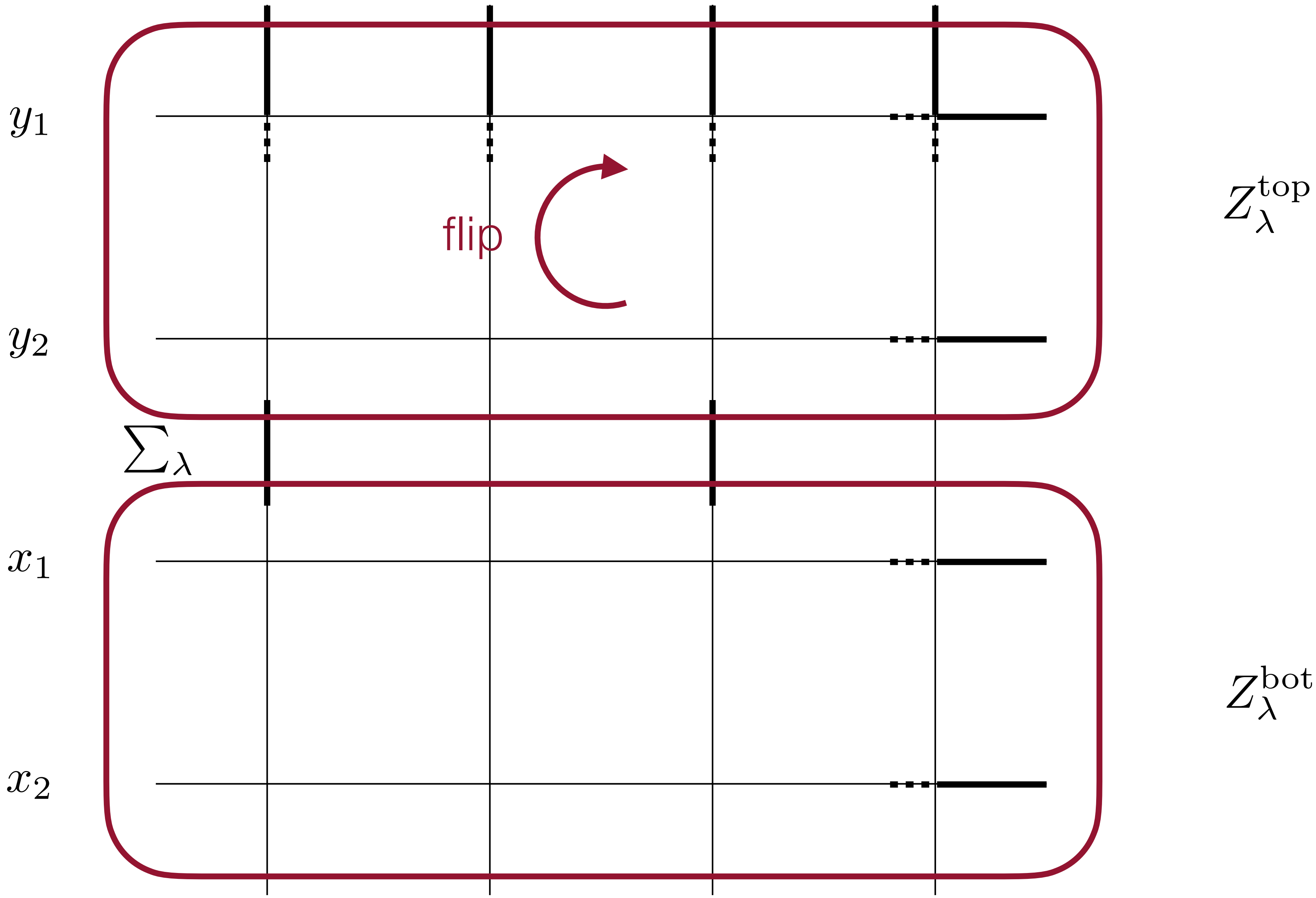
Cauchy identity



$$Z = \prod_{i<j}(x_i + x_j) \cdot \prod_{i<j}(y_i + y_j) \cdot \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + y_j)$$

[Bump–McNamara–Nakasuji 2014]

Cauchy identity



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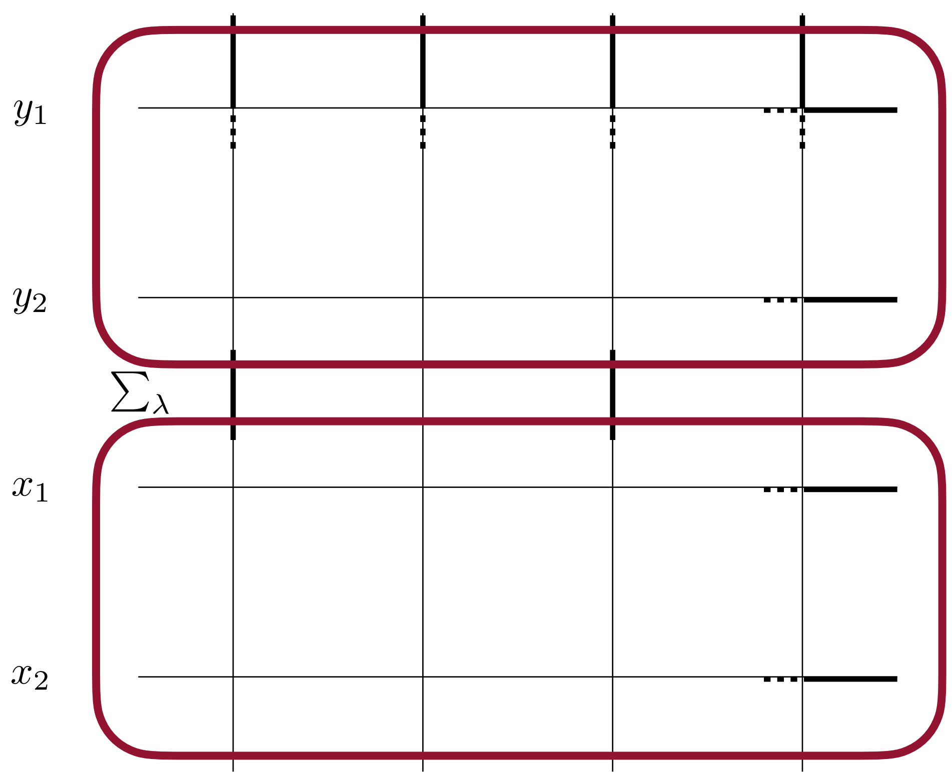


$$Z_\lambda^{\text{top}} = \cancel{\prod_{i<j}(y_i + y_j)}\, s_{\hat{\lambda}'}(\mathbf{y})$$

$$Z_\lambda^{\text{bot}} = \cancel{\prod_{i<j}(x_i + x_j)}\, s_{\lambda}(\mathbf{x})$$

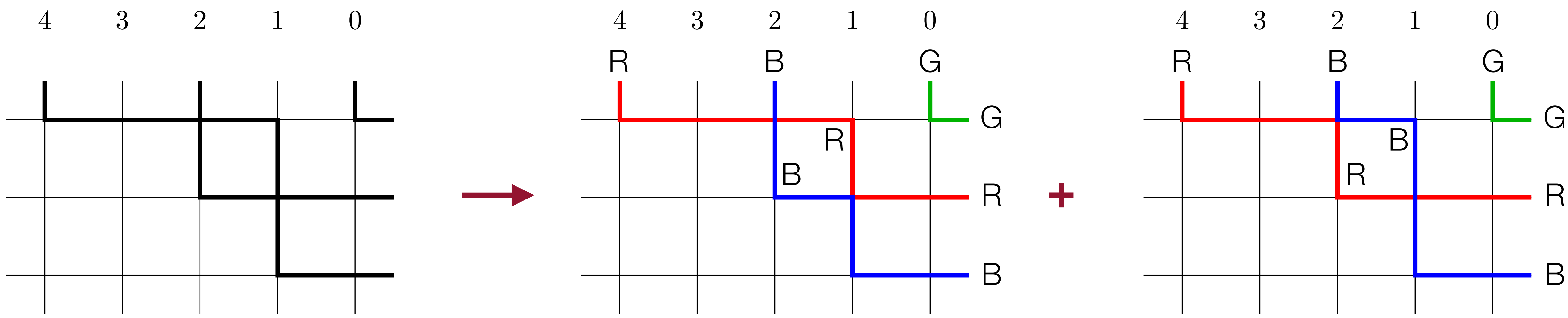
$$Z = \cancel{\prod_{i<j}(x_i + x_j)} \cdot \cancel{\prod_{i<j}(y_i + y_j)} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + y_j) = \sum_{\lambda} Z_\lambda^{\mathsf{top}} \cdot Z_\lambda^{\mathsf{bot}}$$

$$\sum_{\lambda} s_\lambda(\mathbf{x}) s_{\hat{\lambda}'}(\mathbf{y}) = \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + y_j)$$

[Bump–McNamara–Nakasuji 2014]

Colored lattice paths

Ordered palette of r colors: R $>$ B $>$ G



New right boundary data: permutation $w \in S_r$ of (R, B, G)

Have constructed vertex configuration weights such that the partition function is refined to:

uncolored $$Z(\lambda; \mathbf{z}) = \sum_{w \in S_r} Z(\lambda, w; \mathbf{z})$$ colored

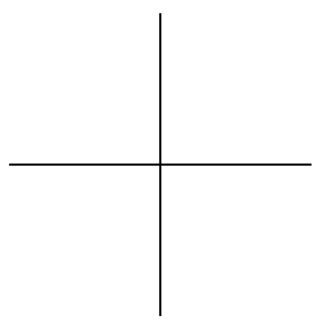
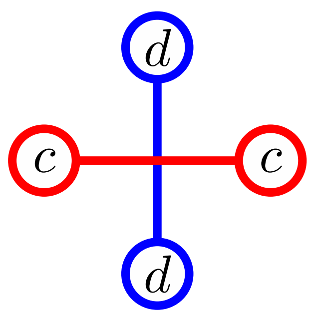
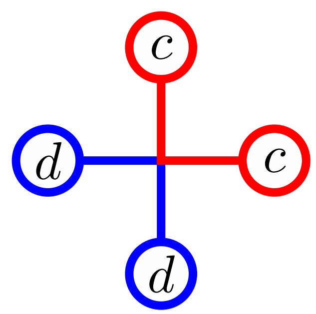
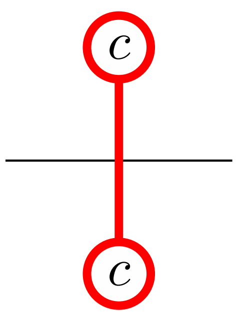
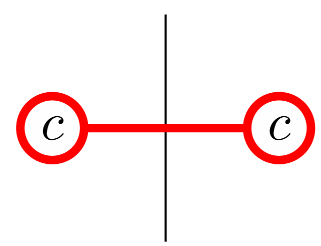
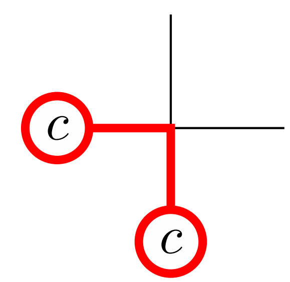
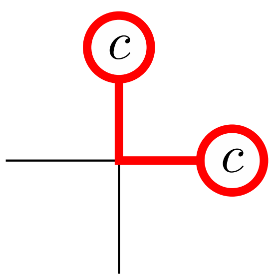
Concept based on [Borodin–Wheeler 2018]

[Brubaker–Buciumas–Bump–HG JCTA 2021 and arXiv:1906.04140]

Colored lattice paths

$$\mathbf{z}^{\rho} s_{\lambda}(\mathbf{z}) = Z(\lambda; \mathbf{z}) = \sum_{w \in S_r} Z(\lambda, w; \mathbf{z})$$

$(v = 0)$

	 				
1	$\begin{cases} z_i & \text{if } c > d \\ 0 & \text{if } c < d \end{cases}$	0	z_i	z_i	1

Theorem: [Brubaker–Buciumas–Bump–HG JCTA 2021]

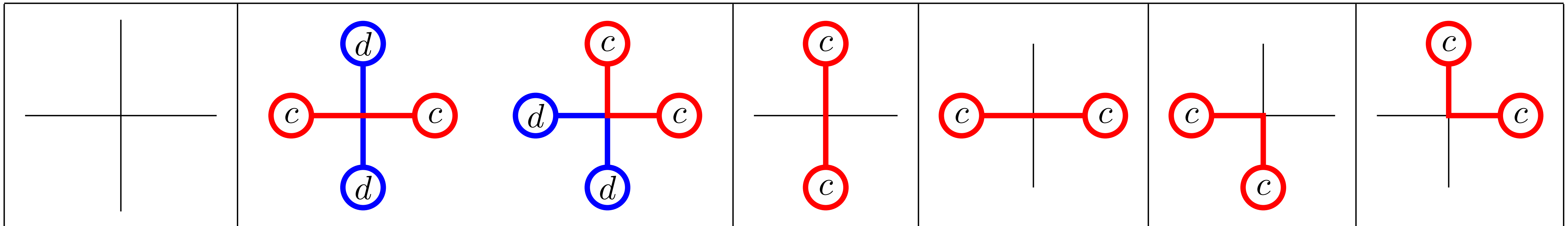
$Z(\lambda, w; \mathbf{z}) =$ Demazure atoms

$\sum_{w \leqslant y} Z(\lambda, w; \mathbf{z})$

Demazure atoms decompose Demazure characters, also known as key polynomials, (of which Schur polynomials are a specific example) into their smallest non-intersecting pieces.

Colored lattice paths

$(v = 0)$



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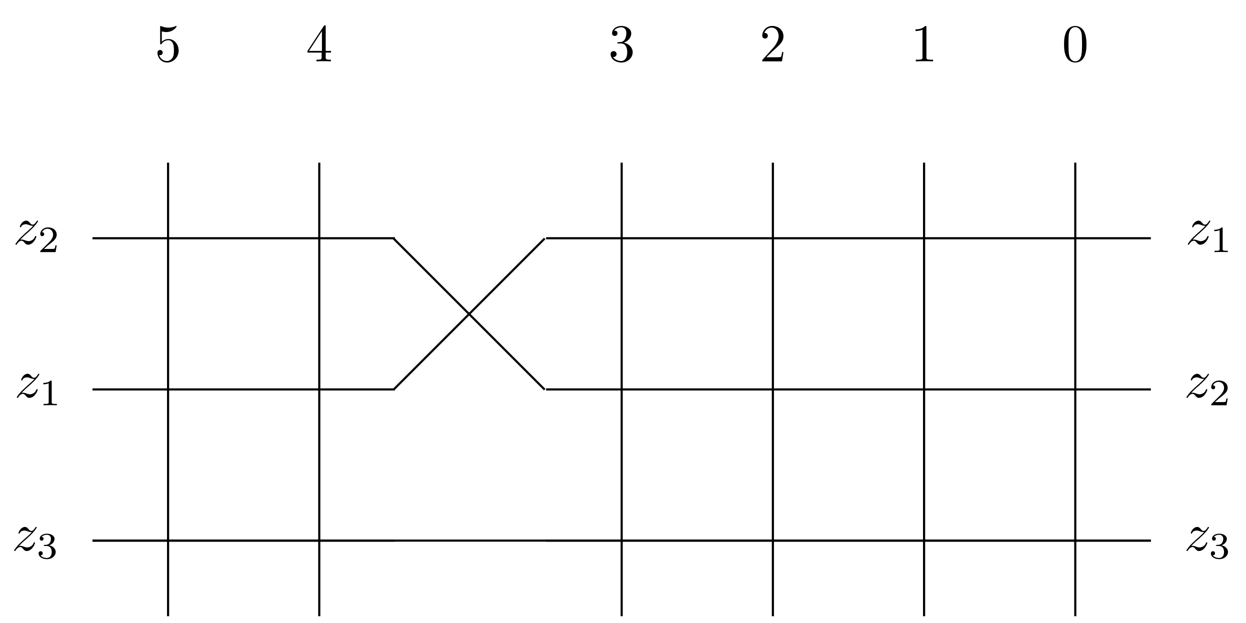
New algorithms for computing so-called Lascoux–Schützenberger key tableaux (related to Kashiwara crystals of Young tableaux).

Yang-Baxter equations

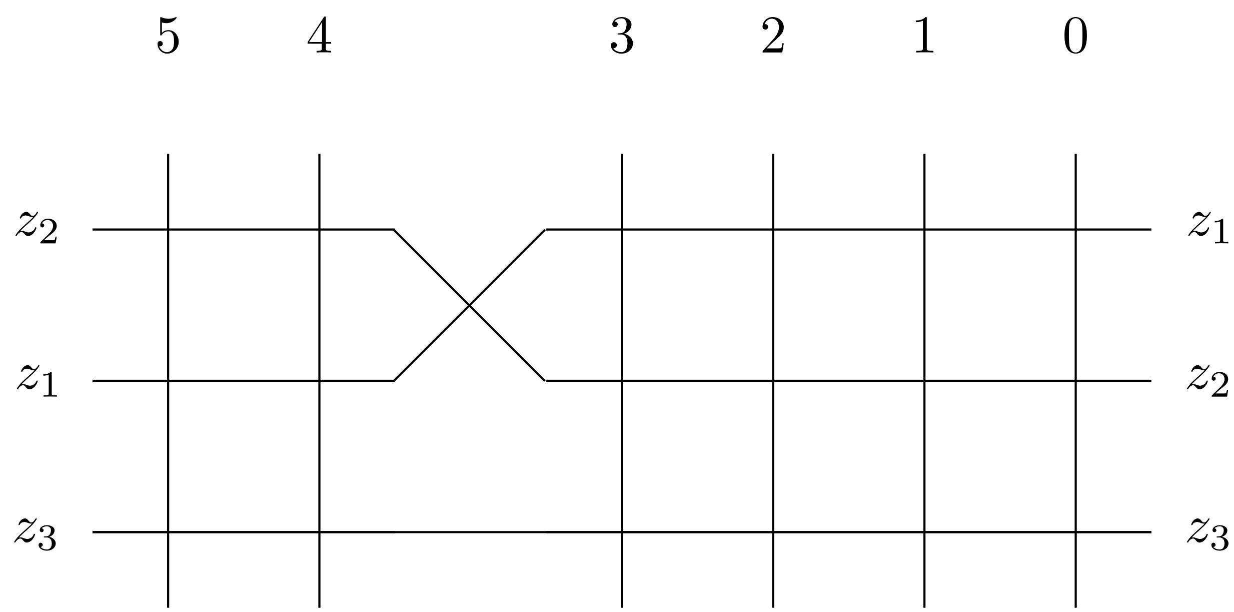
The idea of the proof is to obtain functional equations for the partition functions with respect to length of the permutation w.

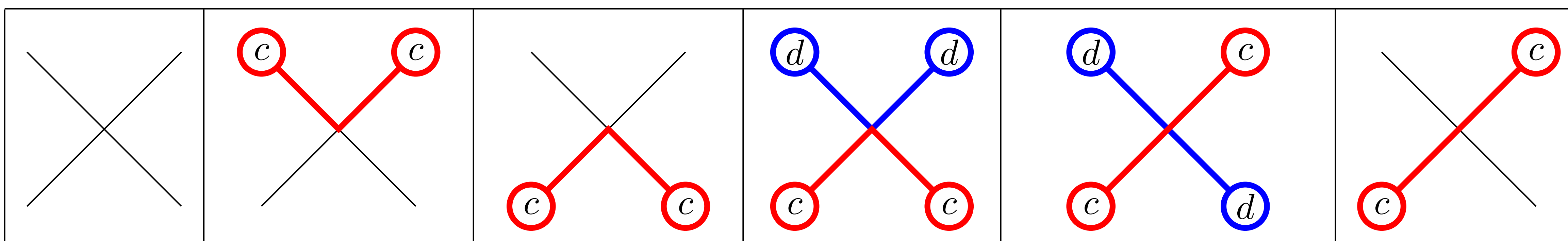
Satisfies a Yang–Baxter equation $\longleftrightarrow$ solvable lattice model

Introduce a new type of vertices, called R-vertices.



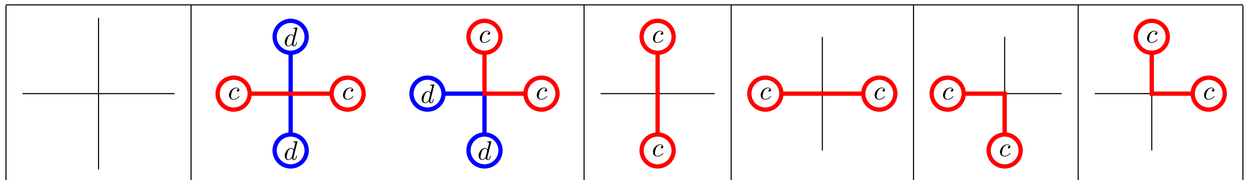
Yang-Baxter equations





z_2	z_2	z_1	$\begin{cases} z_1 & \text{if } c < d \\ z_2 & \text{if } c > d \end{cases}$	$\begin{cases} z_1 - z_2 & \text{if } c > d \\ 0 & \text{if } c < d \end{cases}$	$z_1 - z_2$

Yang-Baxter equations



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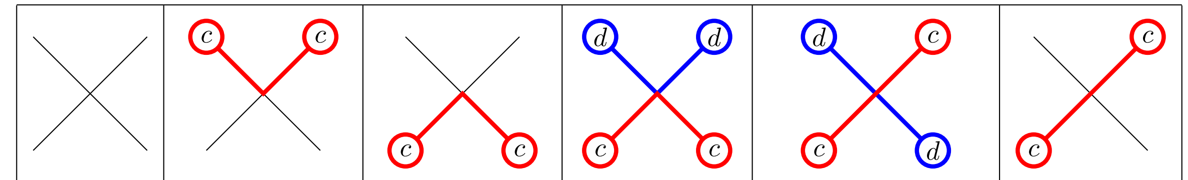
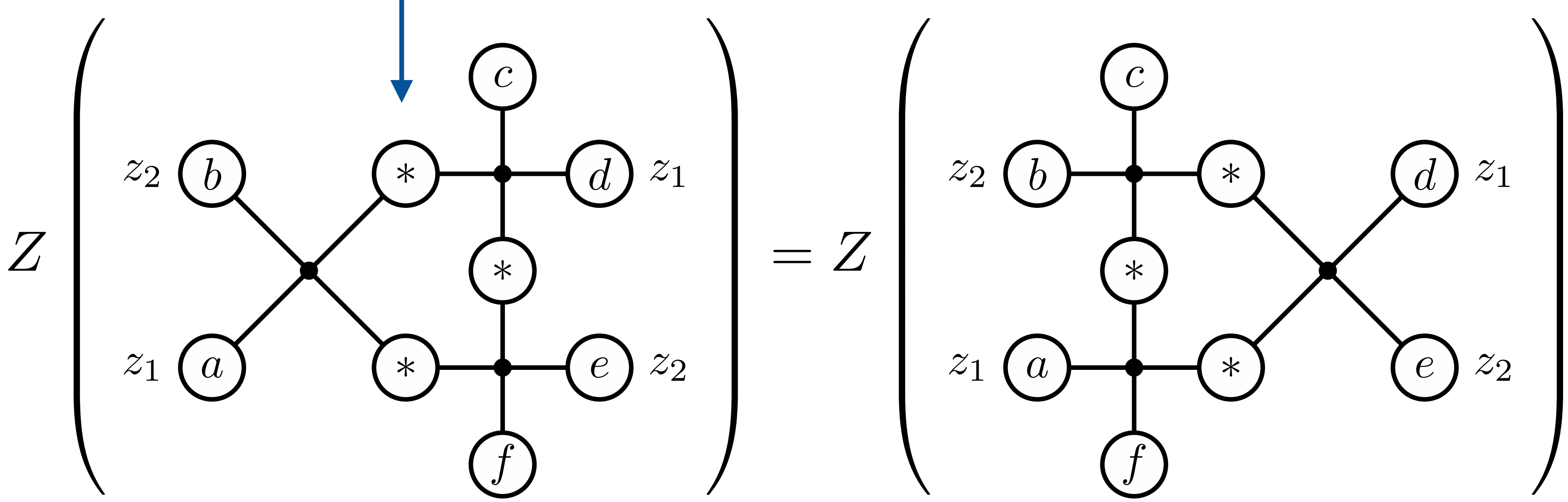


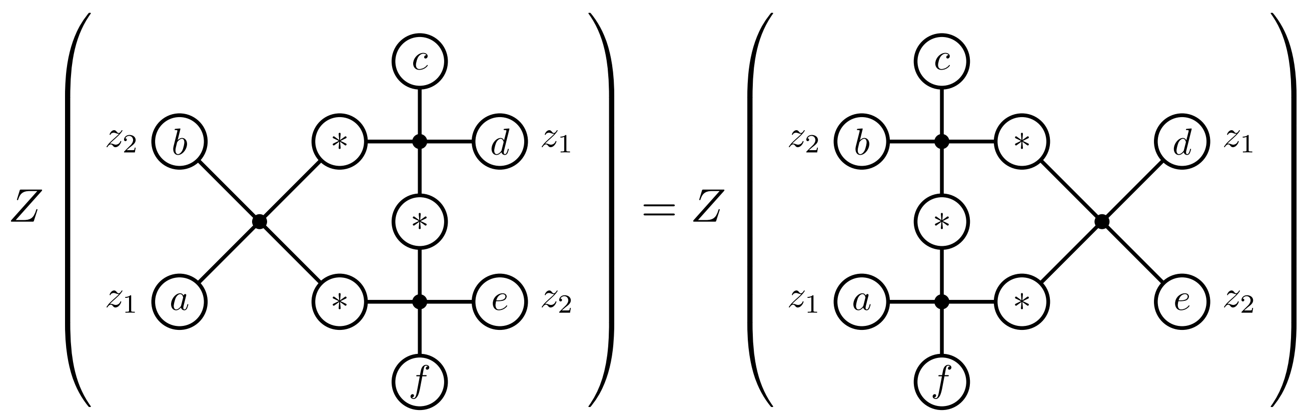
z_2	z_2	z_1	$\begin{cases} z_1 & \text{if } c < d \\ z_2 & \text{if } c > d \end{cases}$	$\begin{cases} z_1 - z_2 & \text{if } c > d \\ 0 & \text{if } c < d \end{cases}$	$z_1 - z_2$

sum over internal edge configurations

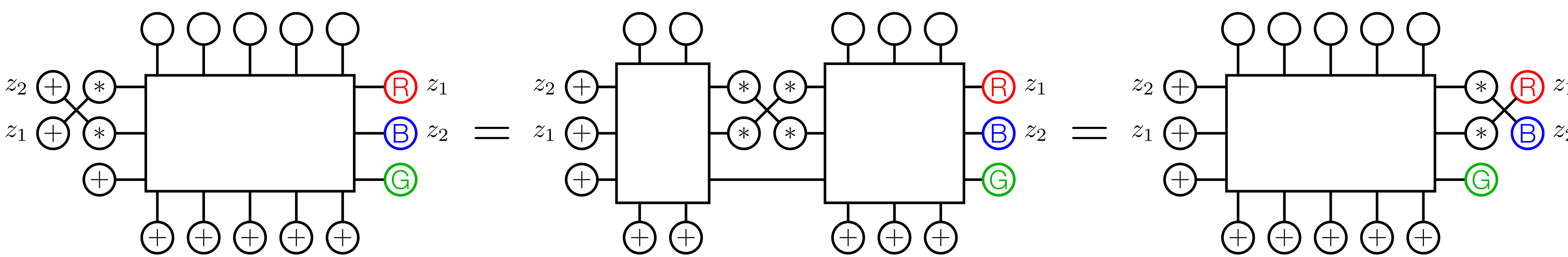


Denote uncolored edge by $\oplus$

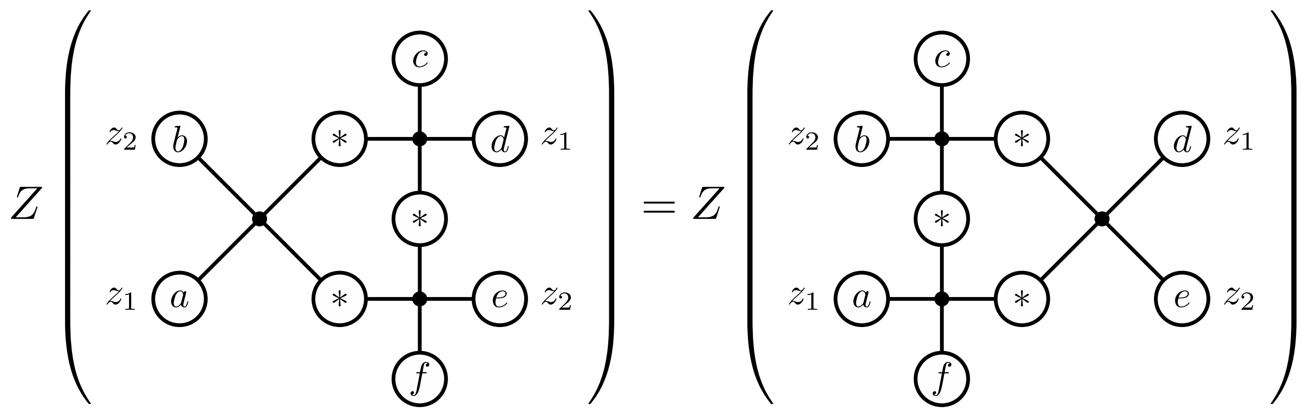
Yang-Baxter equations

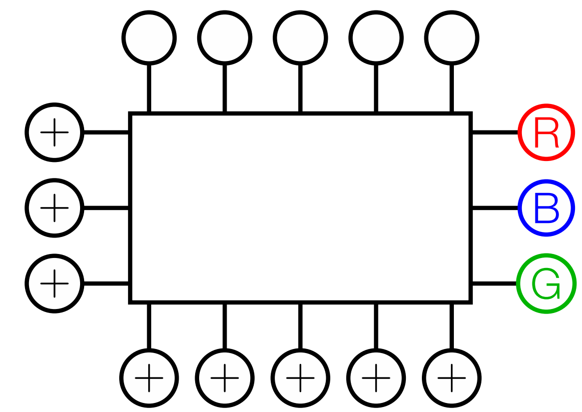


Train argument



Yang-Baxter equations



$Z(\lambda, w; \mathbf{z}) =$ 

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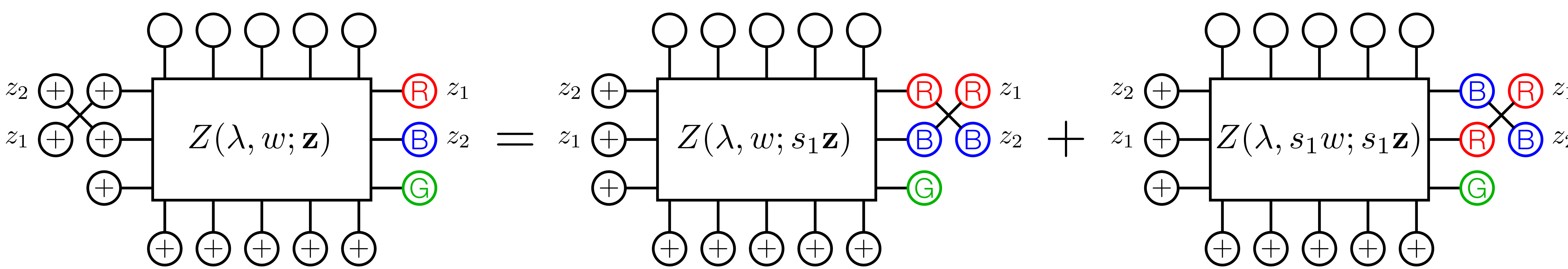


We can solve for $Z(\lambda, s_1 w; s_1\mathbf{z})$ in terms of $Z(\lambda, w; s_1\mathbf{z})$ and $Z(\lambda, w; \mathbf{z})$

Recursion relation in terms of Demazure (divided difference) operators

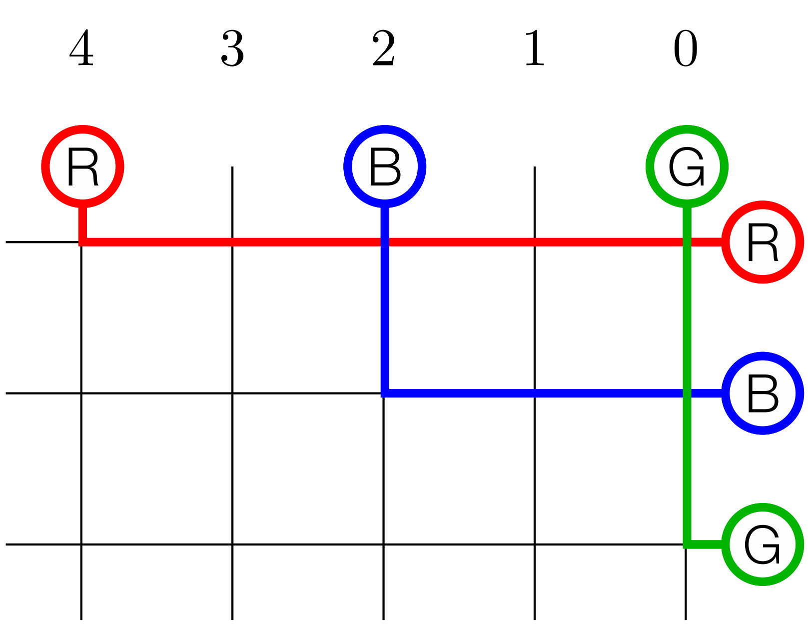
Yang-Baxter equations

Recursion relation in terms of Demazure (divided difference) operators

$$Z(\lambda, s_i w; \mathbf{z}) = \begin{cases} D_i \; Z(\lambda, w; \mathbf{z}) & \text{if } s_i w > w \\ D_i^{-1} Z(\lambda, w; \mathbf{z}) & \text{if } s_i w < w \end{cases} \qquad D_i f(\mathbf{z}) = -\frac{f(\mathbf{z}) - \frac{z_i}{z_{i+1}} f(s_i \mathbf{z})}{1 - \frac{z_i}{z_{i+1}}}$$

Thus, writing any w as a reduced word $s_{i_1} s_{i_2} \cdots s_{i_k}$ we get that

$$Z(\lambda, w; \mathbf{z}) = D_{i_1} D_{i_2} \cdots D_{i_k} \; Z(\lambda, 1; \mathbf{z})$$



$$Z(\lambda, 1; \mathbf{z}) = \mathbf{z}^{\lambda+\rho}$$

Representation theory

$(v \neq 0)$

Representation theory

Disclaimer: there is already some representation theory for $v = 0$.

The Schur polynomial s_λ is the character of a highest weight representation (π_λ, V_λ) of $\mathrm{GL}_r(\mathbb{C})$.

$$s_\lambda(\mathbf{z}) = \mathrm{tr}_{V_\lambda}(\pi_\lambda(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}))$$

My foremost interest is not in complex groups, but in groups defined over other fields of interest in number theory. E.g. $\mathrm{GL}_r(\mathbb{Q}_p)$

Number theory $\longleftrightarrow$ Representation theory

Number theory

A powerful way of studying prime numbers is using p-adic numbers which one can think of as power series in a prime p.

Alternatively,

$3,\ 3.1,\ 3.14,\ 3.1415, \ldots \qquad \pi$

Completion of
Cauchy sequences

$$\mathbb{Q} \xrightarrow[\text{Standard norm } |\cdot|]{} \mathbb{R} \qquad\qquad \mathbb{Q} \xrightarrow[p\text{-adic norm } |\cdot|_p]{} \mathbb{Q}_p$$

The "closeness" of two p-adic numbers is measured by how many powers of p their difference is divisible by.

Ostrowski's theorem states that these are the only possible (non-trivial) absolute values and completions of $\mathbb{Q}$.

(Can even do p-adic calculus)

Representation theory

$\mathrm{GL}_r(\mathbb{Q}_p)$ is the group of invertible $r \times r$ matrices with elements in $\mathbb{Q}_p$.
$\mathrm{SL}_r, \mathrm{SO}_\mathrm{n}, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$

We will mainly consider representations consisting of functions $f : \mathrm{GL}_r(\mathbb{Q}_p) \to \mathbb{C}$ and the right-regular action $\pi(g)f : h \mapsto f(hg)$ for $g, h \in \mathrm{GL}_r(\mathbb{Q}_p)$.

In particular, the principal series representation $\pi_{\mathbf{z}}$ for $\mathbf{z} \in (\mathbb{C}^\times)^r$.

Crucial tool for studying representations:	Whittaker functions
$\updownarrow$ analogy	$\updownarrow$
periodic functions:	Fourier series basis

General idea: embed the former into the space of the latter and determine the support of the image

Results

The previously constructed partition functions with $v = 1/p$ are Whittaker functions for $\pi_{\mathbf{z}}$

Lattice model	Whittaker function for
uncolored	spherical vectors $\pi(g)f = f$ for $g \in \mathrm{GL}_r(\mathbb{Z}_p)$ [Brubaker–Bump–Friedberg 2009, Tokuyama 1988]
colored	Iwahori vectors $\pi(g)f = f$ for $g \in \mathrm{GL}_r(\mathbb{Z}_p) \cap \{$lower triang. $\mathrm{mod}\ p\}$ [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]
	[Brubaker–Buciumas–Bump–HG arXiv:2112.14670]
charged	metaplectic spherical vectors metaplectic n-cover of $\mathrm{GL}_r(\mathbb{Q}_p)$ [Brubaker–Bump–Chinta–Friedberg–Gunnells 2012, Brubaker–Buciumas–Bump 2019]

Results

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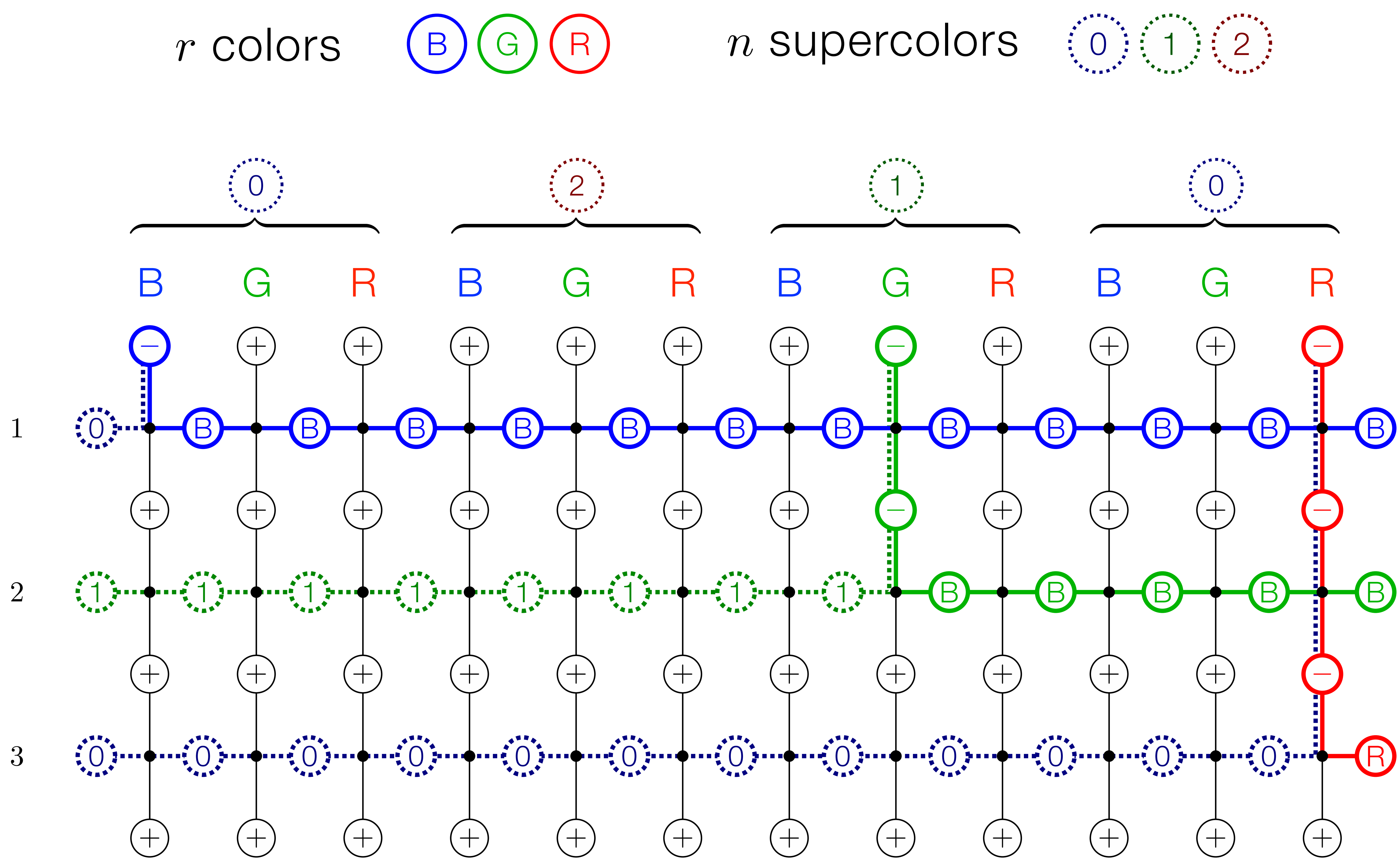
charged	[Brubaker–Buciumas–Bump–HG arXiv:2112.14670] metaplectic spherical vectors metaplectic n-cover of $\mathrm{GL}_r(\mathbb{Q}_p)$ [Brubaker–Bump–Chinta–Friedberg–Gunnells 2012, Brubaker–Buciumas–Bump 2019]
colored + supercolored	metaplectic Iwahori vectors [Brubaker–Buciumas–Bump–HG arXiv:2012.15778]

Results

The previously constructed partition functions with $v = 1/p$ are Whittaker functions for $\pi_{\mathbf{z}}$

Lattice model	Whittaker function for
uncolored	spherical vectors Schur polynomials Hall-Littlewood polynomials
colored	Iwahori vectors Non-symmetric Macdonald polynomials (limits of) Parahoric: Macdonald pol. with prescribed symmetry
charged	metaplectic spherical vectors p-parts of Weyl group multiple Dirichlet series
colored + supercolored	metaplectic Iwahori vectors

Metaplectic version



Research areas

Number theory ⟷ Representation theory

Theoretical physics | Quantum algebra | Combinatorics

NT	RT	TP	QA	C	
●	●	●			Automorphic forms and representations
●	●				Langlands program
●	●	●			Whittaker functions for local representations
●	●		●	●	Solvable lattice models
	●		●		$\boxed{\text{Quantum groups}}$
	●			●	(Non)-symmetric special polynomials
	●	●	●		Vertex operator algebras

this talk

https://hgustafsson.se

Quantum groups

Solutions to the Yang-Baxter equations arise from quantum groups. These are q-deformations of universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$.

	$\mathrm{GL}_r(F)$	metaplectic n-cover of $\mathrm{GL}_r(F)$
spherical	$U_q\big(\widehat{\mathfrak{gl}}(1\vert 1)\big)$	$U_q\big(\widehat{\mathfrak{gl}}(1\vert n)\big)$
Iwahori	$U_q\big(\widehat{\mathfrak{gl}}(r\vert 1)\big)$	$U_q\big(\widehat{\mathfrak{gl}}(r\vert n)\big)$

If the quantum group modules are known for the horizontal and vertical edge configurations then one can compute the Boltzmann weights and the R-matrix directly from the quantum group.

They then automatically satisfy the Yang-Baxter equations.

The module for the vertical edges is not known for any of the lattice models in this talk. The weights had to be constructed, and the Yang-Baxter equations had to be checked by hand.

Thank you!

Slides will be made available at

https://hgustafsson.se

Special polynomials

Table 2. Relations between different Whittaker functions and associated special polynomials.

Whittaker function		**Special polynomial**
Spherical Whittaker function $\sum_{w\in W}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Schur polynomial $\prod_{\alpha\in\Delta^+}(1-v\mathbf{z}^{-\alpha})s_\lambda(\mathbf{z})$
Li's Whittaker function $\sum_{w\in W}(-v)^{-\ell(w)}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Hall-Littlewood polynomial $\mathbf{z}^{-\rho}P_{\lambda+\rho}(\mathbf{z},v^{-1})$
Iwahori Whittaker function $\phi_{w_1}(\mathbf{z};\varpi^{-\lambda})$	$=$	Non-symmetric Macdonald polynomial $(-v)^{\ell(w)}\mathbf{z}^{-\rho}w_0E_{w_0w(\lambda+\rho)}(\mathbf{z};\infty,v)$
Parahoric Whittaker function $\psi_1^{\mathbf{J}}(\mathbf{z};\varpi^{-\lambda})$	$=$	Macdonald polynomial with prescribed symmetry $\mathbf{z}^{-\rho}S_{\lambda+\rho}^{(\emptyset,\mathbf{J})}(\mathbf{z};0,v^{-1})a_{\lambda+\rho}^{(\emptyset,\mathbf{J})}$

References for each row is found in Table 1 in the same paper: arXiv:1906.04140

Representation theory deluxe

Setup

$\mathbb{Q}_p$ $\mathbb{Z}_p$

F non-archimedean local field, $\mathfrak{o}$ ring of integers

$p\mathbb{Z}_p$ p

$\mathfrak{p}$ maximal ideal with uniformizer $\varpi \in \mathfrak{p}$

p^{-1}

reciprocal of the residue field cardinality $v = |\mathfrak{o}/\mathfrak{p}|^{-1}$

While the following representation-theory-results hold for any split reductive group $\mathbf{G}$, we will restrict to $\mathbf{G} = \mathrm{GL}_r$ for which we can compare with the lattice model just described.

Let $\mathbf{G} = \mathrm{GL}_r$, $G = \mathbf{G}(F)$. Standard maximal split torus $\mathbf{T}$.

Details in [arXiv:1906.04140]

Principal series representation

Character $\tau_{\mathbf{z}}$ of $\mathbf{T}(F)$ trivial on $\mathbf{T}(\mathfrak{o})$ parametrized by $\mathbf{z} \in (\mathbb{C}^\times)^r$

$\mathbf{T}(F)/\mathbf{T}(\mathfrak{o})$ representatives: $\varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \ddots & \\ & & & \varpi^{\lambda_r} \end{pmatrix} \quad \lambda \in \mathbb{Z}^r$

$\tau_{\mathbf{z}}\big(\varpi^\lambda \mathbf{T}(\mathfrak{o})\big) = \mathbf{z}^\lambda = \prod_i z_i^{\lambda_i}$. Inflated to Borel subgroup $B = \mathbf{B}(F)$.

Upper triangular

Principal series representation $I(\mathbf{z}) = \mathrm{Ind}_B^G(\delta^{1/2}\tau_{\mathbf{z}})$

Modular quasicharacter

Basis of Iwahori fixed vectors

Iwahori subgroup $J \subset \mathbf{G}(\mathfrak{o})$ of lower triangular matrices $\bmod \mathfrak{p}$

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \qquad \begin{aligned} \mathfrak{o} &= \mathbb{Z}_p \\ \mathfrak{p} &= p\mathbb{Z}_p \end{aligned}$$

Basis of Iwahori fixed vectors in $I(\mathbf{z})$ $\quad \{\Phi_w^{\mathbf{z}}(g)\}_{w \in W = S_r}$

Bruhat decomposition + Iwahori factorization: $G = \bigsqcup_{w \in W} BwJ$

$$\Phi_w^{\mathbf{z}}(bw'k) = \begin{cases} \delta^{1/2}\tau_{\mathbf{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases} \qquad b \in B, w' \in W, k \in J$$

Basis of Iwahori fixed vectors

Spherical vector right-invariant under $K = \mathbf{G}(\mathfrak{o})$

Maximal compact

$$\Phi^{\mathbf{z}}_{\circ} = \sum_{w \in W} \Phi^{\mathbf{z}}_{w}$$

Vectors invariant under parahoric subgroup

$$\sum_{\tilde{w} \in W_P} \Phi_{w\tilde{w}}$$

Parabolic (block-triangular) mod $\mathfrak{p}$

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix} \quad \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

Iwahori $\subseteq$ parahoric $\subseteq$ spherical

Whittaker functions

$N_- = \mathbf{N}_-(F)$

$\mathbf{B}_-$ opposite Borel (lower triangular), unipotent radical $\mathbf{N}_-$

Character $\chi : F \to \mathbb{C}^\times$ trivial on $\mathfrak{o}$ but no larger fractional ideal.

Fix character $\psi : N_- \to \mathbb{C}^\times$ such that $\psi(n) = \chi\Big(\sum_{i=1}^{\text{r-1}} n_{i+1,i}\Big)$

$$\psi\left(\begin{pmatrix} 1 & & & \\ a & 1 & & \\ * & b & 1 & \\ * & * & c & 1 \end{pmatrix}\right) = \chi(a+b+c)$$

Whittaker functional $\Omega_{\mathbf{z}} : I(\mathbf{z}) \to \mathrm{Ind}_{N_-}^{G}(\psi)$

$$f \mapsto \int_{N_-} f(n)\psi(n)^{-1} dn$$

Normalization

Right-translation

Iwahori Whittaker function $\phi_w(\mathbf{z}; g) = \delta^{1/2}(g)\Omega_{\mathbf{z}^{-1}}\left(\pi(g)\Phi_w^{\mathbf{z}^{-1}}\right)$

Base case

$\phi_w(\mathbf{z}; g)$ is determined by its values on $g = \varpi^{-\lambda} w'$ with $\lambda \in \mathbb{Z}^r_{\geq 0}$ and $w' \in W = S_r$ such that

positive roots

$$\lambda_i - \lambda_{i+1} \geqslant \begin{cases} 0 & \text{if } (w')^{-1}\alpha_i \in \Delta^+ \\ -1 & \text{if } (w')^{-1}\alpha_i \in \Delta^- \end{cases}$$

dominant weight

w'-almost dominant weight λ $\xleftrightarrow[\rho = (r-1, \ldots, 1, 0)]{w'\mu = \lambda + \rho}$ $\mu \in \mathbb{Z}^r_{\geqslant 0}$

Bijection between data determining the values for Iwahori Whittaker functions and the boundary data for the lattice model.

Base case $\quad w' = w$

$$\phi_{w'}(\mathbf{z}; \varpi^{-\lambda} w') = v^{\ell(w')} \mathbf{z}^{\lambda}$$

Compare: $Z_{\mu, w'}(\mathbf{z}) = v^{\ell(w')} \mathbf{z}^{w'\mu} = v^{\ell(w')} \mathbf{z}^{\lambda+\rho}$

Recursion relations

Intertwining integral $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z}) \to I(w\mathbf{z})$ $\quad \mathcal{A}_w^{\mathbf{z}}\Phi(g) = \int_{N \cap wN_- w^{-1}} \Phi(w^{-1}ng)\,dn$

[Casselman–Shalika 80,
Brubaker–Bump–Licata 15]

$$\Omega_{\mathbf{z}} \longleftarrow \mathcal{A}_{s_i}^{\mathbf{z}} \longrightarrow \Phi_w^{\mathbf{z}}$$

[Casselman 80,
Brubaker–Bump–Licata 15]

Leads to recursion relations

equivalent to Yang–Baxter equation!

$$\mathbf{z}^\rho \phi_{s_i w}(\mathbf{z}; g) = \begin{cases} T_i \ \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) > \ell(w), \\ T_i^{-1} \ \mathbf{z}^\rho \phi_w(\mathbf{z}; g) & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

Demazure operators

$$T_i = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} s_i + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}}$$