Fourier coefficients of automorphic forms

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Yale, Colloquium – Nov 13, 2019

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Outline

- Why study these Fourier coefficients?
- From modular forms to automorphic forms
- Fourier expanding periodic functions on groups
- Different kinds of Fourier coefficients
- How to relate them to each other
- Simplification for small automorphic representations
- Recent work*
- String theory applications

*) Including joint work with Gourevitch, Kleinschmidt, Persson and Sahi

Motivation

for studying Fourier coefficients

Let $r_k(n) = \#(\text{integer solutions to } n = x_1^2 + x_2^2 + \cdots + x_k^2)$

Theorem: (Jacobi four-square theorem) [Jacobi 1829]

$$r_4(n) = 8 \sum_{\substack{0<d|n \\ 4\nmid d}} d, \qquad n \geq 1$$

Proof:

Consider generating function $f_k(z) = \sum_{n=0}^{\infty} r_k(n) e^{2\pi i n z}$

See [Diamond–Shurman, §1.2] for review.

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Consider generating function $f_k(z) = \sum_{n=0}^{\infty} r_k(n) e^{2\pi i n z}$

$$r_{k_1+k_2}(n) = \sum_{n_1+n_2=n} r_{k_1}(n_1) r_{k_2}(n_2) \implies f_{k_1+k_2}(z) = f_{k_1}(z) f_{k_2}(z)$$

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$$f_k(z) = \big(f_1(z)\big)^k \qquad r_1(n) = \begin{cases} 2 & \text{if square} \\ 0 & \text{oterhwise} \end{cases} \qquad n = x_1^2$$

$$f_1(z) = \vartheta(z) := \sum_{d \in \mathbb{Z}} e^{2\pi i d^2 z} \longleftarrow \text{Jacobi theta function}$$

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Satisfy:

$$\left.\begin{array}{l} \text{(1).}\ \ f_k(z+1) = f_k(z) \\[2ex] \text{(2).}\ \ f_k(\frac{z}{4z+1}) = (4z+1)^{k/2} f_k(z) \end{array}\right\} \text{Will come back to this later}$$

See [Diamond–Shurman, §1.2] for review.

Motivation

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The space of functions satisfying (1) and (2) is finite-dimensional for each k after also adding a condition on polynomial growth.

For $k = 4$ the space is two-dimensional. (Riemman-Roch)

[Diamond–Shurman, Theorem 3.5.1]

Motivation

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The space of functions satisfying (1) and (2) is finite-dimensional for each k after also adding a condition on polynomial growth.

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Basis of Eisenstein series whose Fourier coefficients are known

By matching first two coefficients: $$r_4(n) = 8 \sum_{\substack{0<d|n \\ 4\nmid d}} d, \qquad n \geq 1$$

$\square$

See [Diamond–Shurman, §1.2] for review.

Modular forms

Function $f:\mathcal{H}\to\mathbb{C}$ on the upper half plane $\mathcal{H}=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$

$$\left.\begin{array}{l} (1).\ \ f_{2k}(z+1)=f_{2k}(z) \\[1em] (2).\ \ f_{2k}(\tfrac{z}{4z+1})=(4z+1)^k f_{2k}(z)\end{array}\right\} \iff f_{2k}(\gamma(z))=(cz+d)^k f_{2k}(z) \qquad \begin{array}{l}\gamma=\pm(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})\\[1em] \gamma=\pm(\begin{smallmatrix}1&0\\4&1\end{smallmatrix})\end{array}$$

$$\mathrm{SL}_2(\mathbb{R})\circlearrowleft\mathcal{H}: \qquad \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R}) \qquad \gamma(z)=\frac{az+b}{cz+d}$$

Generate
$\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$
of finite index

Satisfying:

- $f(\gamma(z))=(cz+d)^k f(z)$ for all $\gamma\in\underset{\Gamma}{\cancel{\mathrm{SL}_2(\mathbb{Z})}}$ for some weight $k\in\mathbb{N}$
- Holomorphic
- Polynomial growth

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- Polynomial growth

Typical example: holomorphic Eisenstein series

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

Goal

v0.1

Compute Fourier coefficients of automorphic forms

Modular forms $\rightsquigarrow$ Automorphic forms

SL_2 $\qquad$ G

E_8

E_7

E_6

$\mathrm{SO}_{n,n}$

SL_n

GL_n

Group theory upgrade:

$$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$$

$$z = g(i) \longleftrightarrow g\mathrm{SO}_2(\mathbb{R})$$

$\mathrm{SO}_2(\mathbb{R}) = \mathrm{Stab}(i)$

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Representatives $g = \begin{pmatrix} \sqrt{y} & x \\ 0 & 1/\sqrt{y} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$

$g(i) = x + iy$

$$f(z) \longrightarrow f(g)$$

$$f(\gamma(z)) \longrightarrow f(\gamma g)$$

Automorphic forms

Smooth function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ satisfying:

- Automorphic invariant: $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Z})$.
- Annihilated by polynomials in G-invariant differential operators.

  Compare $\overline{\partial} f = 0$
- K-finiteness: $\mathrm{span}\{g \mapsto \varphi(gk) : k \in K\}$ is finite dimensional.

  Often take $\varphi : G(\mathbb{R})/K \to \mathbb{C}$.

  Maximal compact subgroup.
  For $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2(\mathbb{R})$.
- Polynomial growth

SL₂ Eisenstein series

Typical example of automorphic form on $\mathcal{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$

Non-holomorphic Eisenstein series

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2 \right\}$$

$$E_s(z, \bar{z}) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{(\operatorname{Im} z)^s}{|cz+d|^{2s}} = \sum_{\gamma \in B(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z})} (\operatorname{Im} \gamma(z))^s$$

$s \in \mathbb{C}$

Compare with the holomorphic Eisenstein series, a modular form of weight k

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

$$\bar{\partial} G_k = 0$$

$$\big(\Delta - s(s-1)\big) E_s = 0 \qquad \Delta = 4y(\partial_x^2 + \partial_y^2)$$

E_s invariant under $\mathrm{SL}_2(\mathbb{Z})$ while G_k transforms with weight k.

SL_2 Eisenstein series

Typical example of automorphic form on $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$

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$s \in \mathbb{C}$

To be able to generalize to other groups: Let $\chi_s(g) = \operatorname{Im}\big(g(i)\big)^s$.

$G = \mathrm{SL}_2$: $$E_s(g) = \sum_{\gamma\in B(\mathbb{Z})\backslash\,\mathrm{SL}_2(\mathbb{Z})} \chi_s(\gamma g)$$

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} k$$

$z = x + iy$

$k \in \mathrm{SO}_2(\mathbb{R})$

$G = \mathrm{SL}_n$: $$E_{\vec{s}}(g) = \sum_{\gamma\in B(\mathbb{Z})\backslash\,\mathrm{SL}_n(\mathbb{Z})} \chi_{\vec{s}}(\gamma g)$$

Strictly upper triangular

$g = nak$

Diagonal

$k \in K = \mathrm{SO}_n(\mathbb{R})$

SL_2 Eisenstein series

$G = \mathrm{SL}_2$: $$E_s(g) = \sum_{\gamma \in B(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \chi_s(\gamma g)$$ $$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} k$$ $z = x + iy$, $k \in \mathrm{SO}_2(\mathbb{R})$

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$k \in K = \mathrm{SO}_n(\mathbb{R})$

$\vec{s} \in \mathbb{C}^r \longleftrightarrow \lambda = 2s_1\Lambda_1 + \ldots 2s_r\Lambda_r$ weight

Manifestly invariant under $G(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z})$

For SL_2 this means that $\varphi = E_s$ is periodic in x.

In general: “periodic” in the variables of n.

Core problem

Consider a function φ on the Heisenberg group
$\mathbb{H}_3(\mathbb{R}) = N(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

What does it mean to be periodic?

$$\varphi\Big(\underbrace{\overbrace{\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}}^{\gamma \in N(\mathbb{Z})} \overbrace{\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}}^{g \in N(\mathbb{R})}}_{\begin{pmatrix} 1 & x+a & z+c+ay \\ & 1 & y+b \\ & & 1 \end{pmatrix}}\Big) = \varphi\Big(\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}\Big)$$

How to Fourier expand it?

Core problem

$$\varphi\left(\underbrace{\begin{pmatrix}1&a&c\\&1&b\\&&1\end{pmatrix}\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}}_{\begin{pmatrix}1&x+a&z+c+ay\\&1&y+b\\&&1\end{pmatrix}}\right)=\varphi\left(\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix}\right)$$

Simply expanding φ in x, y and z using the Fourier modes $e^{2\pi i(mx+ny+kz)}$ does not work!

Invariant under $\begin{cases}x\to x+a\\ y\to y+b\\ z\to z+c+\cancel{ay}\end{cases}$ while φ is not.

First step: expand in z.

Core problem

Start with the (abelian) center: $Z(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$

$$\varphi(g) = \sum_{k \in \mathbb{Z}} \mathcal{F}_{Z,k}[\varphi](g) \quad = \sum_{m,n \in \mathbb{Z}} \mathcal{F}_{N,m,n}[\varphi](g) + \sum_{k \neq 0} \mathcal{F}_{Z,k}[\varphi](g)$$

$$\mathcal{F}_{Z,k}[\varphi](g) = \int_{\mathbb{Z} \backslash \mathbb{R}} \varphi\Big(\begin{pmatrix} 1 & & z' \\ & 1 & \\ & & 1 \end{pmatrix} g\Big) e^{-2\pi i k z'} \, dz'$$

$\mathcal{F}_{Z,0}[\varphi](g)$ is periodic in x and y. (Absorb z-shift in z'-integration)

$$\mathcal{F}_{N,m,n}[\varphi](g) = \int_{(\mathbb{Z} \backslash \mathbb{R})^3} \varphi\Big(\begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} g\Big) e^{-2\pi i (m x' + n y')} \, dx' \, dy' \, dz'$$

Core problem

$$\mathcal{F}_{N,m,n}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} g\right) e^{-2\pi i(mx'+ny')}\, dx'dy'dz'$$

$$\Big\downarrow$$

Group theory upgrade

$$\int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(ug)\, \psi_{m,n}(u)^{-1}\, du$$

Character

$\psi(uu') = \psi(u)\psi(u')$ $\psi(u) = 1$ for $u \in N(\mathbb{Z})$

Core problem

Different kinds of Fourier coefficients

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(ug)\,\psi_{m,n}(u)^{-1}\,du = \mathcal{W}_{\psi_{m,n}}$$

Whittaker coefficient

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$$

Different unipotent subgroups:

u is unipotent if $(1-u)^N = 0$ for some N

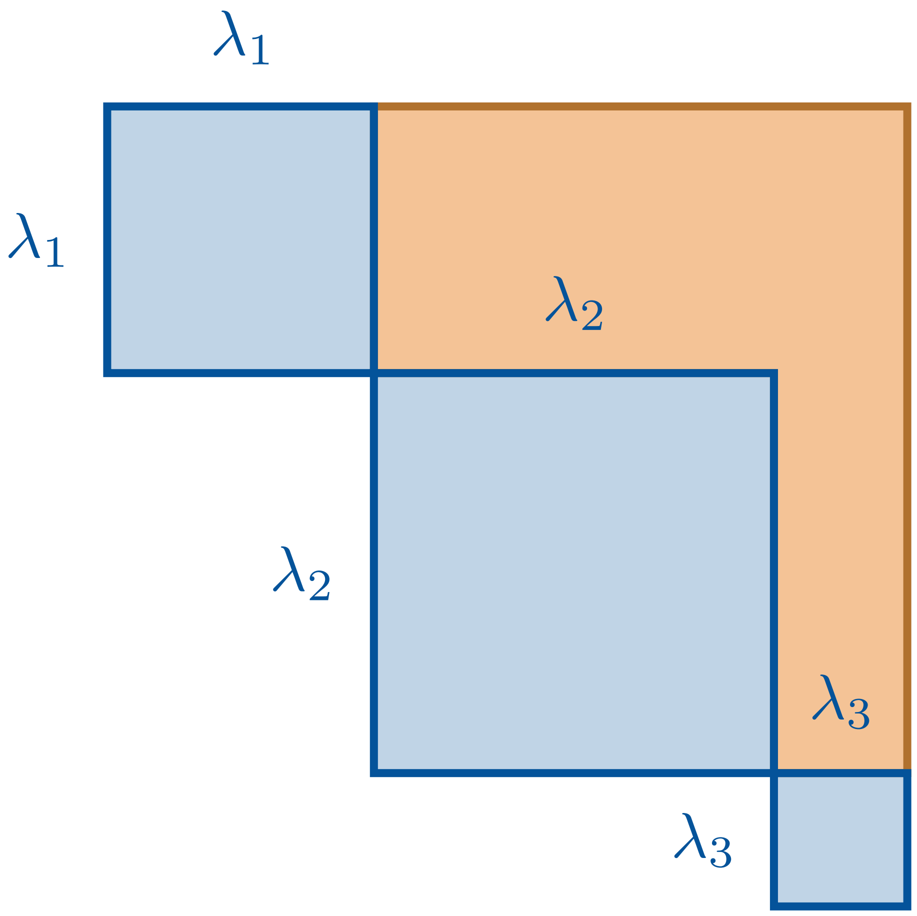
$\log u = -\sum_{k=1}^{N-1} \frac{1}{k}(1-u)^k$

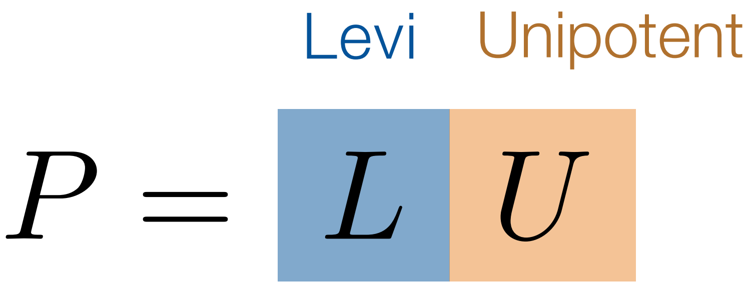
$$U = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$$

Maximal parabolic Fourier coefficients

Parabolic subgroups

For GL_n and SL_n a (standard) parabolic subgroup P is given by an integer partition λ of n.





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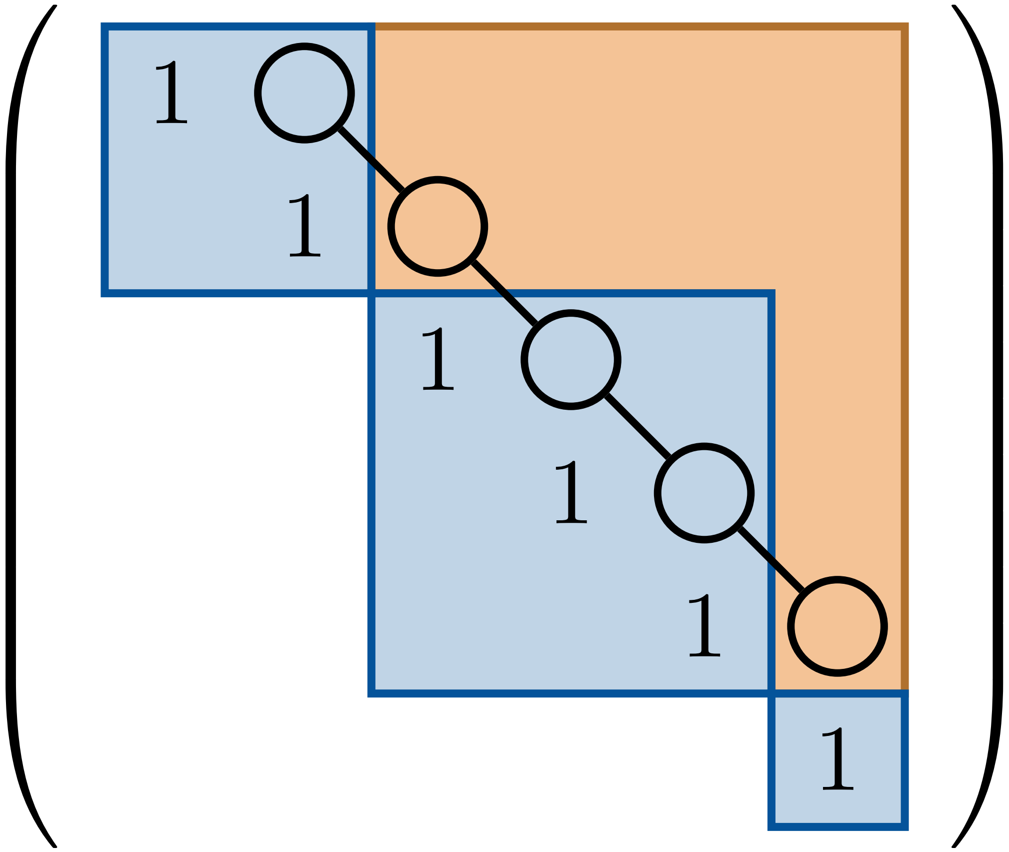
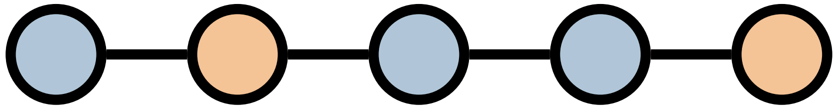


$$P = L\ U$$



In general, described by set of simple roots.

Blue nodes ● in a Dynkin diagram.

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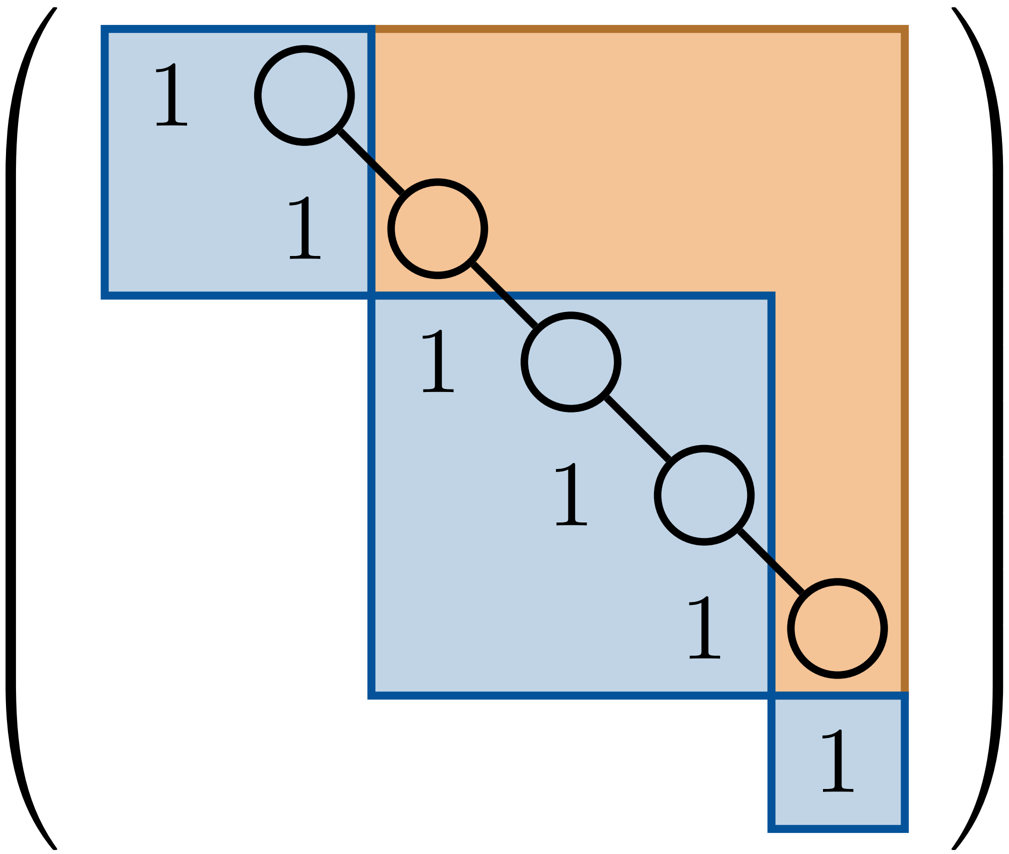

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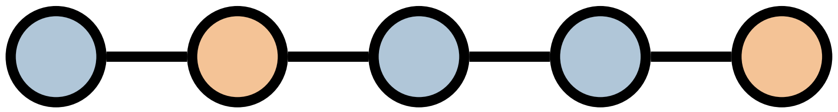

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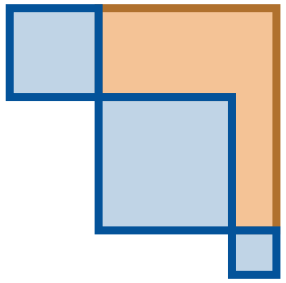
Parabolic subgroups



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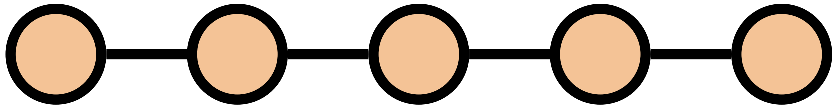
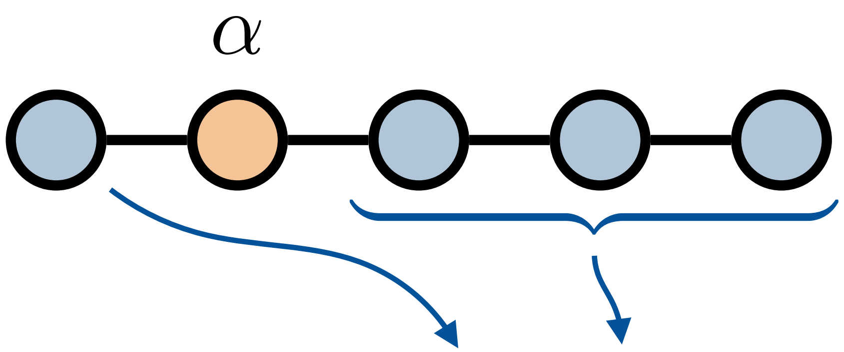
$$P = \boxed{L}\boxed{U}$$

Minimal parabolic (Borel) B



Maximal $U = N$. Small $L =$ torus.

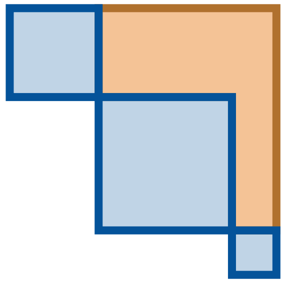
Maximal parabolic P_α



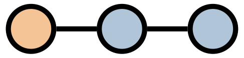
Small U. Large L.

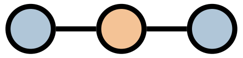
Semi-simple part of L may be read from the Dynkin diagram. Type $A_1 \times A_3$.

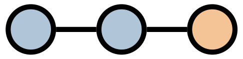
Parabolic Fourier coefficients



SL_4

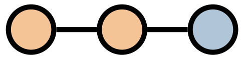


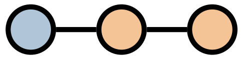




U : $\begin{pmatrix} 1 & * & * & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$

Maximal parabolic





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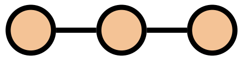


$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$

Borel

Character on U
(Fourier mode)

Fourier coefficient: $$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\psi^{-1}(u)\,du$$

Goal

v1.0

Goal

v1.0

Compute parabolic Fourier coefficients of automorphic forms

Motivation

Motivation

Quantum corrections to graviton scattering amplitudes in string theory are described by automorphic forms on simply-laced groups.

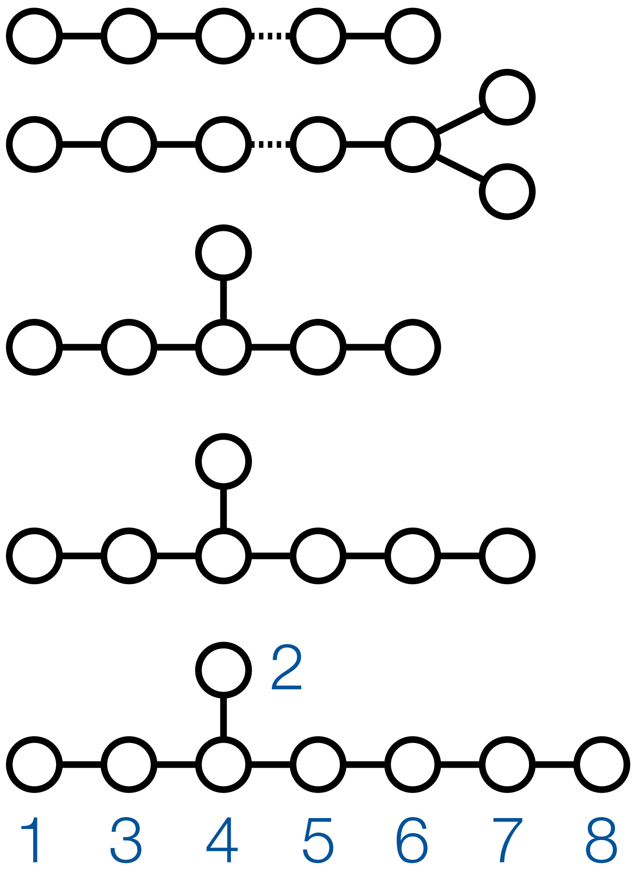
Space-time = $\mathbb{R}^{10-d} \times T^d$ ← d-dimensional torus

Eisenstein series $E_{\vec{s}}$ on $G = E_{d+1}$.

Particular $\vec{s} \leftrightarrow \lambda$ determined by string theory.

↓

Small automorphic representations



[Green–Gutperle 97, Günaydin–Neitzke–Pioline–Waldron 06, Green–Miller–Vanhove 15]
Reviewed in [Fleig–HG–Kleinschmidt–Persson 18]

Motivation

Eisenstein series $E_{\vec{s}}$ on simply-laced groups.

Particular $\vec{s} \leftrightarrow \lambda$ determined by string theory.

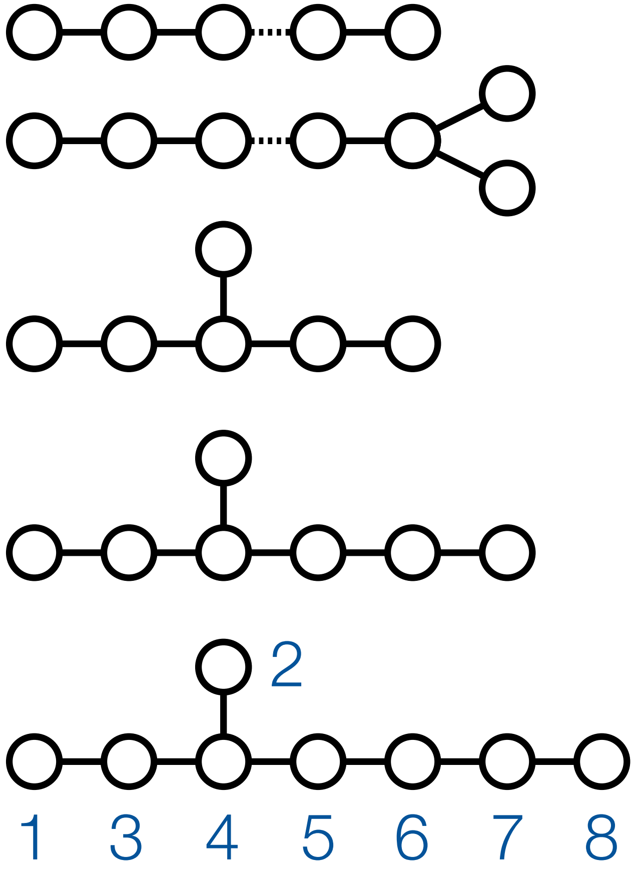
↓

Small automorphic representations

↓

Defining property:

Few non-vanishing Fourier coefficients



Will be made more precise when we consider orbits of Fourier coefficients.

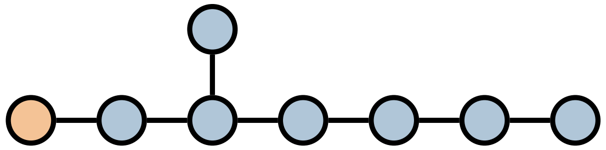
[Green–Gutperle 97, Günaydin–Neitzke–Pioline–Waldron 06, Green–Miller–Vanhove 15]
Reviewed in [Fleig–HG–Kleinschmidt–Persson 18]

Motivation

Different parabolic subgroups study different limits and objects in string theory.

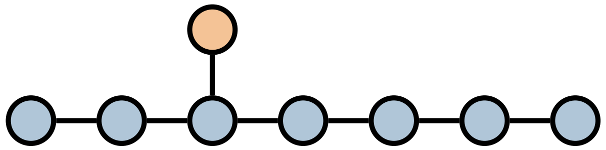
- String perturbation limit

  D-instantons, NS5-instantons



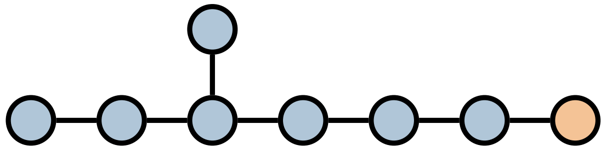
- M-theory limit

  M2, M5-instantons



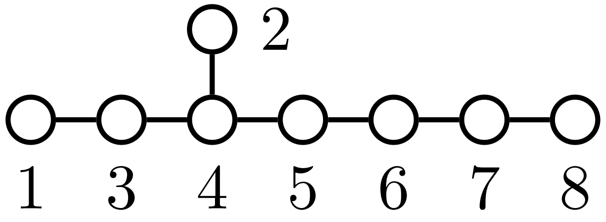
- Decompactification limit

  Higher dimensional BPS states, black holes



$E_{d+1} \to E_d$

What is known?

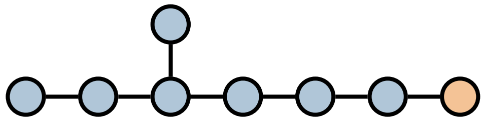


$\operatorname{rank} G \leq 5$:

$\mathrm{SL}_2(\mathbb{R}) \quad \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^+ \quad \mathrm{SL}_3(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \quad \mathrm{SL}_5(\mathbb{R}) \quad \mathrm{Spin}_{5,5}(\mathbb{R})$

Maximal parabolic Fourier coefficients reviewed in [Green–Miller–Vanhove 15]

E_6, E_7, E_8



Decompactification limit computed in [Bossard–Pioline 17, Bossard–Kleinschmidt 16]

Kac–Moody (e.g. E_9, E_{10}, E_{11})

Eisenstein series well-defined [Garland 06] (affine), [Carbone–Lee–Liu 17] (rank 2 hyperbolic)

Constant mode Whittaker coefficient: [Garland 01, Fleig–Kleinschmidt 12]

Generic Whittaker coefficient (*p*-adic): [Patnaik 17] (affine)

Strategy

Many Fourier coefficients of automorphic forms are difficult to compute directly.

Known: Whittaker coefficients.

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \quad \mathcal{W}$$

Determined by its values on $L =$ torus (small)

Casselman–Shalika formula.

Wanted: (maximal) parabolic Fourier coefficients

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \mathcal{F}$$

Large L. Direct computation difficult.

Strategy:

When possible, write the latter $(\mathcal{F})$ in terms of the former $(\mathcal{W})$.

(Theorem I)

Core strategy example

v0.1

Parabolic Fourier coefficient in terms of Whittaker coefficients

Let $G = \mathrm{SL}_4$, $U = \left\{\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$ and $\psi^{-1}\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = e^{2\pi i(m_1x_1+m_2x_2+m_3x_3)}$

$m_1, m_2, m_3 \in \mathbb{Z}$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) e^{2\pi i(x_1+\underline{m_2x_2+m_3x_3})}\, d^3x$$

For now, assume $m_1 = 1$

$$\Big\downarrow \; ?$$

$$\mathcal{W}_{m_1,m_4,m_6}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^6} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) e^{2\pi i(m_1x_1+m_4x_4+m_6x_6)}\, d^6x$$

Core strategy example

v0.1

Parabolic Fourier coefficient in terms of Whittaker coefficients

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For now, assume $m_1 = 1$

Step 1: Conjugation

Let $\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_2 & 1 & 0 \\ 0 & -m_3 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z})$. Automorphic invariance gives: $\varphi(g') = \varphi(\gamma_0 g')$.

$$\varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g \right) = \varphi\left(\gamma_0 \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0^{-1} \gamma_0 g \right) = \varphi\left(\begin{pmatrix} 1 & x_1 + m_2 x_2 + m_3 x_3 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g \right)$$

Core strategy example

v0.1

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} g\right) e^{2\pi i(x_1+\underline{m_2x_2+m_3x_3})}\, d^3x$$

Step 1: Conjugation

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$$\varphi\left(\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} g\right) = \varphi\left(\gamma_0\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\gamma_0^{-1}\gamma_0 g\right) = \varphi\left(\begin{pmatrix}1&x_1+m_2x_2+m_3x_3&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\gamma_0 g\right)$$

Thus, with a shift in the x_1 integration variable

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \gamma_0 g\right) e^{2\pi i x_1}\, d^3x$$

Core strategy example

v0.1

Step 1: Conjugation

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Thus, with a shift in the x_1 integration variable

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Z}\backslash\mathbb{R})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right) e^{2\pi i x_1}\, d^3x$$

To do the same with any $m_1 \neq 0$ would need to conjugate with

$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{m_2}{m_1} & 1 & 0 \\ 0 & -\frac{m_3}{m_1} & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Q})$. But automorphic invariance only for $\mathrm{SL}_4(\mathbb{Z})$.

Tool: Adelic lift

$$\begin{tikzcd}
\text{Adeles } \mathbb{A} \supset \mathbb{R} \arrow[d] & \\
\text{Automorphic forms on } G(\mathbb{A}) \arrow[r, "\text{Compute}"] & \mathbb{Q}\backslash\mathbb{A}\text{-Fourier coefficients} \arrow[d, "\text{Restrict}"'] \\
\text{Automorphic forms on } G(\mathbb{R}) \arrow[u, "\text{Adelic lift}"] & \mathbb{Z}\backslash\mathbb{R}\text{-Fourier coefficients}
\end{tikzcd}$$

For details see [Fleig–Gustafsson–Kleinschmidt–Persson 18, §2, §6]

The ring of adeles

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\text{Completion of Cauchy sequences}", "\text{Standard norm } |\cdot|"'] & \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "p\text{-adic norm } |\cdot|_p"'] & \mathbb{Q}_p \end{tikzcd}$$

For a prime p and $x \in \mathbb{Q}$ prime factorized as $x = p_1^{k_1} \cdots p_n^{k_n}$ we define the p-adic norm

$$|x|_p = \begin{cases} p_i^{-k_i} & \text{if } p = p_i \text{ for any } i \\ 1 & \text{otherwise} \end{cases}$$

Ring of adeles: $\mathbb{A} = \mathbb{R} \times \prod_{\text{prime } p}' \mathbb{Q}_p$

$\mathbb{Q}$ embeds diagonally in $\mathbb{A}$: $\mathbb{Q} \ni q \mapsto (q; q, q, \ldots) \in \mathbb{A}$.

$\mathbb{Q}$ is discrete in $\mathbb{A}$ and $\mathbb{Q} \backslash \mathbb{A}$ is compact.

Dictionary

Fourier expansion on $\mathbb{Z}\backslash\mathbb{R}$ $\longrightarrow$ Fourier expansion on $\mathbb{Q}\backslash\mathbb{A}$

$$f(x) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{Z}\backslash\mathbb{R}} f(x+\xi)e^{2\pi i m\xi}\,d\xi$$

$$f(x) = \sum_{m\in\mathbb{Q}} \int_{\mathbb{Q}\backslash\mathbb{A}} f(x+\xi)\mathbf{e}(m\xi)\,d\xi$$

$$U(\mathbb{Z})\backslash U(\mathbb{R}) \longrightarrow U(\mathbb{Q})\backslash U(\mathbb{A})$$

$G(\mathbb{Z})$-invariant $\longrightarrow$ $G(\mathbb{Q})$-invariant

For details see [Fleig–Gustafsson–Kleinschmidt–Persson 18, §2, §6]

Core strategy example

v1.0

Let $G = \mathrm{SL}_4$, $U = \left\{\begin{pmatrix}1&*&*&*\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$ and $\psi^{-1}\left(\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right) = \mathbf{e}(m_1x_1+m_2x_2+m_3x_3)$

$m_1, m_2, m_3 \in \mathbb{Q}$

$m_1 \neq 0$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} \varphi\left(\begin{pmatrix}1&x_1&x_2&x_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} g\right)\mathbf{e}(m_1x_1+m_2x_2+m_3x_3)\, d^3x$$

Theorem

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4,m_6\in\mathbb{Q}}\sum_{\gamma\in\Gamma_4} \mathcal{W}_{\psi_{m_1,m_4,m_6}}[\varphi](\overbrace{\gamma\gamma_0}^{\in \mathrm{SL}_4(\mathbb{Q})} g)$$

Maximal parabolic
Fourier coefficient

Translated Whittaker coefficients

Core strategy example

Let $G = \mathrm{SL}_4$, $U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $\psi^{-1}\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \mathbf{e}(m_1 x_1 + m_2 x_2 + m_3 x_3)$

$m_1, m_2, m_3 \in \mathbb{Q}$

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$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g\right) \mathbf{e}(m_1 x_1 + m_2 x_2 + m_3 x_3)\, d^3 x$$

Theorem $\quad \mathcal{F}_{U,\psi}[\varphi](g) \;=\; \sum_{m_4, m_6 \in \mathbb{Q}} \sum_{\gamma \in \Gamma_4} \mathcal{W}_{\psi_{m_1, m_4, m_6}}[\varphi](\gamma\gamma_0 g)$

Proof:

Step 1: Conjugation (as before)

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{m_2}{m_1} & 1 & 0 \\ 0 & -\frac{m_3}{m_1} & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Q}).$$

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \gamma_0 g\right) \mathbf{e}(m_1 x_1)\, d^3 x$$

Core strategy example

Theorem $$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4, m_6 \in \mathbb{Q}} \sum_{\gamma \in \Gamma_4} \mathcal{W}_{\psi_{m_1, m_4, m_6}}[\varphi](\gamma\gamma_0 g)$$

Proof:

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$$\mathcal{F}_{U,\psi}[\varphi](g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^3} \varphi\Bigg(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\gamma_0 g\Bigg)\mathbf{e}(m_1 x_1)\, d^3x$$

Step 2: Further expansion along next row

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4, m_5 \in \mathbb{Q}} \int_{(\mathbb{Q}\backslash\mathbb{A})^5} \varphi\Bigg(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\gamma_0 g\Bigg)\mathbf{e}(m_1 x_1 + m_4 x_4 + m_5 x_5)\, d^5x$$

Core strategy example

Step 2: Further expansion along next row

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Step 3: Conjugation for each m_4

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4\in\mathbb{Q}} \sum_{\gamma\in\Gamma_4} \int_{(\mathbb{Q}\backslash\mathbb{A})^5} \varphi\Big(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\gamma\gamma_0 g\Big)\mathbf{e}(m_1x_1{+}m_4x_4)\, d^5x$$

$\Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&-1&m \end{pmatrix} : m \in \mathbb{Q}\right\}$ if $m_4 \neq 0$. $\Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}\right\}$ otherwise.

Step 4: Expand further

Core strategy example

Step 3: Conjugation for each m_4

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4\in\mathbb{Q}} \sum_{\gamma\in\Gamma_4} \int_{(\mathbb{Q}\backslash\mathbb{A})^5} \varphi\Big(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\gamma\gamma_0 g\Big)\mathbf{e}(m_1x_1+m_4x_4)\, d^5x$$

$\Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&m\end{pmatrix} : m \in \mathbb{Q}\right\}$ if $m_4 \neq 0$. $\Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$ otherwise.

Step 4: Expand further

$$\mathcal{F}_{U,\psi}[\varphi](g) = \sum_{m_4,m_6\in\mathbb{Q}} \sum_{\gamma\in\Gamma_4} \int_{(\mathbb{Q}\backslash\mathbb{A})^6} \varphi\Big(\overbrace{\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{N}\gamma\gamma_0 g\Big)\mathbf{e}(m_1x_1+m_4x_4+m_6x_6)\, d^6x$$

$$= \sum_{m_4,m_6\in\mathbb{Q}} \sum_{\gamma\in\Gamma_4} \mathcal{W}_{\psi_{m_1,m_4,m_6}}[\varphi](\gamma\gamma_0 g)$$

Core strategy example

$$\mathcal{F}_{U,\psi}[\varphi](g) \quad = \sum_{m_4,m_6\in\mathbb{Q}}\ \sum_{\gamma\in\Gamma_4} \mathcal{W}_{\psi_{m_1,m_4,m_6}}[\varphi](\gamma\gamma_0 g)$$

□

Compare with [Piatetski-Shapiro 79, Shalika 74] for cusp form.

Other simplifications for small automorphic representations.

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{m_2}{m_1} & 1 & 0 \\ 0 & -\frac{m_3}{m_1} & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Q}).$$

$\Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&m\end{pmatrix} : m\in\mathbb{Q}\right\}$ if $m_4 \neq 0$. $\quad \Gamma_4 = \left\{\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\}$ otherwise.

Nilpotent orbits

Character ψ on $U(\mathbb{A})$ $\longrightarrow$ Nilpotent element $y \in \mathfrak{g}(\mathbb{Q})$

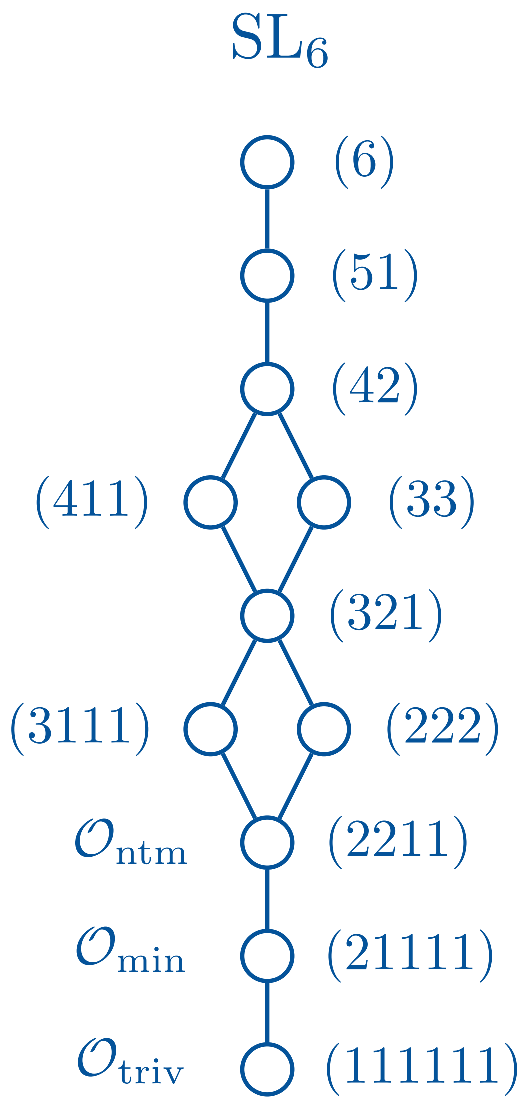
$\psi_y(u) = \mathbf{e}(\langle y, \log u \rangle)$ $\langle \cdot, \cdot \rangle$ Killing form

$$\mathcal{F}_{U,\psi_y}[\varphi](g) = \mathcal{F}_{\gamma U \gamma^{-1}, \psi_{\gamma y \gamma^{-1}}}[\varphi](\gamma g) \qquad \gamma \in G(\mathbb{Q})$$

$$\mathcal{F}_{U,\psi_y}[\varphi] \equiv 0 \iff \mathcal{F}_{\gamma U \gamma^{-1}, \psi_{\gamma y \gamma^{-1}}}[\varphi] \equiv 0$$

Nilpotent orbit $\mathcal{O}_y = \{\gamma y \gamma^{-1} : \gamma \in G(\underset{\mathbb{C}}{\cancel{\mathbb{Q}}})\}$

Partial ordering



Small automorphic representations

Size of orbit ⟷ Size of automorphic representation

An automorphic representation π is characterized by a set of nilpotent orbits $\mathrm{WF}(\pi)$ called its wave-front set.

Minimal automorphic representation:

$\mathrm{WF}(\pi_{\mathrm{min}})$ contains $\mathcal{O}_{\mathrm{min}}$ but no larger orbit.

Next-to-minimal automorphic representation:

$\mathrm{WF}(\pi_{\mathrm{ntm}})$ contains $\mathcal{O}_{\mathrm{ntm}}$ but no larger orbit.

If $\mathcal{O}_y \notin \mathrm{WF}(\pi)$ then $\mathcal{F}_{U,\psi_y}[\varphi] \equiv 0$ for $\varphi \in \pi$ [Gomez–Gourevitch–Sahi 17]

(Similar local statements by Matumoto and Mœglin–Waldspurger)

Small automorphic representations

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Small automorphic representations

↓

Defining property:

Few non-vanishing Fourier coefficients

Small automorphic representations

Defining property:

Few non-vanishing Fourier coefficients

SL_4 example: Whittaker coefficients specified by character

$$\psi_{m_1,m_2,m_3}\left(\begin{pmatrix} 1 & x_1 & * & * \\ 0 & 1 & x_2 & * \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \mathbf{e}(m_1x_1 + m_2x_2 + m_3x_3)$$

Maximally degenerate

Minimal representation: only $\mathcal{W}$ with characters $\psi_{m_1,0,0}$, $\psi_{0,m_2,0}$, $\psi_{0,0,m_3}$ survive.

Next-to-minimal representation: also $\psi_{m_1,0,m_3}$ survive.

Realizations

$$E_{\vec{s}}(g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\vec{s}}(\gamma g)$$

$$\vec{s} \in \mathbb{C}^r \longleftrightarrow \lambda = 2s_1\Lambda_1 + \ldots 2s_r\Lambda_r$$

$$\chi_{\vec{s}}(g) = \chi_{\vec{s}}(nak) = a^\lambda$$

For SL_n, $(n > 3)$:

$E_{(s,0,\ldots,0)}(g)$ is in a minimal automorphic representation.

$E_{(0,s,\ldots,0)}(g)$ is in a next-to-minimal automorphic representation.

For E_6, E_7, E_8:

$E_{(3/2,0,\ldots,0)}(g)$ is in a minimal automorphic representation.

$E_{(5/2,0,\ldots,0)}(g)$ is in a next-to-minimal automorphic representation.

[Fleig–HG–Kleinschmidt–Persson 18, Table 6.2]

BPS-orbits and character orbits

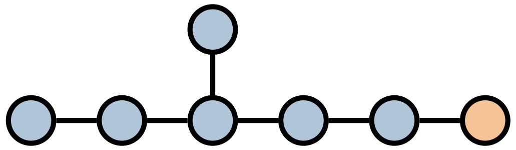
$$\mathcal{F}_{U,\psi_y}[\varphi](g) = \mathcal{F}_{\gamma U \gamma^{-1},\psi_{\gamma y \gamma^{-1}}}[\varphi](\gamma g) = \mathcal{F}_{U,\psi_{\gamma y \gamma^{-1}}}[\varphi](\gamma g)$$

$P = \boxed{L}\boxed{U}$ $\gamma U \gamma^{-1} = U$ for $\gamma \in L(\mathbb{Q})$

Character orbit

- Decompactification limit

Higher dimensional BPS states, black holes



$G = E_7,\ L \cong \mathrm{GL}_1 \times E_6$

E_7 automorphic form in 4-dimensional scattering amplitudes.

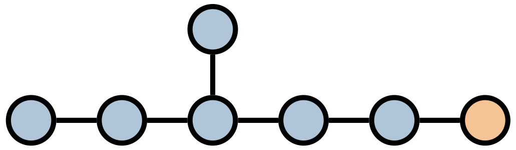
E_6 parabolic contains information about particle states in 5-dimensions that are protected supersymmetry: BPS states.

(Short representations of a $\mathbb{Z}_2$-graded algebra.)

BPS-orbits and character orbits

- Decompactification limit

  Higher dimensional BPS states, black holes



$G = E_7, \; L \cong \mathrm{GL}_1 \times E_6$

BPS states specified by 27 electromagnetic charges. ⟷ Fourier modes on U (characters) are specified by 27 integers.

Classified by their $(\mathrm{GL}_1 \times E_6)$-orbits:

With L acting in the same way.

Name	Dimension	G-orbit	Vanishes for:
$\frac{1}{8}$-BPS	27	$\mathcal{O}_{(3A_1)''}$	$\pi_{\mathrm{triv}}, \pi_{\mathrm{min}}, \pi_{\mathrm{ntm}}$
$\frac{1}{4}$-BPS	26	$\mathcal{O}_{\mathrm{ntm}}$	$\pi_{\mathrm{triv}}, \pi_{\mathrm{min}}$
$\frac{1}{2}$-BPS	17	$\mathcal{O}_{\mathrm{min}}$	π_{triv}

Vanishing properties from intersection with G-orbits in $\mathrm{WF}(\pi)$

Reduction principle

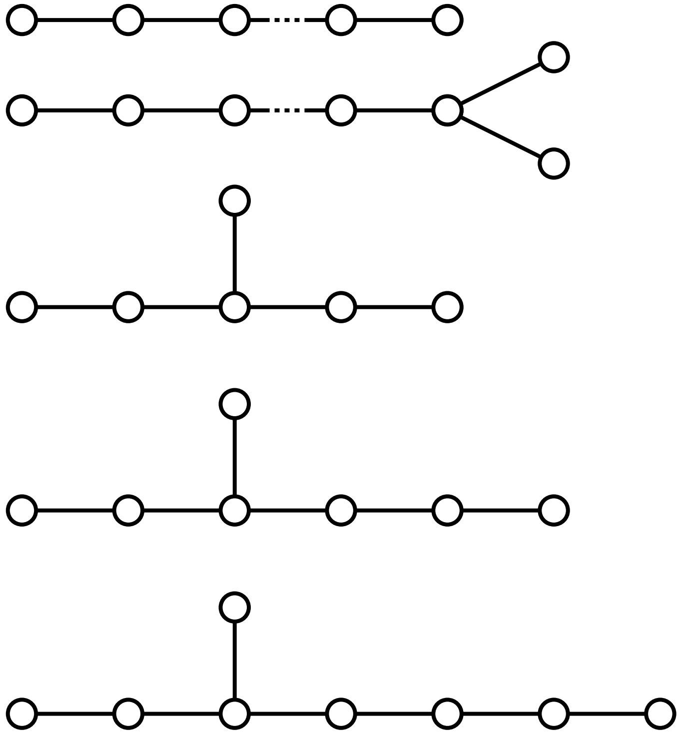
When possible, write parabolic Fourier coefficient in terms of Whittaker coefficients.

$\mathcal{F}$ $\mathcal{W}$

Theorem I [Gourevitch–HG–Kleinschmidt–Persson–Sahi 18].

This is possible for automorphic forms in minimal and next-to-minimal automorphic representations of simply-laced groups.

For other cases we give the coefficients that would replace $\mathcal{W}$ in this statement: so called Levi-distinguished coefficients.



Any number field, any central extension of reductive group, any representation.

Explicit formulas

Theorem II [Gourevitch–HG–Kleinschmidt–Persson–Sahi 19].

Formulas for expressing maximal parabolic Fourier coefficients, and φ itself, in terms of Whittaker coefficients for minimal and next-to-minimal representations of simply-laced groups.

Example $G = \mathrm{SO}_{5,5}$:

φ next-to-minimal, U_{α_1} analogous to first row

Character $\psi = \psi_y$ with $y \in \mathfrak{g}^{\times}_{-\alpha_1}(\mathbb{Q})$ in a minimal orbit.

$$\mathcal{F}_{U_{\alpha_1},\psi}[\varphi](g) = \mathcal{W}_{\psi}[\varphi](g) + \sum_{i=3}^{5} \sum_{\gamma \in \Gamma_i} \sum_{y' \in \mathfrak{g}^{\times}_{-\alpha_i}(\mathbb{Q})} \mathcal{W}_{\psi_{y+y'}}[\varphi](\gamma g)$$

Maximally degenerate

Certain coset representatives in $G(\mathbb{Q})$ specified in paper.

Summary

- Solved Jacobi's four-squares-problem by Fourier expanding modular forms
- From modular forms to automorphic forms
- How to Fourier expand periodic functions on groups
- Small automorphic representations have few non-vanishing Fourier coefficients
- Write parabolic Fourier coefficients in terms of Whittaker coefficients

Thank you!

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