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Lie Group / Quantum Mathematics Seminar

# Multiple Dirichlet series — Whittaker functions — Lattice models

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Based on work joint with
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# 1. Multiple Dirichlet series

↑ variable

Single variable: $\sum_{n=1}^{\infty} a_n n^{-s}$ $\quad a_n, s \in \mathbb{C}$

Ex: Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

- Converges for $Re(s) > 1$
- Functional eq. $\xi(s) = \xi(1-s)$ $\quad \xi(s) = \pi^{-s/2}\Gamma(s)\zeta(s)$

↳ meromorphic continuation

Multiple Dirichlet series:

$$Z(\vec{s}) = \sum_{n_1 \dots n_r} A_{n_1,\dots n_r} n_1^{-s_1} \cdots n_r^{-s_r} \qquad \vec{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$$

Ex: Moments of quadratic Dirichlet L-functions

Quadratic field $\mathbb{Q}(\sqrt{d})$

$$Z(\vec{s}) = \sum_{d} L(s_1, \chi_d) \cdots L(s_{r-1}, \chi_d)\, d^{-s_r}$$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$\chi_d$ quadratic charcter associated to $\mathbb{Q}(\sqrt{d})$

Goldfeld-Hoffstein 85, Diaconu-Goldfeld-Hoffstein 03

Fisher-Friedberg 03, 04

**Goal:** Define families of multiple Dirichlet series that can be meromorphically continued and capture many known special cases in number theory (including above example).

**Some observed common properties:**

- Coefficients $A_{n_1 \ldots n_r}$ should satisfy certain "twisted" multiplicativity properties
- Coefficients contain Gauss sums ($n$-th order)

Hint for "metaplectic covers".

## Meromorphic continuation

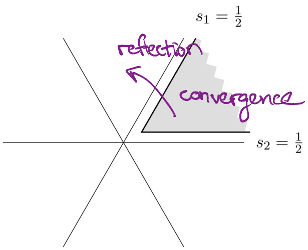
$Z(\vec{s})$ has some domain of convergence

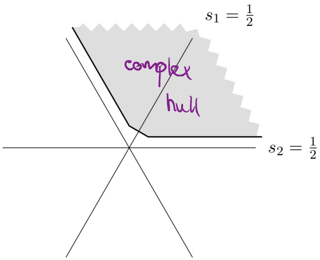
<u>Want</u>: $A_{n_1 \sim n_r}$ such that $Z(\vec{s})$ satisfies appropriate functional equations

## Weyl group multiple Dirichlet series:

Group of functional equations $\cong$ Weyl group $W$ of some root sys. of rank $r$

Ex: $A_2$




Meromorphic continuation to complex hull: Bochner 38

$\Longrightarrow$ $A_{n_1 - n_r}$ defined using a $W$-action on rational functions

Chinta-Gunnels action

# Defining the coefficients

<u>Setup</u>: $k$ algebraic number field $\supset \mu_n$ ← $n$-th roots of unity

For convenience: $\mu_{2n} \subset k$ so that $-1$ is an $n$-th power

Fix finite set of places $S \supset \{$ all archimedean, those ramified $/\mathbb{Q}\}$ and large enough so $\mathcal{O}_S := \{x \in k : |x|_\nu \leq 1 \text{ for all } \nu \notin S\}$ is a principal ideal domain. ← unique prime factorization

<u>Data</u> defining Weyl group multiple Dirichlet series of order $n$

- Reduced root system of rank $r$
- $\vec{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$
- "twisting parameters" $\vec{m} \in (\mathcal{O}_S)^r$
- $\Psi$ in a finite dimensional vector space of functions $(k_S^\times)^r \to \mathbb{C}$ with certain transformation properties.

will not be important

to be defined

$|c_i| = |\mathcal{O}_S / c_i \mathcal{O}_S|$

$$Z(\vec{s}, \vec{m}, \Psi) = \sum_{\vec{c}} \underline{H(\vec{c}; \vec{m}) \Psi(\vec{c})} \, |c_1|^{-2s_1} \cdots |c_r|^{-2s_r}$$

$\vec{c} = (c_1, \ldots, c_r)$

$c_i$ is summed over $(\mathcal{O}_S \setminus \{0\}) / \mathcal{O}_S^\times$

Twisted multiplicativity:

If $\gcd(c_1 \cdots c_r, c_1' \cdots c_r') = 1$ then

$$H(c_1c_1', \dots, c_rc_r'; \vec{m}) = \varepsilon_{\vec{c},\vec{c}'} \cdot H(\vec{c}; \vec{m})\, H(\vec{c}'; \vec{m})$$

explicit product of $n$-th power residue symbols

Similar property for $\vec{m}$.

Remains to define $H$ for $\vec{c} = (p^{k_1}, \dots, p^{k_r})$, $\vec{m} = (p^{l_1}, \dots, p^{l_r})$

$p$ prime

Create generating series:

$$\sum_{k_i = 0}^{\infty} H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})\, |p|^{-2(k_1 s_1 + \dots + k_r s_r)}$$

called $p$-part and are related to metaplectic Whittaker functions

Several ways to define the p-parts

Developed in parallel

- Weyl group action average on rational functions Chinta-Gunnels 10
- Sums over crystal bases or Gelfand-Tsetlin patterns (type A)

  Brubaker-Bump-Friedberg 11a

  Two versions: $\Gamma$ and $\Delta$

  summing over different patterns weighted differently.

$$\left\{ \begin{matrix} a_{11} & & a_{12} & & a_{13} \\ & a_{22} & & a_{23} & \\ & & a_{33} & & \end{matrix} \right\}$$

$a_{ij} \geq a_{i+1,j+1} \geq a_{i,j+1}$

- p-adic Whittaker functions on metaplectic covers BBF 11a, Chinta-Offen 13, McNamara 11,16

  Global Whittaker coefficient of Eisenstein series on metaplectic covers of $GL_{r+1}$ = Type A Weyl group multiple Dirichlet series BBF 11a

- Partition functions of lattice models (type A & C)

  Brubaker-Bump-Chinta-Friedberg-Gunnels 12, Ivanov 12

  Gelfand-Tsetlin patterns $\longleftrightarrow$ lattice model states

At the beginning it was not clear if the different methods produced the same objects.

In fact, a major part of the book Brubaker-Bump-Friedberg 11b is devoted to a combinatorial proof of $T = \Delta$

Reinterpreting in terms of lattice models this can instead be proven with a simple application of a Yang-Baxter equation. Brubaker-Buciumas-Bump-Gray 19

At the end everything was nicely tied tied together by McNamara 11,16 showing that they all agree with metaplectic spherical Whittaker functions.

## 2. Whittaker functions

to start with: non-metaplectic

$F$ non-archimedean local field $\quad \mathbb{Q}_p$

$\mathcal{O}$ ring of integers $\quad \mathbb{Z}_p$

$\mathfrak{p}$ maximal ideal with generator $p$ $\quad p\mathbb{Z}_p$

We will for simplicity restrict to $G = GL_r(F)$
but similar for any reductive group.

$T$ diagonal matrices

$B$ upper triangular

$N$ —"— with unit diagonal

$B_-$, $N_-$ lower

A space of Whittaker functions is determined by

- Representation $\pi$
- Whittaker functional $\Omega : \pi \longrightarrow \mathbb{C}$ with character $\psi : N_- \longrightarrow \mathbb{C}$ such that

$$\Omega(\pi(n) \cdot f) = \psi(n)\, \Omega(f) \quad f \in \pi \quad n \in N_-$$

We will consider subspaces of the principal series representation $I(z) := \mathrm{Ind}_B^G(\chi_z)$

$z \in (\mathbb{C}^\times)^r$

character on $T$ specified by $z$

- Spherical vectors: $I(z)^K$ ← right-invariant under $K = GL_r(\mathcal{O})$

1-dimensional

- Iwahori fixed vectors: $I(z)^J$ ← $J \subset K$ lower triangular mod $\mathfrak{p}$

Basis $\Phi_w^z$ enumerated by Weyl group $W$

$$\sum_w \Phi_w = \text{spherical}$$

We will take Whittaker functional with $\psi$ such that $\psi(n) = \chi_F\left(\sum_{i=1}^{r-1} n_{i+1,i}\right)$ ← non-degenerate

where $\chi_F : F \to \mathbb{C}^\times$ is some fixed character trivial on $\mathcal{O}$ but no larger fractional ideal.

$$\Omega(f) = \int_{N_-} f(n)\, \psi(n)^{-1}\, dn$$

Whittaker function

$$\phi_w(z;g) = \Omega\left(\pi(g)\cdot\Phi_w^{z^{-1}}\right) \quad \times \text{ normalization}$$

Determined by values at $g = \begin{pmatrix} p^{\lambda_1} & & & \\ & p^{\lambda_2} & & \\ & & \ddots & \\ & & & p^{\lambda_r} \end{pmatrix} w'$ ⟶ 1 for spherical

$\underbrace{\qquad\qquad}_{p^\lambda}$ $\lambda$ partition

Recursion relations in terms of $\ell(w)$

Intertwining integral $A_w : I(z) \longrightarrow I(wz)$

$$(A_w f)(g) = \int_{N \cap wN_-w^{-1}} f(w^{-1}ng)\,dn$$

- $\Omega A_{s_i} = (\text{scalar})\ \Omega$ Casselman-Shalika 80
- $A_{s_i}\Phi_w^z = \sum_{w'} (\text{matrix})_{ww'}\ \Phi_{w'}^z$ Casselman 80

↳ R-matrix for $U_q(\widehat{gl}(r))$

Leads to $\phi_{s_iw}(z;g) = T_i^{\pm 1}\ \phi_w(z;g)$

↳ Demazure type (divided difference) operators on rational functions in $z_i$

Brubaker-Bump-Licata 15

Brubaker-Buciumas-Bump-HG (19) Finite Iwahori Hecke alg

# Metaplectic case – distilled version

Assume $\mu_{2n} \subset F$. The metaplectic $n$-fold cover of $G$ is defined by a central extension

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \xrightarrow{pr} G \longrightarrow 1$$

Let $\tilde{T} = pr^{-1}(T)$.

not abelian

$$[t_1, t_2] \in \mu_n \qquad t_1, t_2 \in \tilde{T}$$

We pick a convenient cover such that the group commutator has a nice form.

$T$ abelian $\rightarrow$ irreducible repr. $\cong$ characters $\chi_z$ 1 dim

which we then extend to $B$ and induce to $G$

$I(z) = \mathrm{Ind}_B^G(\chi_z)$

The irreducible representations of $\tilde{T}$ are $n^r$ - dimensional

$\Longrightarrow$ Principal series is vector valued

This means that the Whittaker functional $\Omega : I(z) \longrightarrow \mathbb{C}$ is not only an integral

$$\int_{N_-} f(ng)\,\psi(n)^{-1}\,dn$$

but must also project to a vector-component of $f$ denoted by $\Omega_\mu$. $\mu \in \Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^r$

We get similar recursion relations using

R-matrix for $U_q(\hat{\mathfrak{gl}}(n))$

- $\Omega_\mu A_{s_i} = \sum_\nu (\text{matrix})_{\mu,\nu}\, \Omega_\nu$ Kazhdan-Patterson 84

- $A_{s_i} \Phi_w = \sum_{w'} (\text{matrix})_{w,w'}\, \Phi_{w'}$

R-matrix for $U_q(\hat{\mathfrak{gl}}(r))$

$U_q(\hat{\mathfrak{gl}}(n))$ R-matrix is a Drinfeld twist which introduces Gauss sums

See Brubaker-Buciumas-Bump-Gray 19
Brubaker-Buciumas-Bump-HG (20)

# Dictionary

| projected component of Metaplectic Whittaker function | p-part for Weyl group multiple Dirichlet series |
|---|---|
| $\chi_z$ $\quad z \in (\mathbb{C}^\times)^r$ | $\vec{s}$ $\quad s_1, \ldots, s_{r-1}$ |
| $g = \begin{pmatrix} p^{\lambda_1} & & & \\ & p^{\lambda_2} & & \\ & & \ddots & \\ & & & p^{\lambda_r} \end{pmatrix}$ | $\vec{m} \leftarrow$ twist |

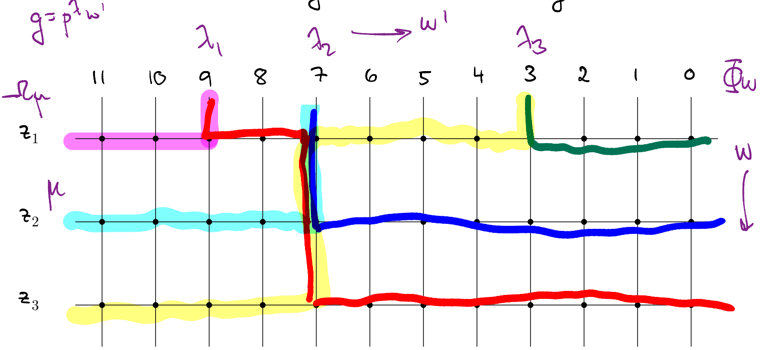
McNamara 11, 16 $\qquad$ Brubaker - Bump - Chinta - Friedberg - Gunnells 12

Chinta - Offen 13

## Lattice model (non-metaplectic)

Grid of $r$ rows and sufficiently many columns

State of model: assignment of $r$ distinct colored paths moving downwards and rightwards




$\mu \in \Lambda / n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^r$ $\quad$ $\mu = (\mu_1, \ldots, \mu_r)$

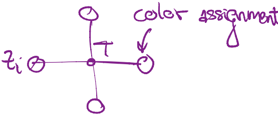
Metaplectic: add mirrored paths of $n$ "super" colors

Partition function

$$Z_{\lambda,\mu,w}(z) = \sum_{\substack{\text{states } s \text{ with} \\ \text{given boundary}}} \mathrm{weight}(s) = \phi_{\mu,w}(z; p^{\lambda} w')$$

Whittaker function

$$\mathrm{weight}(s) = \prod_{\text{vertex } T} \mathrm{weight}(s|_T)$$

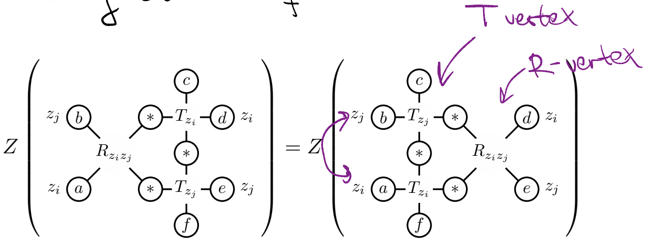



Brubaker-Buciumas-Bump-HG (19, 20)

Recursion relations = Yang-Baxter equations

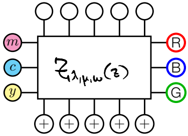
makes lattice model solvable

Yang-Baxter equation:




n super colors →

$\mu$ in $\Omega_\mu$

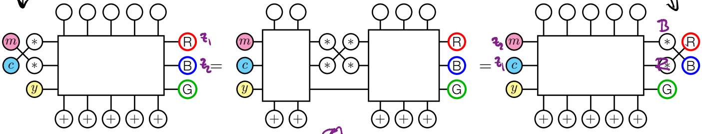



← r distinct colors

order: $w$ in $\Phi_w$

$\Omega_\mu A_{s_i}$

$A_{s_i} \Phi_w$




$U_q(\hat{gl}(n))$ $U_q(\hat{gl}(n|r))$ $U_q(\hat{gl}(r))$

## (Super) symmetry between $r$ & $n$

rank  degree of cover

colors $\longleftrightarrow$ super colors

$U_q(\hat{\mathfrak{gl}}(n|r))$

Historically: model for spherical metaplectic looked very different

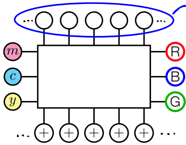
BBCFG 12

Remarkable that it could be absorbed into similar language of colored paths used in Brubaker-Bump-Buciumas-HG (19)

$q = |\mathcal{O}/\mathfrak{p}|^{\pm 1}$  $P$

Fock space interpretation:

Kashiwara-Miwa-Stern module for $U_{q^{\pm}}(\hat{\mathfrak{gl}}(n))$




element of Drinfeld twisted quantum Fockspace

only super colors

BBBG 20: spherical metaplectic

Row transfer matrix $= \exp(H)$ ← $\Sigma$ step operators

only colors

In progress: Iwahori non-metaplectic

Row transfer matrix $= \exp(H)$ on Fock space with a different Drinfeld twist.