Multiple Dirichlet series and metaplectic Whittaker functions

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Multiple Dirichlet series $\longleftrightarrow$ Metaplectic Whittaker functions $\longleftrightarrow$ Solvable lattice models

The goal of this talk is to tell you how we use p-adic representation theory to say something about number theory, and, if time permits, how we use solvable lattice models to say something about p-adic representation theory.

1 Multiple Dirichlet series

A multiple Dirichlet series is a Dirichlet series in several complex variables

$$\sum_{n_1 \ldots n_r} A_{n_1 \ldots n_r} n_1^{-s_1} \cdots n_r^{-s_r} \qquad A_{n_1 \ldots n_r}, s_1, \ldots s_r \in \mathbb{C} \tag{1}$$

converging in some domain with large enough $\mathrm{Re}(s_i)$. We are interested in the case where this Dirichlet series can be meromorphically continued to $\mathbb{C}^r$ the result of which is called an L-function. This puts some complicated restrictions on the coefficients $A_{n_1 \ldots n_r}$ and a lot of work has been put into both verifying that series appearing in various examples can be meromorphically continued as well as constructing new such multiple Dirichlet series from the ground up.

A standard way multiple Dirichlet series may appear in nature is to start with a Dirichlet series in a single variable $\sum_n a_n n^{-s}$ and let a_n itself be an L-function in another complex variable. This is useful, since using standard Tauberian arguments one can draw conclusions about a_n from the behavior of the Dirichlet series $\sum_n a_n n^{-s}$. This was used for example in [Goldfeld–Hoffstein 85] to obtain estimates for the mean central values of quadratic L-functions. See also [Diaconu–Goldfeld–Hoffstein 03] and [Fisher–Friedberg 03, 04] for higher moments. However, we run into problems with meromorphic continuation beyond the third moment if we continue in the same manner.

We therefore turn to the construction of a certain class of multiple Dirichlet series called Weyl group multiple Dirichlet series which manifestly have a meromorphic continuation. They are defined with respect to a reduced root system Φ and an integer order n. This introduction of an order n allows for example the multiple Dirichle series to be related to the distribution of n-th order twisted L-functions studied by [Friedberg–Hoffstein–Lieman 03] as shown by [Brubaker–Bump 06].

We will first start with the rank 1 case which are the Kubota Dirichlet series. For $n=2$ the Kubota Dirichlet series recover the quadratic L-functions $L\big(2s - \frac{1}{2}, \chi_d\big)$.

1.1 Kubota Dirichlet series

To define them, we first need some setup. The same setup will later be reused for multiple Dirichlet series of general rank.

Let K be a number field $\supset \mu_n$ the group of n-th roots of unity. For convenience assume also $K \supset \mu_{2n}$ so that -1 is an n-th power.

Fix finite set of places $S \supset \{\text{all archimedean and those ramified over } \mathbb{Q}\}$ large enough so $\mathcal{O}_S := \{x \in K : |x|_v \leqslant 1 \text{ for all } v \in S\}$ is a principal ideal domain.

Let ψ_K be an additive character on K which is trivial on $\mathcal{O}_S$ but no larger fractional ideal. If $a, c \in \mathcal{O}_S$ are nonzero we define the n-th order Gauss sum as these particular sums over roots of unity:

$$g(a, c) := \sum_{d \bmod c} \left(\frac{d}{c}\right)_n \psi_K\left(\frac{a\,d}{c}\right)$$

where $\left(\frac{d}{c}\right)_n$ is the n-th power residue symbol which is multiplicative in c, and for a prime p is defined as the unique n-th root of unity satisfying $\left(\frac{d}{p}\right)_n \equiv d^{(\mathrm{N}p-1)/n} \bmod p$ where $\mathrm{N}p := |\mathcal{O}_S / p\mathcal{O}_S|$ which we will sloppily denote as $|p|$ from here on.

The Kubota Dirichlet series is then for $m \in \mathbb{Z}$, $s \in \mathbb{C}$ essentially[1] defined as

$$\mathcal{D}(m; s) := \sum_{0 \neq c \in \mathcal{O}_S / \mathcal{O}_S^\times} g(m, c)\, |c|^{-2s} \qquad |c| := |\mathcal{O}_S / c\mathcal{O}_S|$$

where $g(m, c)$ are particular sums over roots of unity called Gauss sums.

[Kubota 69] showed that $\mathcal{D}(m; s)$ is the m-th Fourier coefficients of a (non-holomorphic) Eisenstein series $E(g; s)$ on the metaplectic n-fold cover of $\mathrm{SL}_2(K)$. These metaplecic covers, or central extensions, will be discussed later. As a consequence they satisfy a functional equation swapping s and $1 - s$ and can be meromorphically continued to the whole complex plane.

Unlike, for example, the Riemann zeta function, the Kubota Dirichlet series cannot be expressed as an Euler product. Its coefficients are not multiplicative but instead satisfy a *twisted multiplicativity*

$$g(m, cc') = \left(\frac{c}{c'}\right)_n \left(\frac{c'}{c}\right)_n g(m, c) g(m, c') \quad \text{if } \gcd(c, c') = 1.$$

There is a similar property for m.

1.2 A first look at Weyl group multiple Dirichlet series

We use the same number field setup as for Kubota Dirichlet series of order n.
Further input data:

- Reduced root system Φ of rank r with Weyl group W
- $\vec{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$
- Twisting parameters $\vec{m} \in (\mathcal{O}_S)^r$

Then the corresponding Weyl group multiple Dirichlet series looks like this[1]

$$Z(\vec{m}; \vec{s}) := \sum_{\vec{c}} H(\vec{c}; \vec{m}) |c_1|^{-2s_1} \cdots |c_r|^{-2s_r} \tag{2}$$

where $\vec{c} = (c_1, \ldots c_r)$ and each c_i ranges over $(\mathcal{O}_S \setminus \{0\}) / \mathcal{O}_S^\times$. It remains to define the coefficients $H(\vec{c}; \vec{m})$. We will make two requirements:

1. We need $Z(\vec{m}; \vec{s})$ to have meromorphic continuation to $\vec{s} \in \mathbb{C}^r$.

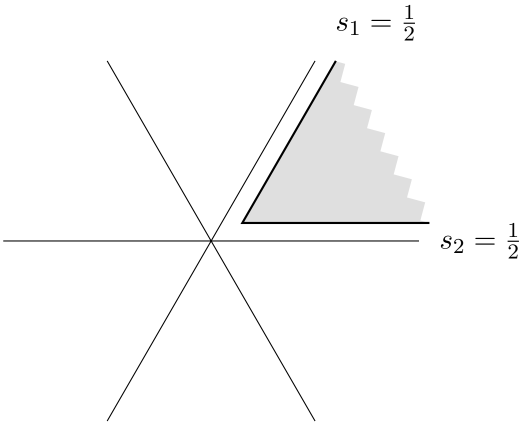
2. The coefficients $H(\vec{c}; \vec{m})$ should satisfy a twisted multiplicativity

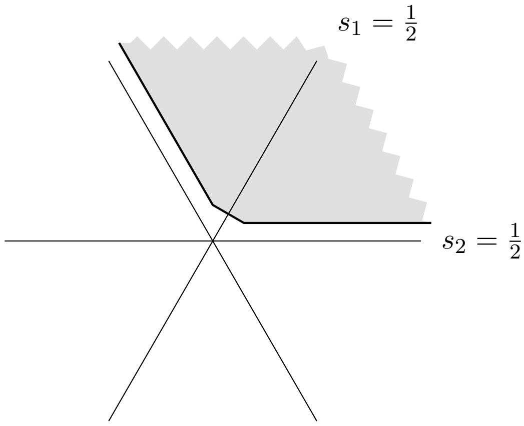
1.2.1 Meromorphic continuation

Example: $\Phi = A_2$, $r = 2$.

1. I am hiding some technicalities related to the invariance of c under units. See [Bump 12] for details.

Assume (2) is convergent the tube domain $\{\mathrm{Re}(s_1) > 1 \text{ and } \mathrm{Re}(s_2) > 1\}$ drawn suggestively like this where we project to the real parts:





We also assume that if we collect the coefficients of $|c_2|^{-s_2}$ for each c_2 we get a set of single variable Dirichlet series (e.g. Kubota Dirichlet series) which have functional equations relating them to the reflections in the critical line $s_1 = \frac{1}{2}$. Using a theorem by [Bochner 38] we can then obtain a meromorphic continuation to the complex hull.

Thus, if we have a group of functional equations which is isomorphic to the Weyl group of the root system Φ we can in this way obtain a meromorphic continuation to $\mathbb{C}^r$.

group of functional equations	$\cong$	Weyl group

1.2.2 Twisted multiplicativity

Like the Kubota Dirichlet series we will also impose twisted multiplicativity for the coefficients H. It then remains to determine the coefficients for prime powers. For a prime p we create a generating series called the ***p*-part** of the multiple Dirichlet series

$$\sum_{k_i=0}^{\infty} H(p^{k_1}, \ldots, p^{k_r}; \vec{m}) z_1^{k_1} \cdots z_r^{k_r}$$

where the z_i are formal parameters.

1.3 Different definitions of the *p*-parts

We are left with constructing such p-parts for which the multiple Dirichlet series has a meromorphic continuation using the above Weyl group method. This is a complicated combinatorial problem and several constructions were developed in parallel.

1. Averaging method using a complicated Weyl group action on rational functions $\mathbb{C}(z_1, \ldots, z_r)$ [Chinta–Gunnells 10]. For type A, $\lambda \in \mathbb{Z}^r$, $h \in \mathbb{C}(z_1^n, \ldots, z_r^n)$ and simple reflection s_i this action is

$$s_i \star \vec{z}^{\lambda} h = \frac{\vec{z}^{s_i \lambda} s_i(h)}{1 - |p|^{-1} \vec{z}^{-n\alpha_i}} ((1 - |p|^{-1}) \vec{z}^{\lfloor \lambda_i - \lambda_{i+1} \rfloor_n \alpha_i} - g(\lambda_{i+1} - \lambda_i - 1) \vec{z}^{(n-1)\alpha_i} (1 - \vec{z}^{-n\alpha_i}))$$

where $g(a) := |p|^{-a} g(p^{a-1}, p^a)$. The expression follows from working backwards from the imposed functional equations and twisted multiplicativity, but it took quite some work to show that it actually forms a W-action.

2. Sums over a Kashiwara–Luzstig crystal basis or over Gelfand–Tsetlin patterns. For type A [Brubaker–Bump–Friedberg 11a], for other cases see [Bump–Friedberg–Goldfeld 12]. In fact there were two versions called Γ and Δ which sum over different Gelfand–Tsetlin patterns weighted differently.

3. Partition functions of solvable lattice models (mainly type A and C) [Brubaker–Bump–Chinta–Friedberg–Gunnells 12, Ivanov 12].

$$\text{Gelfand–Tsetlin patterns} \longleftrightarrow \text{lattice model configurations}$$

4. p-adic spherical Whittaker functions on the metaplectic n-fold cover of a split, simple group G whose coroots form the reduced root system Φ [Brubaker–Bump–Friedberg 11a, Chinta–Offen 13, McNamara 11,16].

   The whole Weyl group multiple Dirichlet series has long been expected to be global Whittaker coefficients of Eisenstein series on these covers. This was shown for type A in [BBF 11a] by induction on rows. Note that these Whittaker coefficients satisfy the same twisted multiplicativity because of the metaplectic cover.

At the beginning it was not at all clear if the different methods produced the same objects. In fact, a major part of a book by [Brubaker–Bump–Friedberg 11b] is devoted to a combinatorial proof of the fact that the two crystal bases versions Γ and Δ agree. A later reinterpretation in terms of lattice models shortened the proof to a simple application of a Yang–Baxter equation.

Historically, 1 and 2 were developed simultaneously and 3 has a natural connection to 2. The relation to Whittaker functions 4 tied everything together.

$$\boxed{1} \longleftrightarrow \boxed{4} \longleftrightarrow \boxed{2} \longleftrightarrow \boxed{3}$$

2 Whittaker functions

I will now discuss some of my recent research on metaplectic Whittaker functions which is joint work with Brubaker, Buciumas and Bump. Our representation theory results are for any reductive group G. but we also have some results connecting to solvable lattice models which is for GL_{r+1} only. For simplicity, I will focus only on GL_{r+1} here as well.

Whittaker functions are a useful tool to realize representations in a concrete space with some desirable properties.

2.1 Non-metaplectic case

Let F be a non-archimedean local field, for example corresponding to a non-archimedean place of the number field K from before. Let $\mathfrak{o}$ be the ring of integers, $\mathfrak{p}$ the maximal ideal with generator p a prime, and let $\psi_F: F \to \mathbb{C}^\times$ be some fixed additive character on F trivial on $\mathfrak{o}$ but no larger fractional ideal. Let T denote the diagonal matrices of $G = \mathrm{GL}_{r+1}(F)$. Denote by B the upper triangular matrices, N the subgroup of B with unit diagonal, and similarly N_- for the lower triangular version.

A space of Whittaker functions is determined by

- Representation π of G. Here: principal series representation which induces a representation (i.e. a character) on T to G parametrized by $\vec{z} \in (\mathbb{C}^\times)^{r+1}$. They are functions $f: G \to \mathbb{C}$.

- Whittaker functional $\Omega: \pi \to \mathbb{C}$ such that for $f \in \pi$ and $n \in N_-$: $\quad \Omega(\pi(n) \cdot f) = \psi(n)\Omega(f)$ where $\psi(n) = \psi_F(\sum_{i=1}^r n_{i+1,i})$. Here: $\Omega(f) = \int_{N_-} f(n)\psi(n)^{-1} dn$ for $f \in I(\vec{z})$.

Rather we will consider two subspaces of $I(\vec{z})$.

1. $I(\vec{z})^K$ the subspace of spherical vectors in $I(\vec{z})$, that is functions which are right-invariant under $K := \mathrm{GL}_{r+1}(\mathfrak{o})$. This space is one-dimensional and we will denote a basis element with convenient normalization by $\Phi_\circ^{\vec{z}}: G \to \mathbb{C}$.

2. $I(\vec{z})^J$ which are right-invariant under the Iwahori subgroup $J \subset K$ consisting of matrices which are lower triangular mod p. There is a basis enumerated by W which decomposes $\Phi_\circ^{\vec{z}}$ into functions supported on different Bruhat cells.

$$\Phi_w^{\vec{z}}(b w' j) = \begin{cases} \Phi_\circ^{\vec{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases} \qquad b \in B, j \in J \qquad \sum_{w \in W} \Phi_w^{\vec{z}} = \Phi_\circ^{\vec{z}}$$

It is the spherical Whittaker functions (although for the metaplectic case) that are connected to multiple Dirichet series, but the richer structure for the Iwahori Whittaker functions make them easier to compute. We can then recover the spherical one by summation.

The Iwahori Whittaker functions we are interested in are thus (suppressing some normalization)

$$\phi_w(g; \vec{z}) := \Omega(\pi(g) \cdot \Phi_w^{\vec{z}}) = \int_{N_-} \Phi_w^{\vec{z}}(n g) \psi(n)^{-1} dn$$

2.2 Demazure-like operators

Following [Brubaker–Bump–Licata 15], a convenient way to compute these Iwahori Whittaker functions is via recursion in the length of w using particular intertwining operators $\mathcal{A}_i: I(\vec{z})^J \to I(s_i \vec{z})^J$, $f \mapsto \int_{N \cap w N_- w^{-1}} f(w^{-1} n g) dn$. These are G-homomorphism so

$$(\Omega \mathcal{A}_i)(\pi(g) \cdot \Phi_w^{\vec{z}}) = \Omega(\pi(g) \cdot (\mathcal{A}_i \Phi_w^{\vec{z}}))$$

and it is known how $\mathcal{A}_i$ interacts with Ω [Casselman–Shalika 80] and the basis $\Phi_w^{\vec{z}}$ [Casselman 80] (the latter of which gives rise to a $\Phi_{s_i w}^{\vec{z}}$). Solving for the corresponding Whittaker function $\phi_{s_i w}(g; \vec{z}) := \Omega(\pi(g) \cdot \Phi_{s_i w}^{\vec{z}})$ one gets that

$$\phi_{s_i w} = T_i \, \phi_w \qquad \text{for } s_i w > w$$

where the operators T_i are defined as

$$T_i = \frac{1 - s_i - q^{-1}(1 - \vec{z}^{-\alpha_i} s_i)}{\vec{z}^{\alpha_i} - 1} \qquad q := |\mathfrak{o} / \mathfrak{p}|$$

and form a finite Iwahori Hecke algebra.

Using this we computed all values of the non-metaplecic Iwahori and parahoric Whittaker functions in [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]. Previously only values for g on the torus had been computed.

2.3 Metaplectic case – a distilled version

Assume that F contains the group of n-th roots of unity μ_n (or rather μ_{2n} for simplicity). The metaplectic n-fold cover $\tilde{G}$ of G is defined by a central extension

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \xrightarrow{\mathrm{pr}} G \longrightarrow 1$$

Let $\tilde{T} = \mathrm{pr}^{-1}(T)$. This is no longer abelian and its irreducible representations are n^r-dimensional. Since the principal series representation $I(\vec{z})$ was an induced representation from the torus, the metaplectic version of $\Phi_w^{\vec{z}}$ becomes vector valued with n^r components which we label by μ.

Thus, the metaplectic version of the Whittaker functional $\Omega: I(\vec{z}) \to \mathbb{C}$ must now also include a projection to one such component μ. To obtain the Whittaker function used for the multiple Dirichlet series one takes the average over μ.

2.3.1 Results

In [Brubaker–Buciumas–Bump-HG arXiv:2012.15778] we show the following:

1. Using the same method as above with the intertwining operators, we obtain vector-valued metaplectic Demazure operators and **compute all values for the metaplectic Iwahori Whittaker functions for all μ**. Previous work had only focused on the μ-average, as well as arguments g on the torus.

2. The corresponding metaplectic vector Demazure operators form an affine metaplectic[2] Hecke algebra and the μ-averaged operator **recovers the Chinta–Gunnells action from p-adic representation theory in a natural way**.

   This was achieved by comparing the μ-averged Demazure operator with the one in [Chinta–Gunnells–Puskás 17, Patnaik–Puskás 17] obtained by starting from the Chinta–Gunnells action.

   For another recent (non p-adic) representation-theoretic explanation of the Chinta–Gunnells action coming from the double affine Hecke algebra see [Sahi–Stokman–Venkateswaran arXiv:1808.01069].

3. **The metaplectic Iwahori Whittaker functions are equal to the partition functions for a solvable lattice model** constructed in the same paper with corresponding boundary conditions. Furthermore, there is a **bijection between the data determining the Whittaker functions and the boundary conditions** for the lattice model.

   The lattice model configurations consist of colored paths in a grid following certain rules. There are two types of palettes of colors: one associated to the Iwahori basis Φ_w and one to the metaplectic vector component μ. We call these colors and super colors.

   The lattice model seems to exhibit a duality between colors and super colors which we are currently investigating. The way they appear in the lattice model hints of a duality between the cover degree n and the rank r.

Summarize:

- We have seen how representation theory explains the form of the Weyl group multiple Dirichlet series via metaplectic Whittaker functions

- Solvable lattice models are useful for computing and maniuplating Whittaker functions

- Hints of a duality between the cover degree n and the rank r.

Iwahori and parahoric Whittaker functions arXiv:1906.04140

Metaplectic Iwahori Whittaker functions: arXiv:2012.15778

2. The Bernstein relations conerning T_i and shifts in $\vec{z}$ are modified with some powers of n.