Bridges between lattice models, representation theory and number theory

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New connections in number theory and physics

Isaac Newton Institute for Mathematical Sciences

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Slides available at https://hgustafsson.se

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Papers

Joint work with Ben Brubaker, Valentin Buciumas and Daniel Bump

0. Vertex operators, solvable lattice models and metaplectic Whittaker functions
   Communications in Mathematical Physics 380 (Dec, 2020), 535–579
1. Colored five-vertex models and Demazure atoms
   Journal of Combinatorial Theory, Series A 178 (Feb, 2021)
2. Colored vertex models and Iwahori Whittaker functions
   arXiv:1906.04140
3. Metaplectic Iwahori Whittaker functions and supersymmetric lattice models
   arXiv:2012.15778
4. Iwahori-metaplectic duality (recently updated)
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Include both pure representation theoretical and lattice model results

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focus today

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- Generator of ideas and conjectures.

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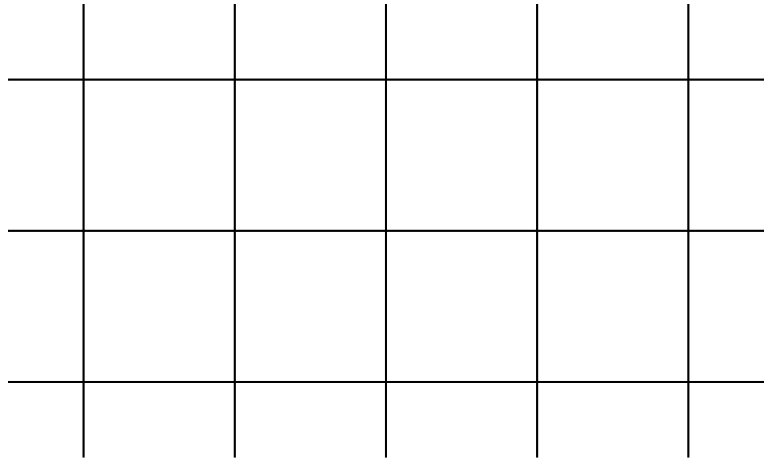
The lattice model consists of a two-dimensional grid with r rows, sufficiently many columns, and each vertex has four adjacent edges.



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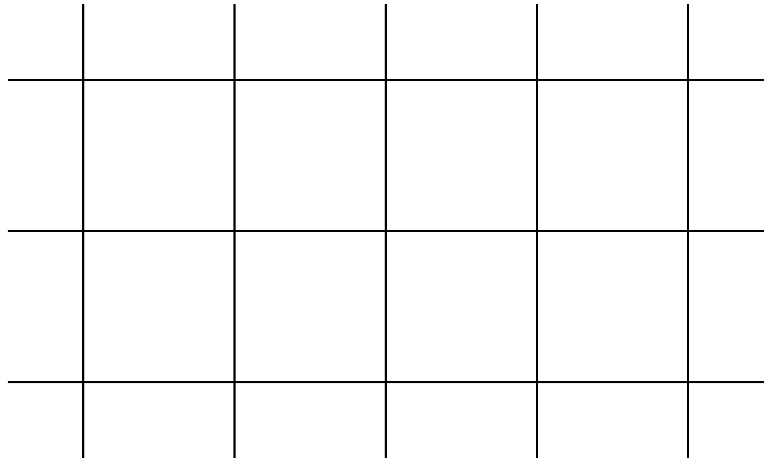
We will assign data to these edges according to certain rules, and in this first example the data is binary: the edge is **filled in**, or not.



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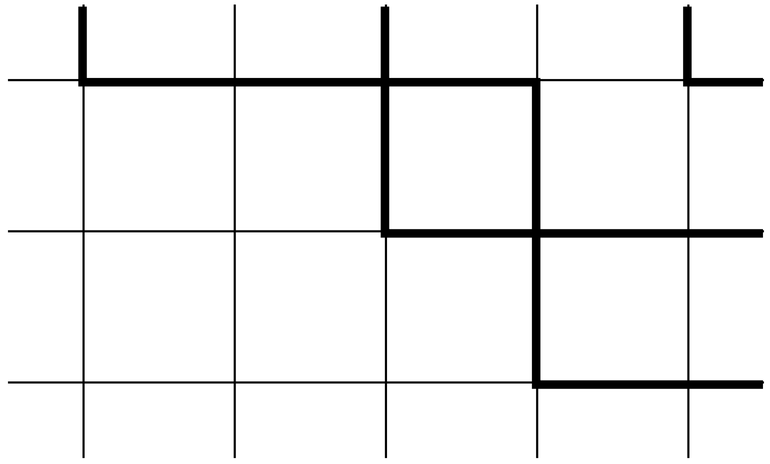

These edges will form paths on the grid, and for given boundary conditions there is a finite number of configurations called states.

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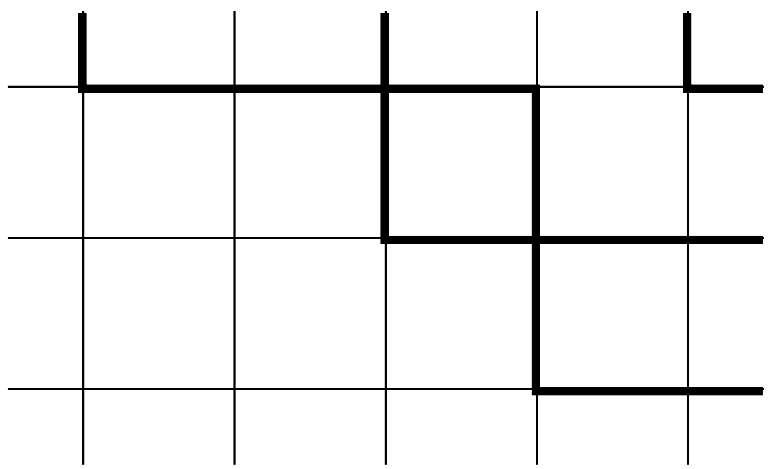

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A state $\mathfrak{s}$ is assigned a Boltzmann weight $\beta(\mathfrak{s}) \in \mathbb{C}[\mathbf{z}]$ depending on parameters $\mathbf{z} = (z_1, z_2, \ldots, z_r) \in \mathbb{C}^r$ (one for each row).

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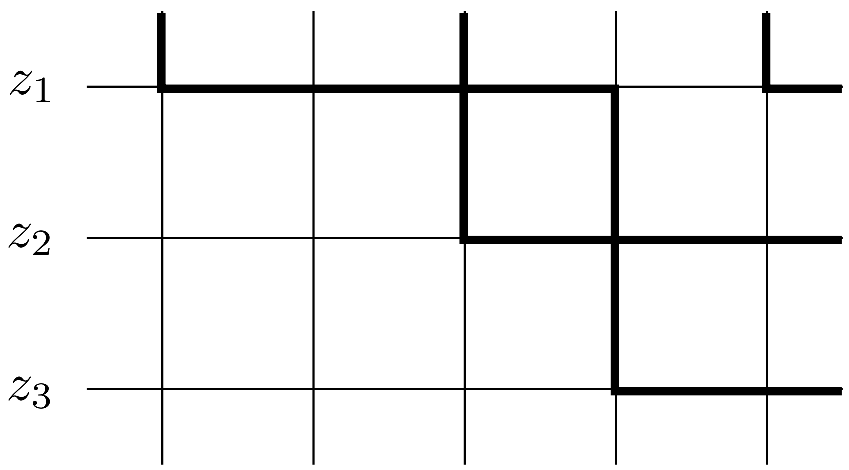

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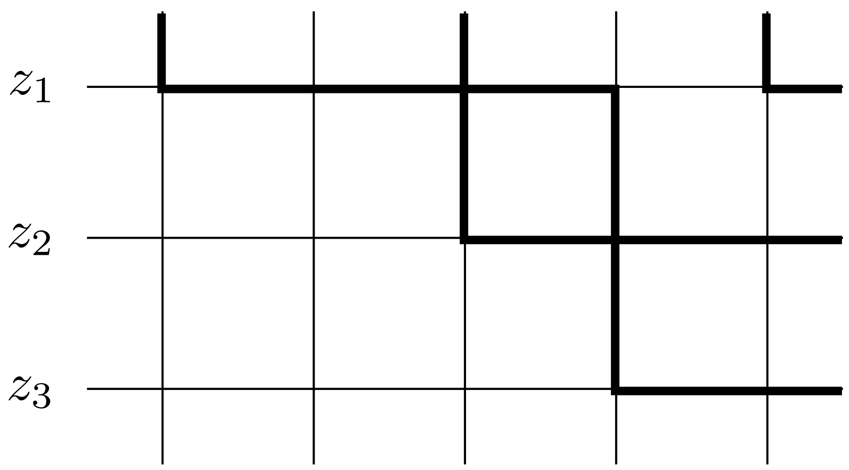

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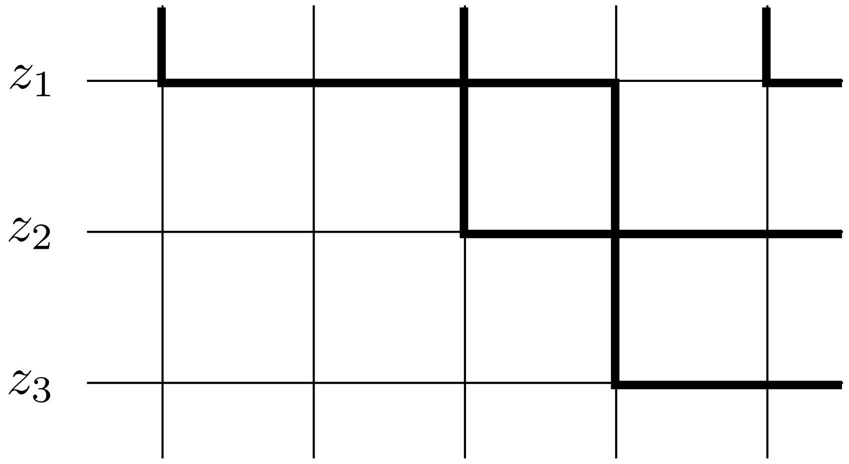

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Goal: any Schur polynomial in $\mathbf{z}$ = such a partition function.

Schur polynomials

Let $\lambda = (\lambda_1, \dots \lambda_r)$ be a partition of r padded with zeroes to length r. We define the Schur polynomial $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by

$$s_\lambda(\mathbf{z}) = \frac{\det(z_i^{(\lambda+\rho)_j})_{ij}}{\det(z_i^{\rho_j})_{ij}}$$

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$\lambda = (3, 1, 1)$ $\quad$ $\mathrm{SSYT}(\lambda) \ni T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$ $\quad$ $\mathrm{wt}(T) = (2, 2, 0, 0, 1)$

(#ones, #twos, #threes, ...)

Lattice paths

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(certain)

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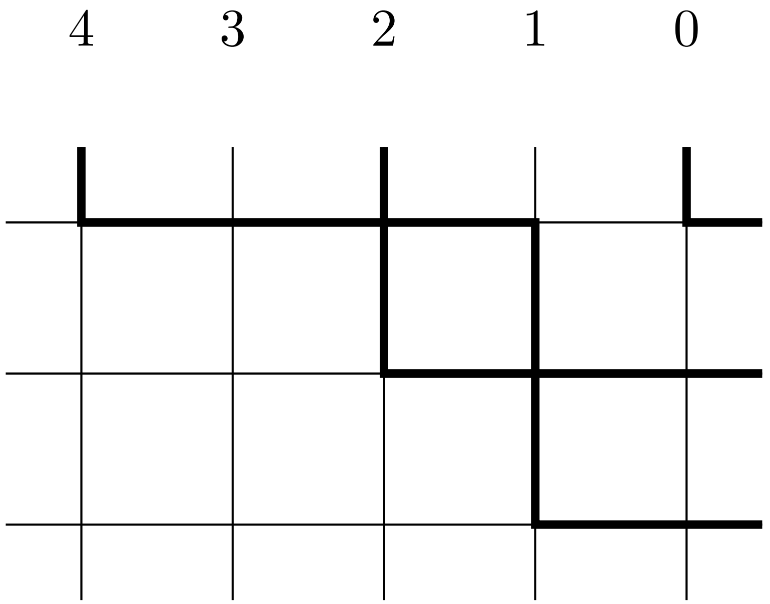

$\lambda \vdash r$

r rows

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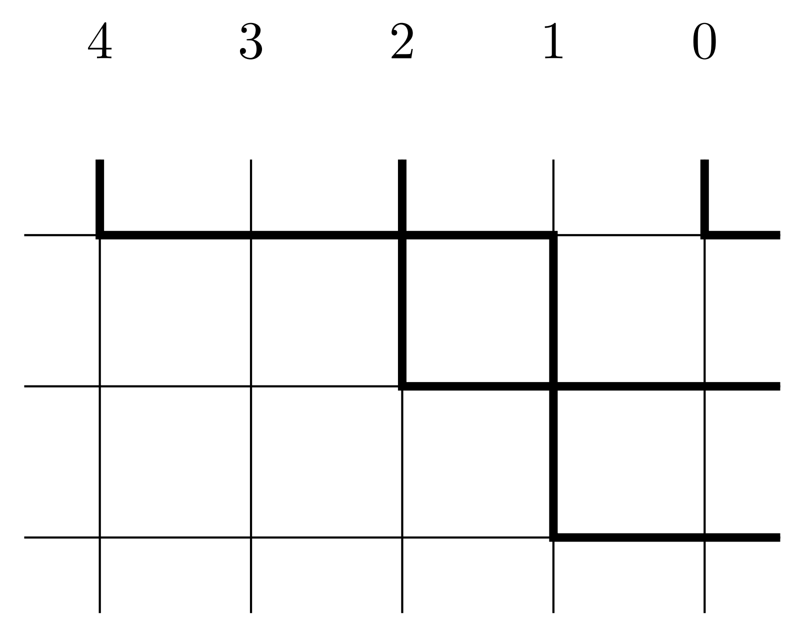

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Let $\lambda^{(i)}(T) \in \mathbb{Z}^i$ be the shape of T after removing labels larger than i

$$\lambda^{(3)}(T) = \mathrm{shape}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = (2,1,0)$$

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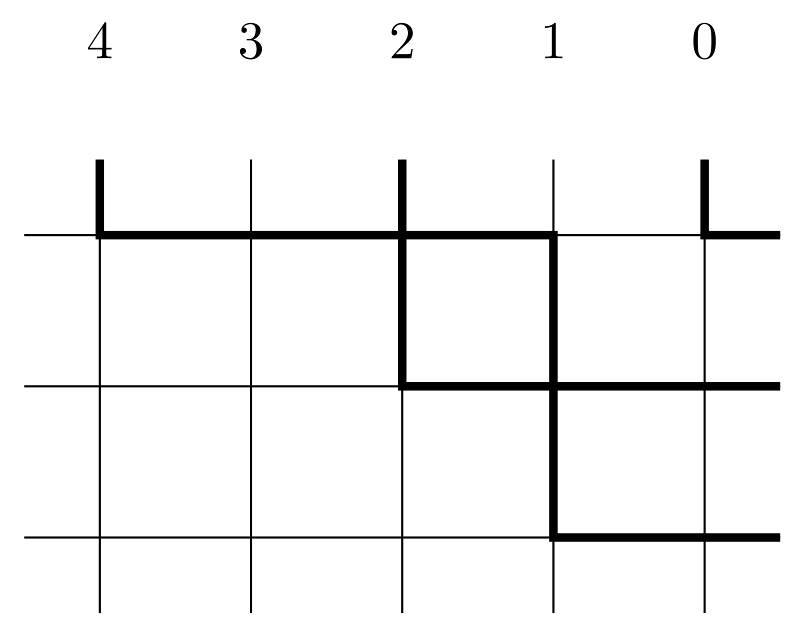

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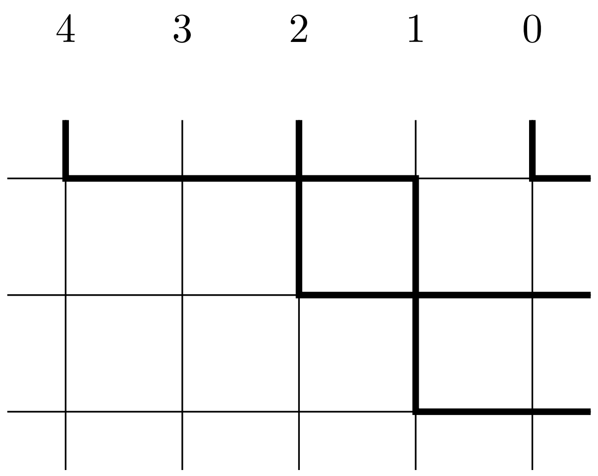

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Gelfand-Tsetlin pattern

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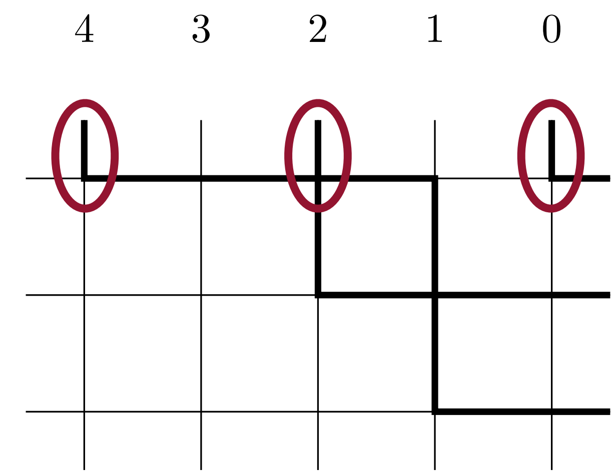

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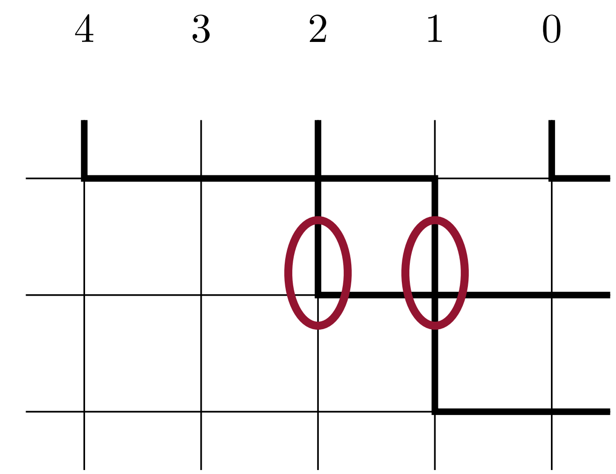

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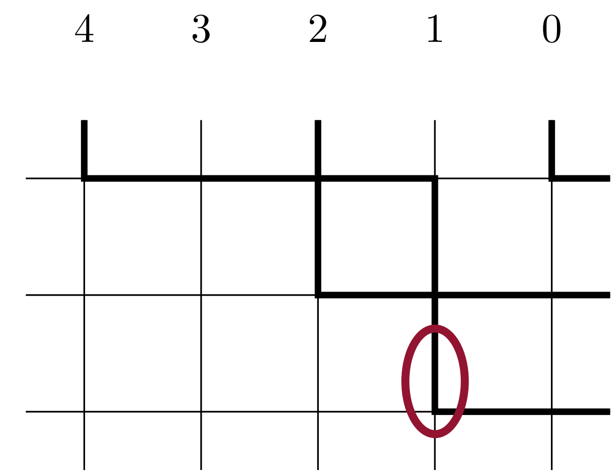

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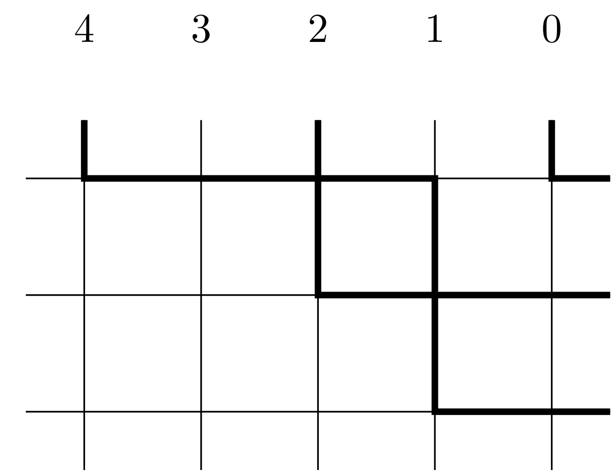
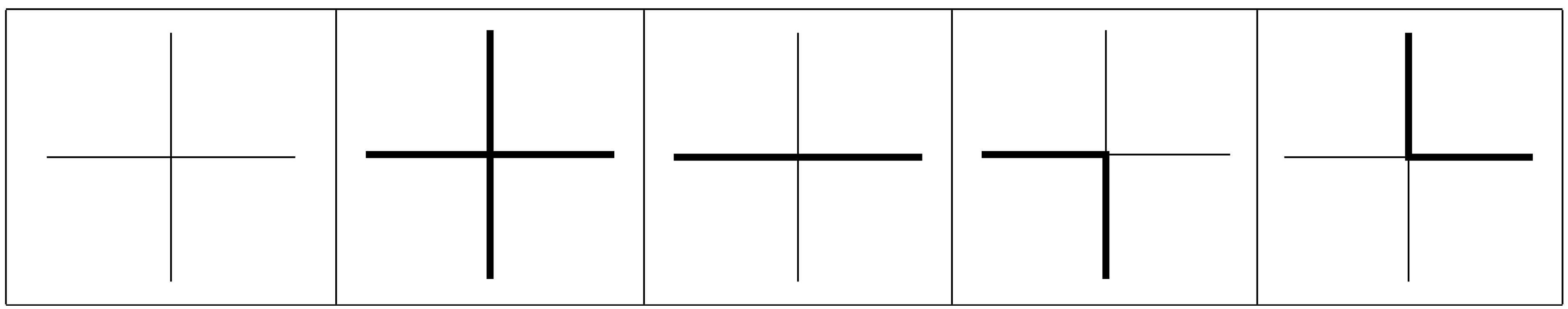


Five different vertex configurations:



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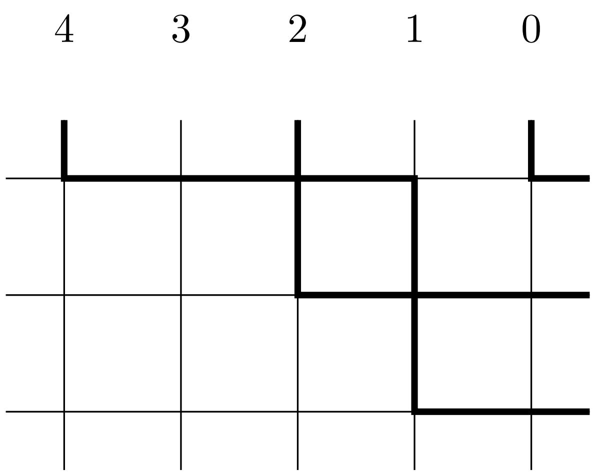

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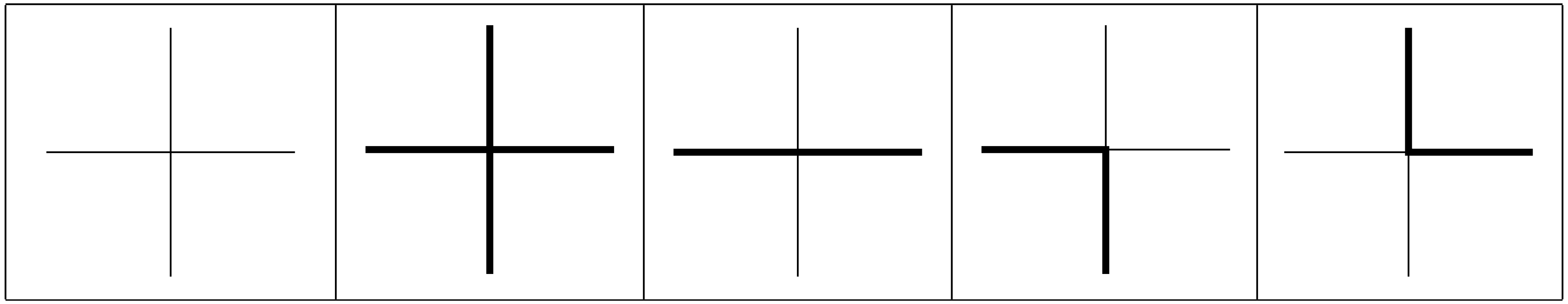
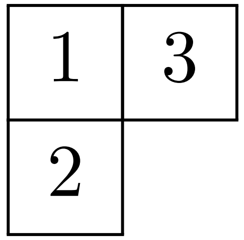


$\mathrm{SSYT} \longleftrightarrow$ lattice paths using these vertex configurations

shape λ filled in top boundary edges at columns $\lambda + \rho$

$T =$ 

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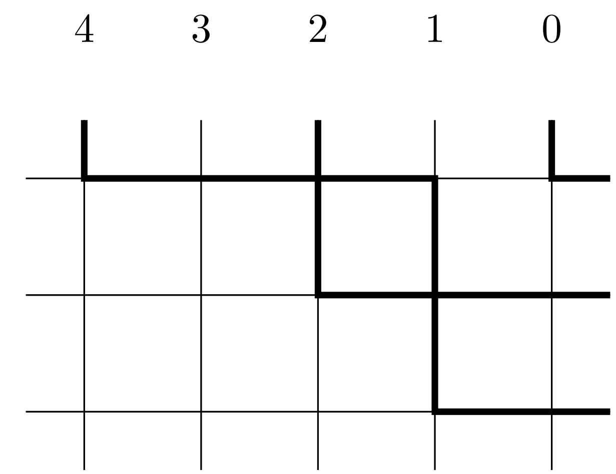

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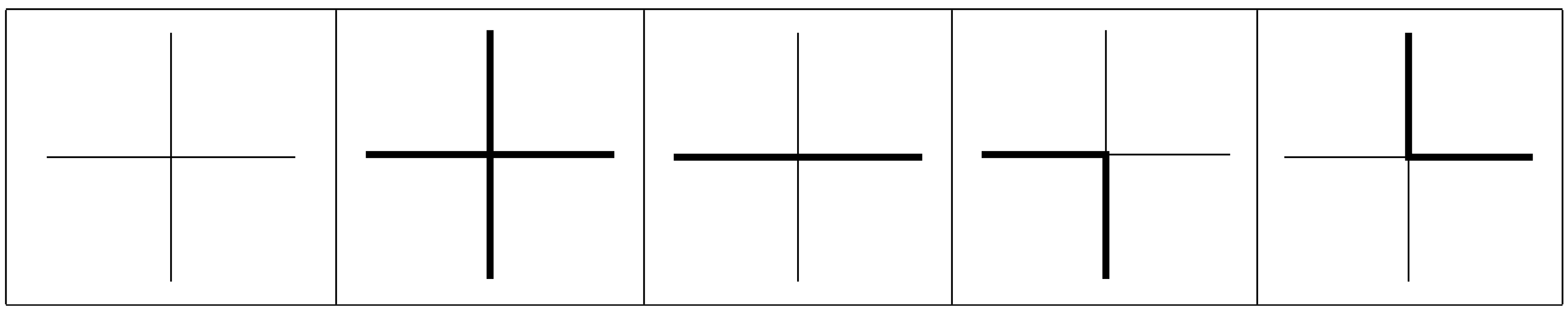

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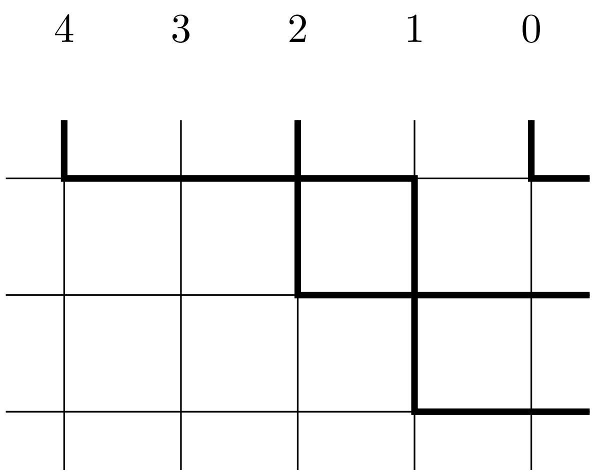

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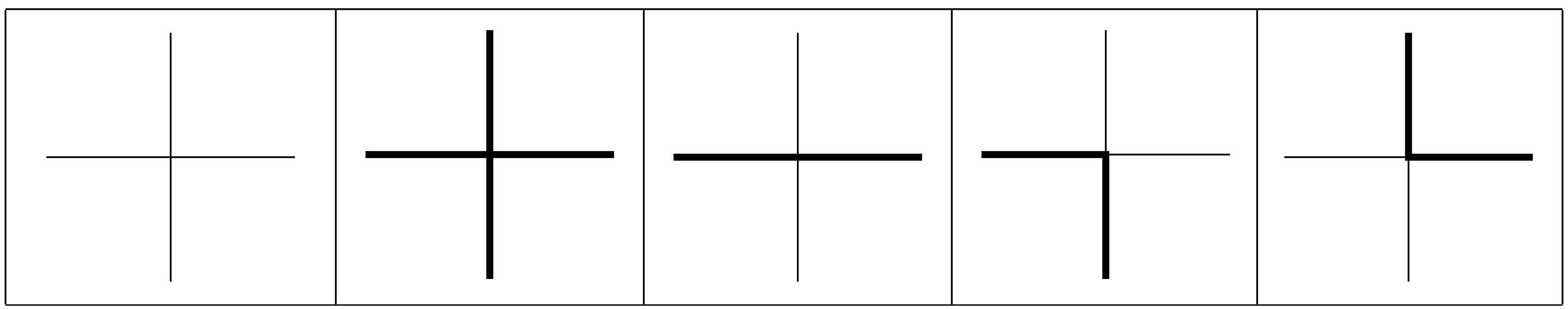

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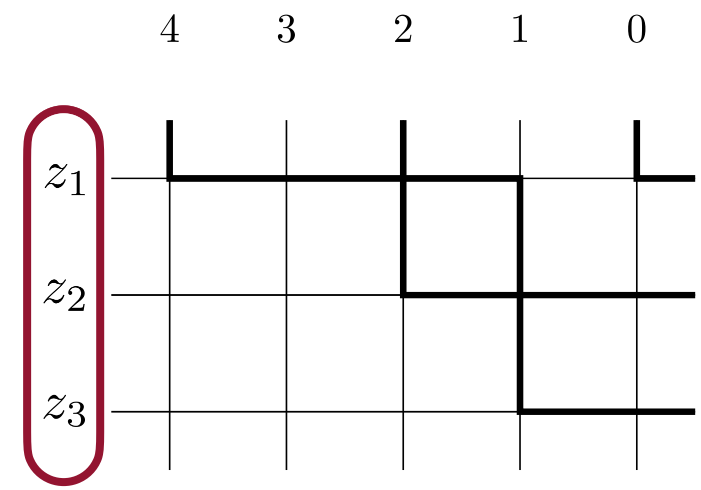
2	



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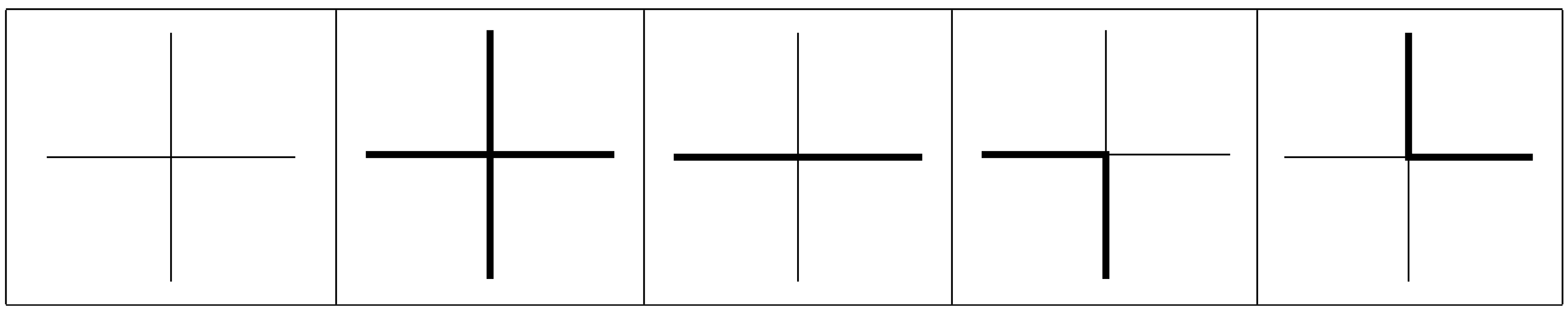

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Introduce row parameters $z_1, \ldots, z_r \in \mathbb{C}$

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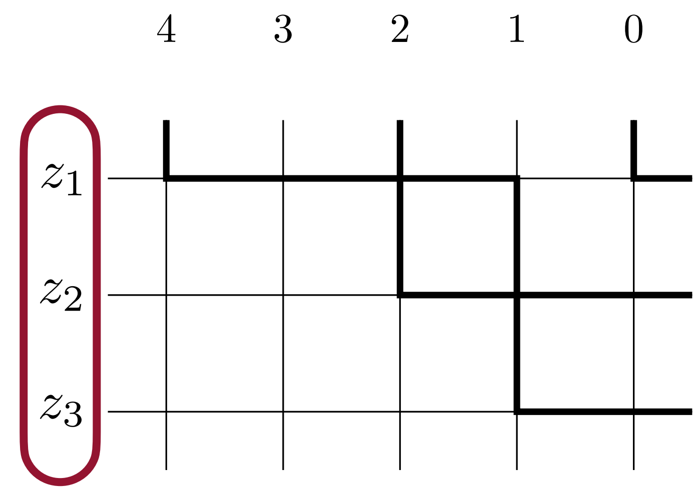
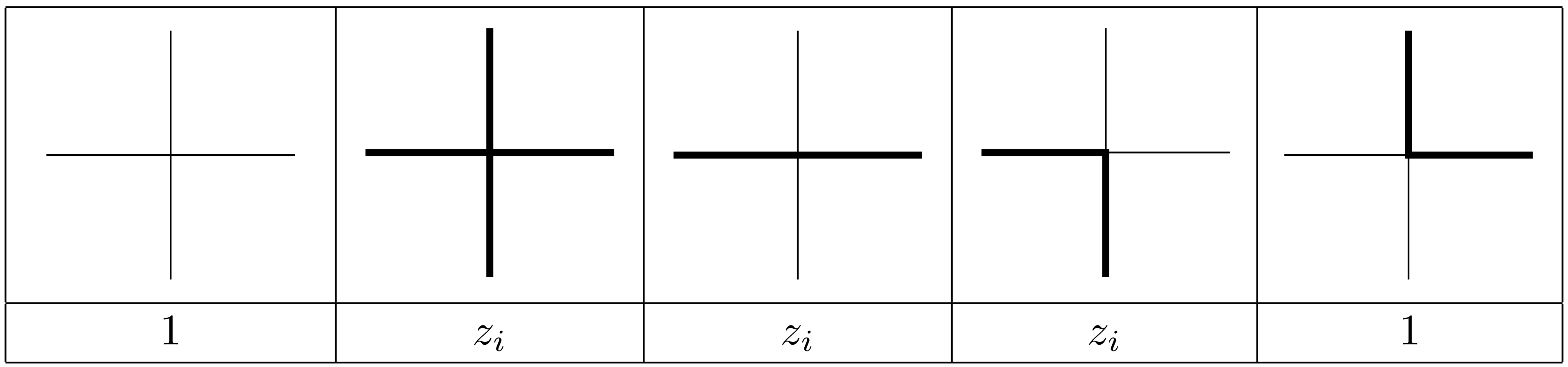
1	3
2	



Five different vertex configurations:



$\mathrm{SSYT} \longleftrightarrow$ lattice paths using these vertex configurations

shape λ filled in top boundary edges at columns $\lambda + \rho$

Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

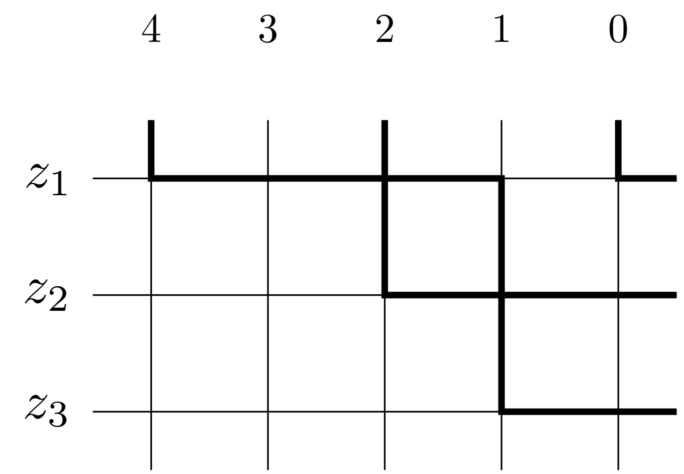
$\mathrm{wt}(T)$ counts the number of filled in left-edges in each row

Introduce row parameters $z_1, \ldots, z_r \in \mathbb{C}$ and vertex weights at row i

$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$

Lattice paths

state $\mathfrak{s}$ →



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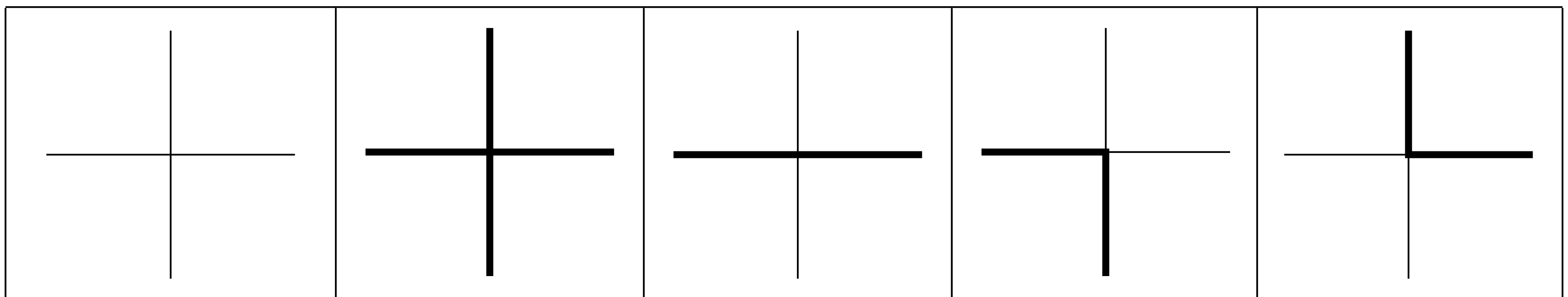


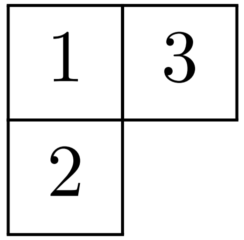
1	z_i	z_i	z_i	1

Goal: capture $\mathbf{z}^{\mathrm{wt}(T)}$ using lattice model data

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

Boltzmann weight $\beta(\mathfrak{s}) := \prod_{\text{vertex}}$ vertex weights

Lattice paths

$T =$ 

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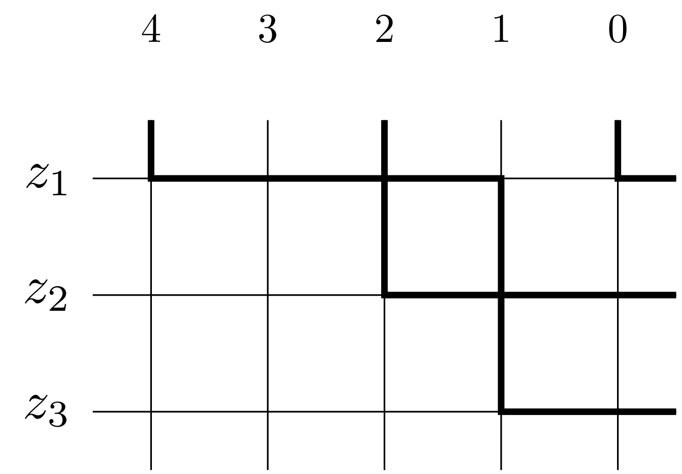
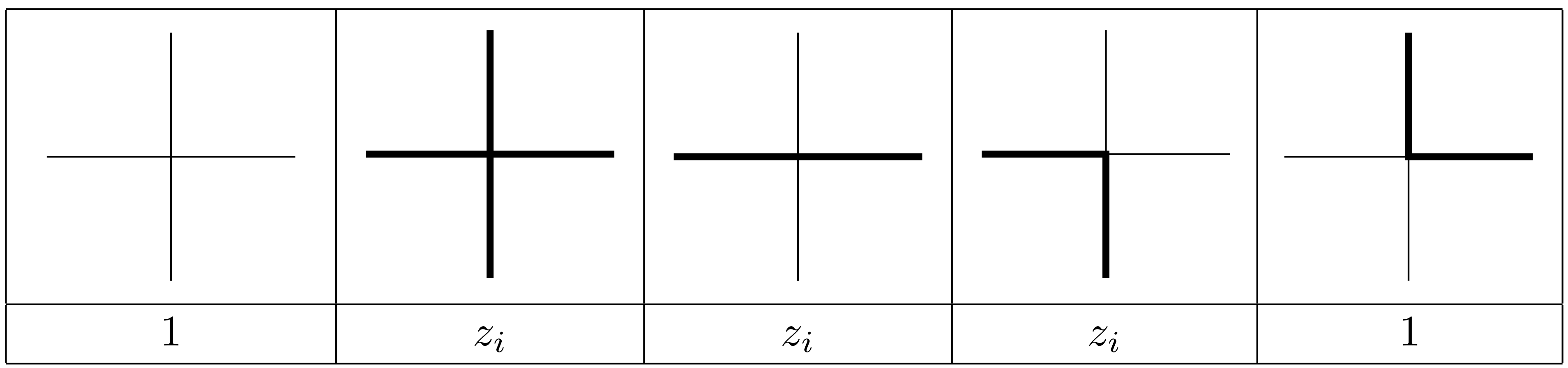


$\beta(\mathfrak{s}) = z_1^3 z_2^2 z_3$

Five different vertex configurations:



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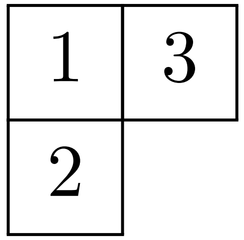
$T =$ 

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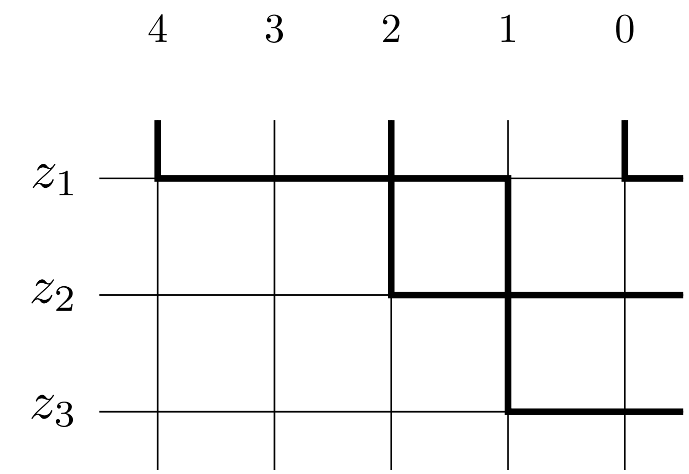

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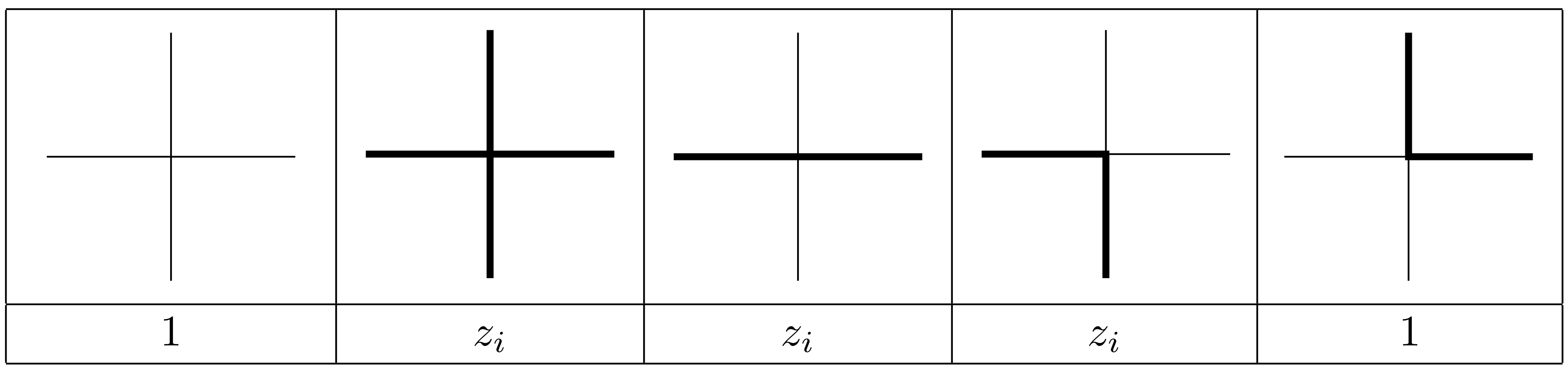

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Boltzmann weight $\beta(\mathfrak{s}) := \prod_{\text{vertex}}$ vertex weights $= \mathbf{z}^\rho \cdot (w_0 \mathbf{z})^{\mathrm{wt}(T)}$

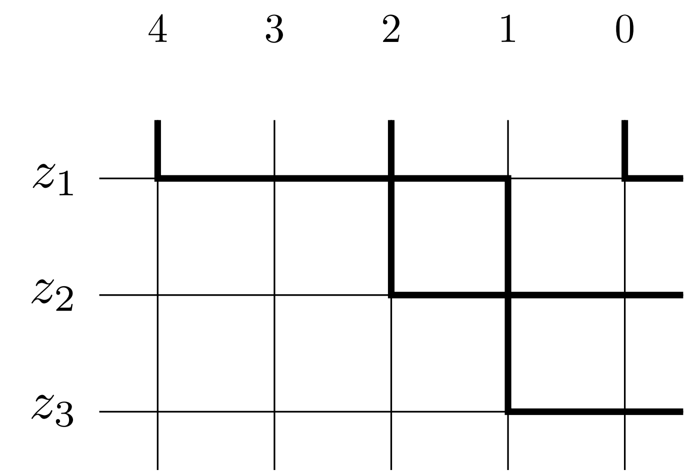
$w_0(z_1, z_2, \ldots, z_r) = (z_r, \ldots, z_2, z_1)$

Lattice paths

$T =$

1	3
2	

state $\mathfrak{s}$ →



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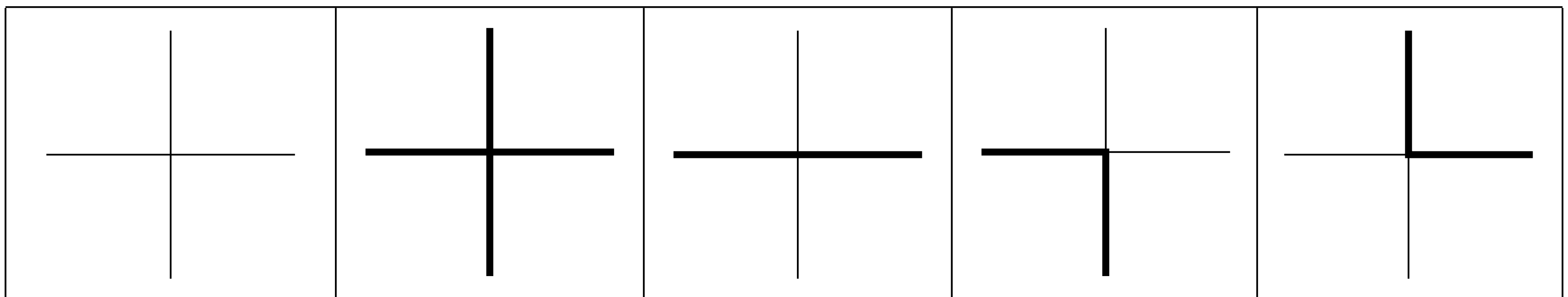

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Partition function $Z(\lambda, \mathbf{z}) := \sum_{\mathfrak{s} \text{ with top } \lambda+\rho} \beta(\mathfrak{s})$

Lattice paths

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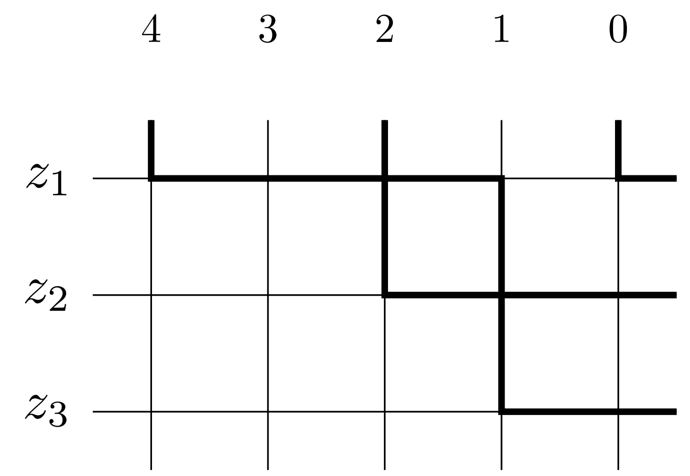
state $\mathfrak{s}$ →



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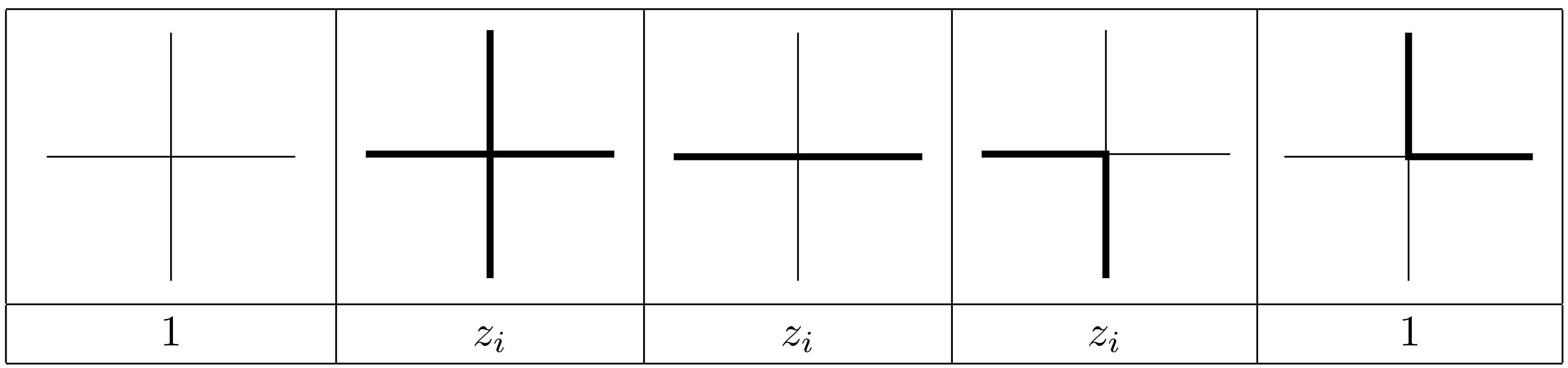

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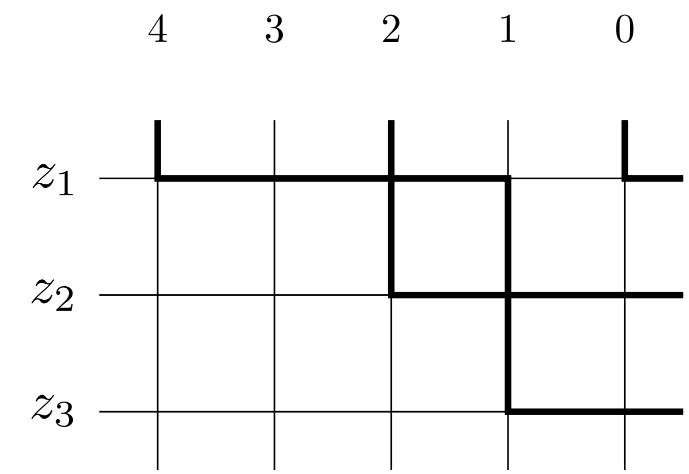

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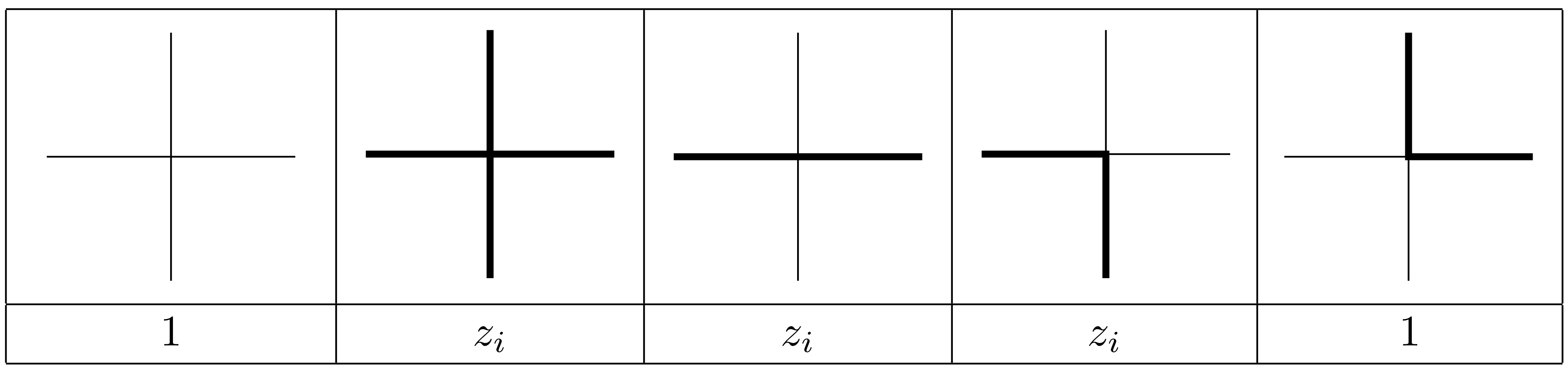

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Cauchy identity

Usefulness of lattice model

5-vertex ⟶ 6-vertex

Cauchy identity

$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\hat{\lambda}'}(\mathbf{y}) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j)$$

[Macdonald 1992 (0.11')]

Cauchy identity

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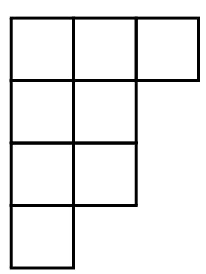
$\lambda =$ 

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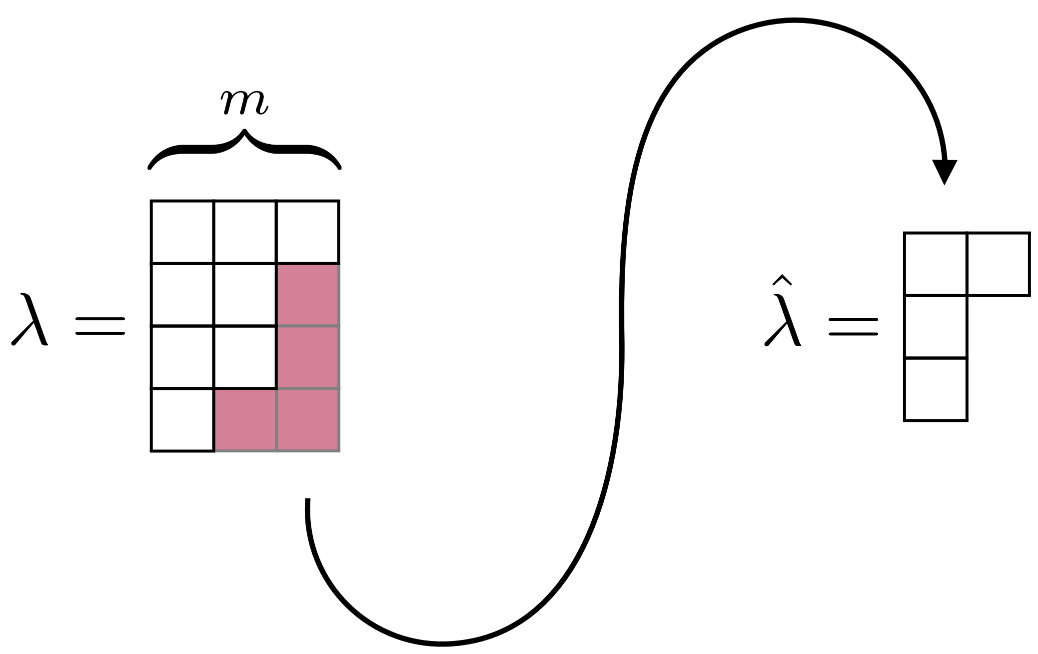

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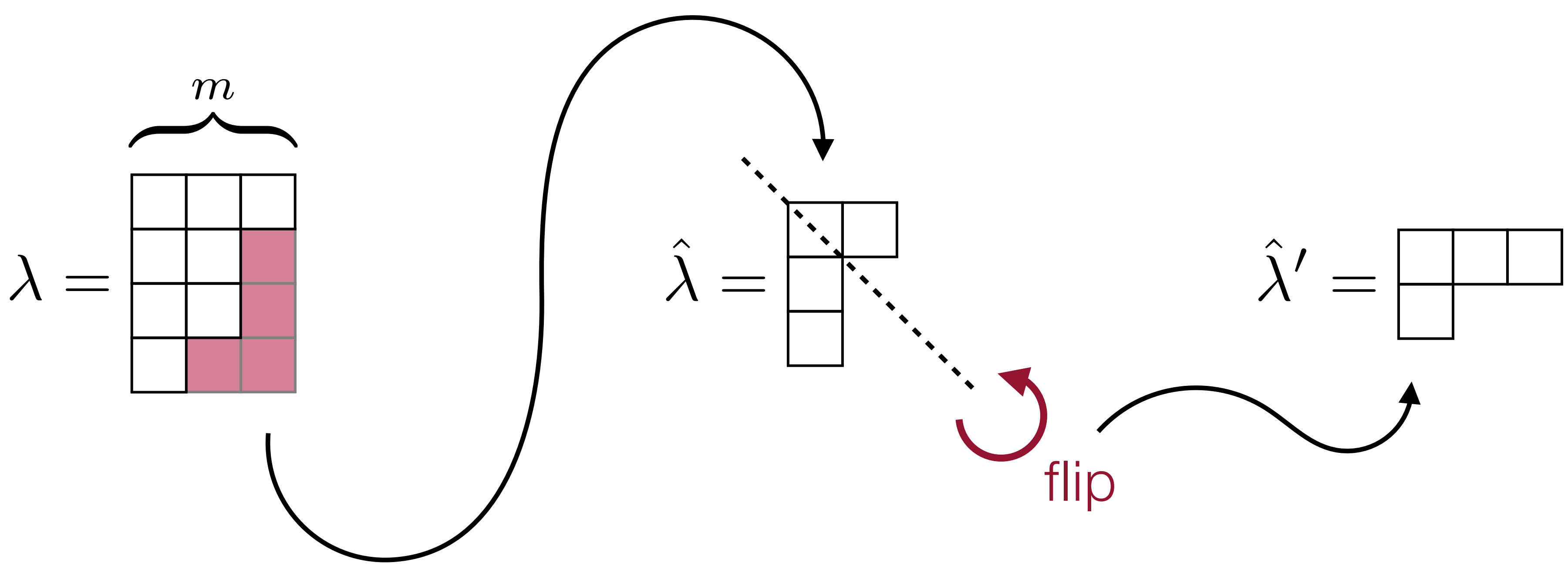
[Macdonald 1992 (0.11')]

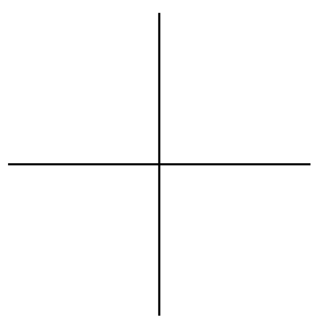
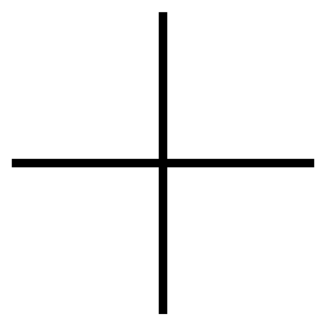
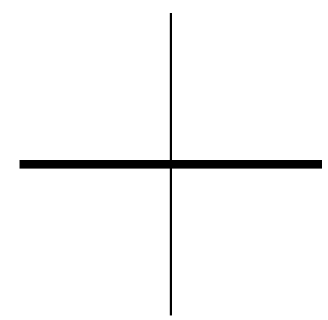
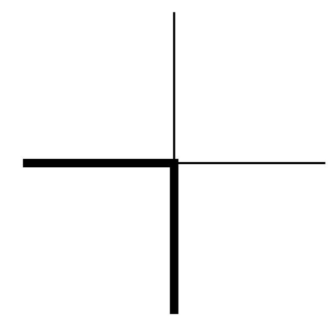
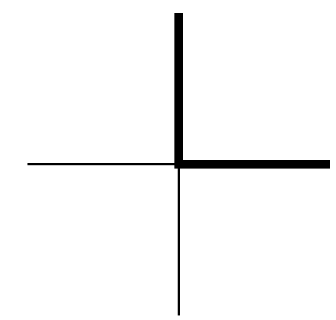
$\hat{\lambda}$ = complement of λ

μ' = conjugate of μ



Cauchy identity

Symmetry of vertex configurations using arrow description

					
1	z		z	z	1

Cauchy identity

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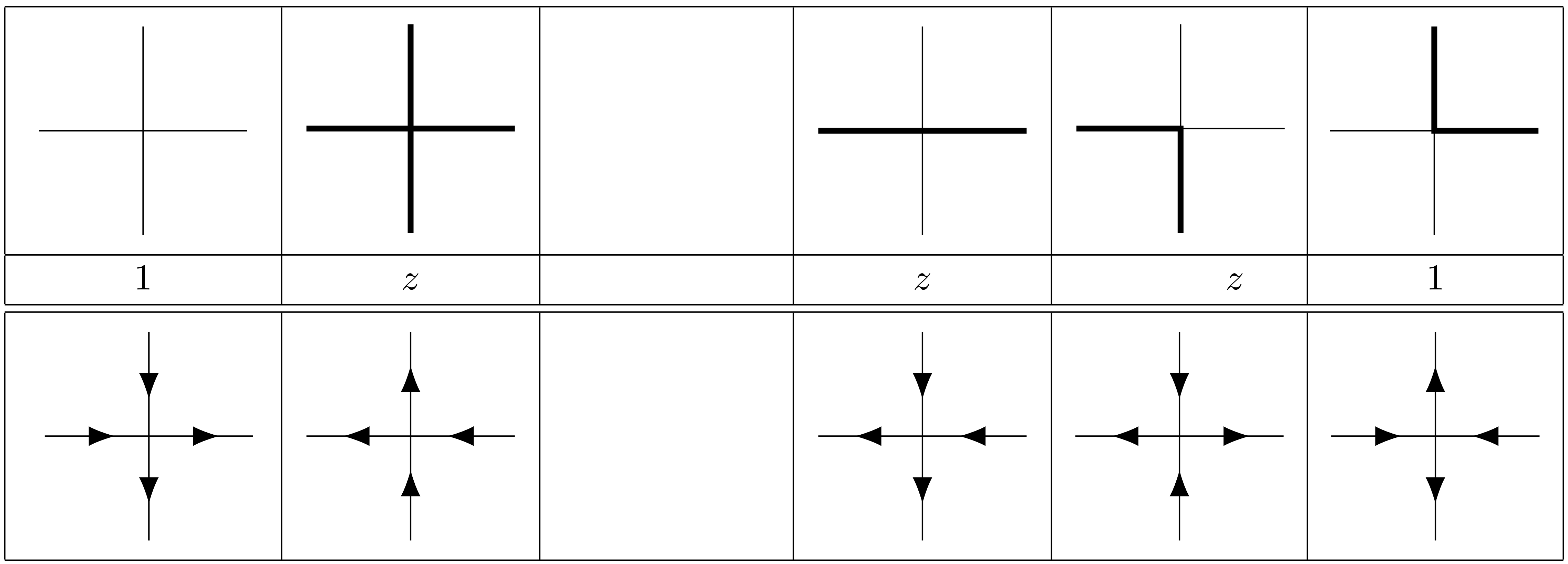

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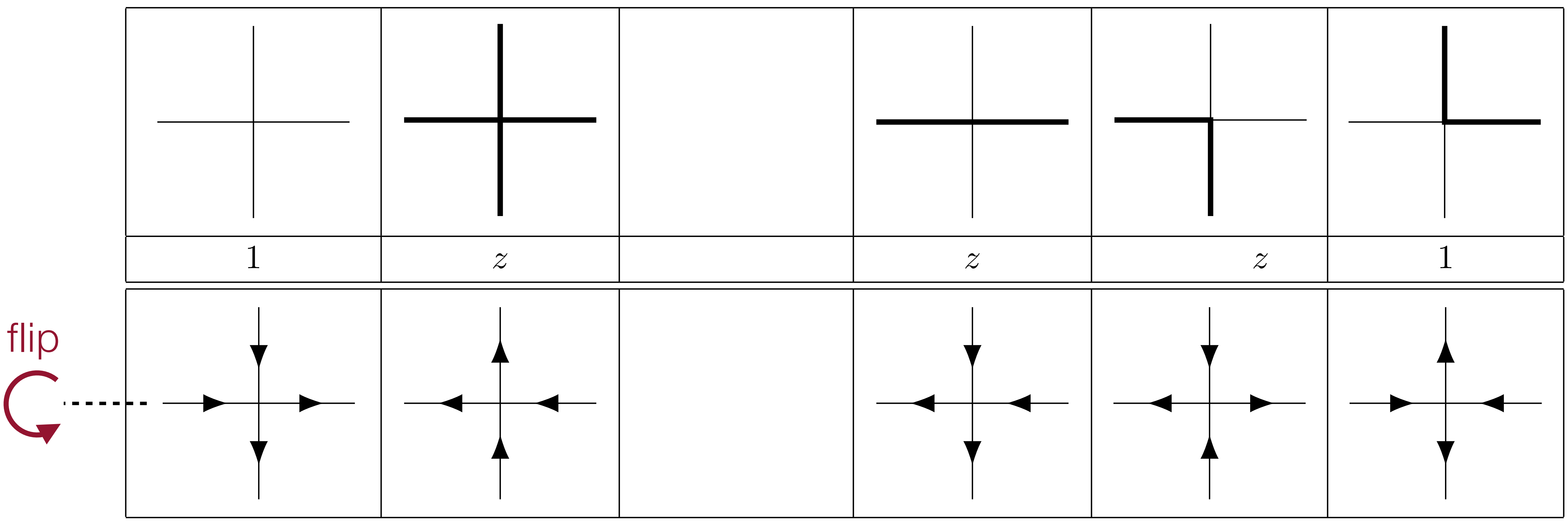

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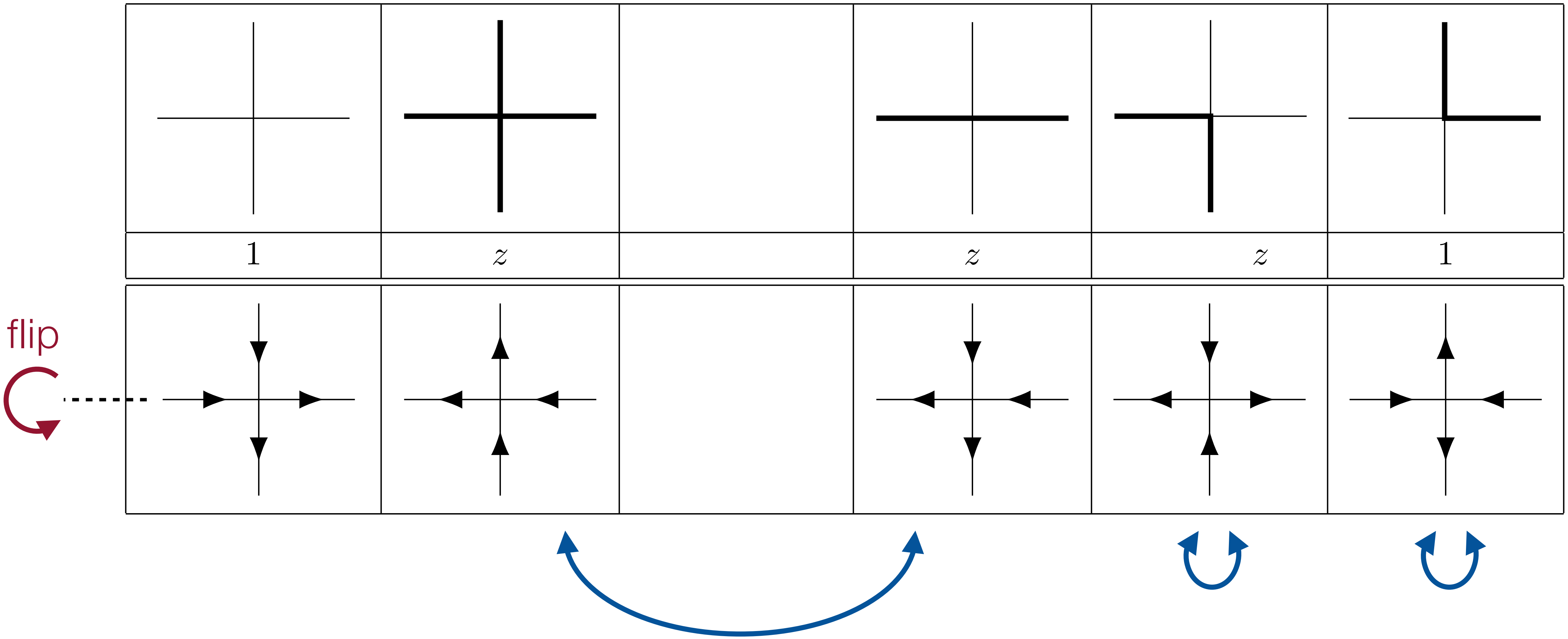

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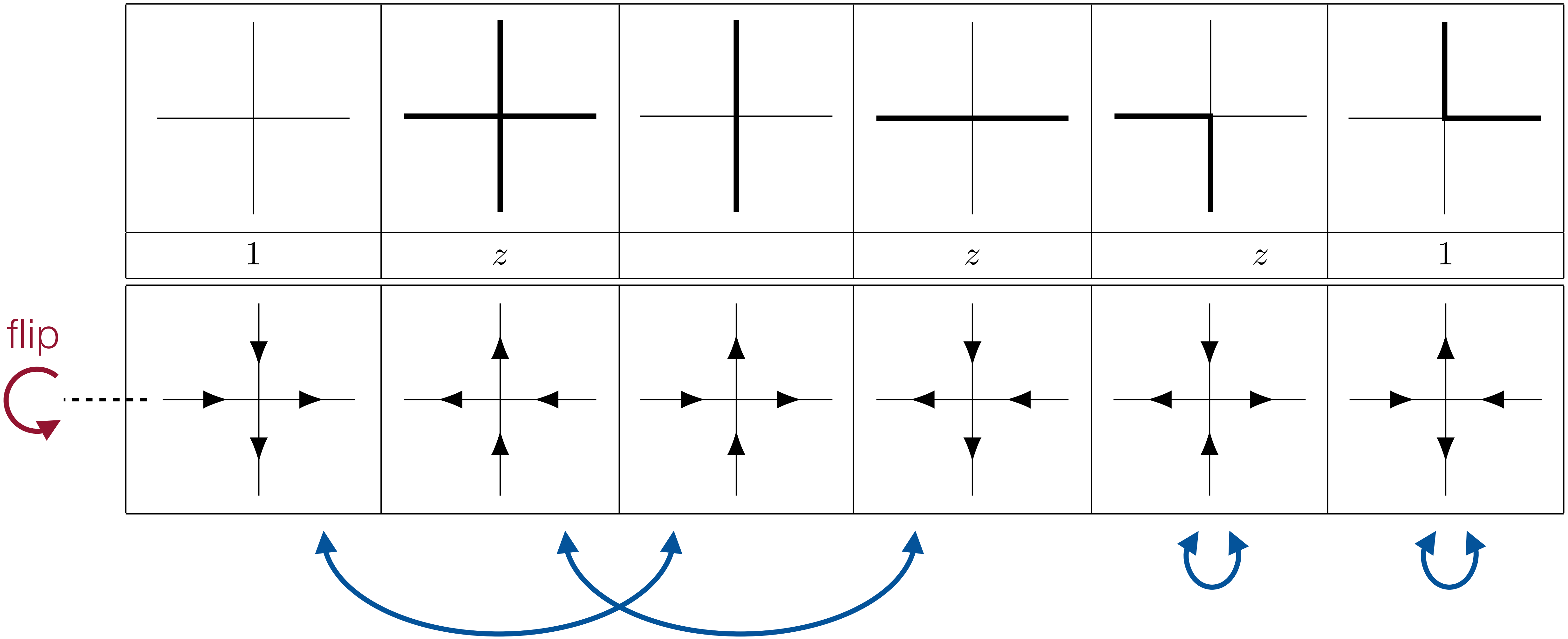

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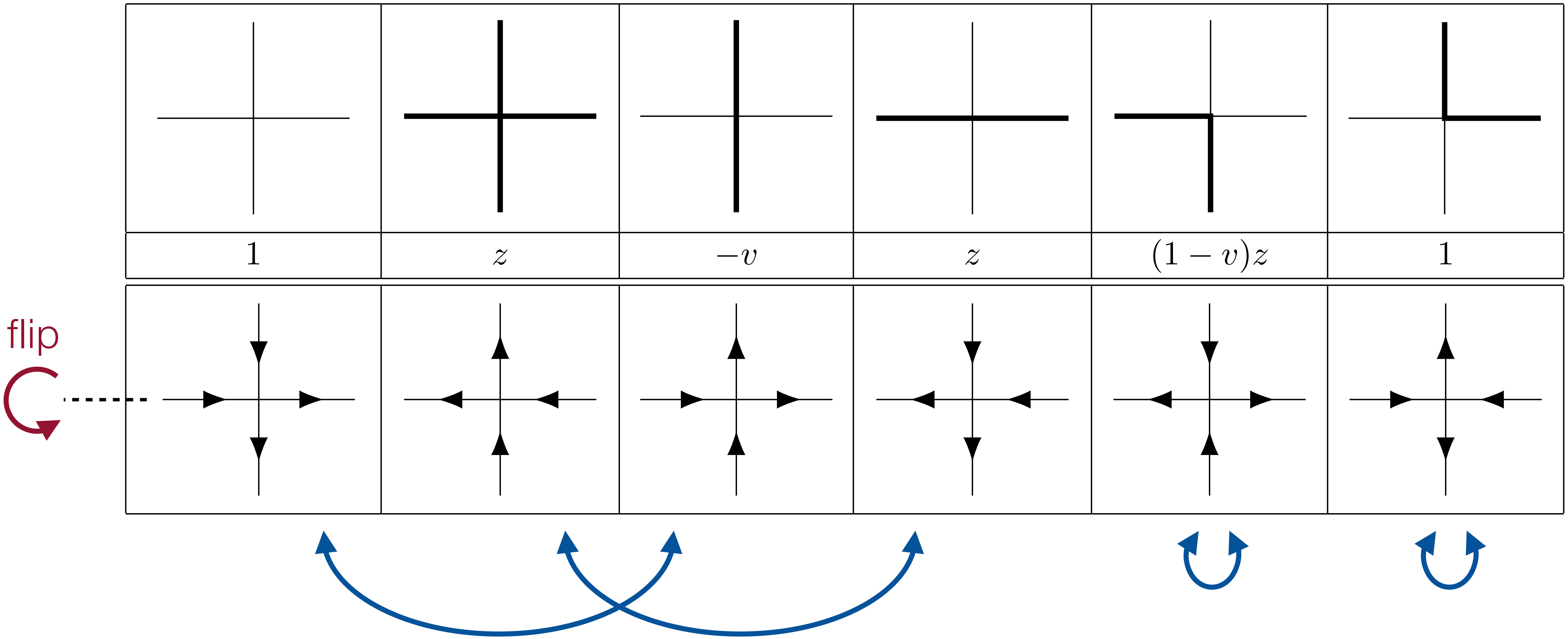

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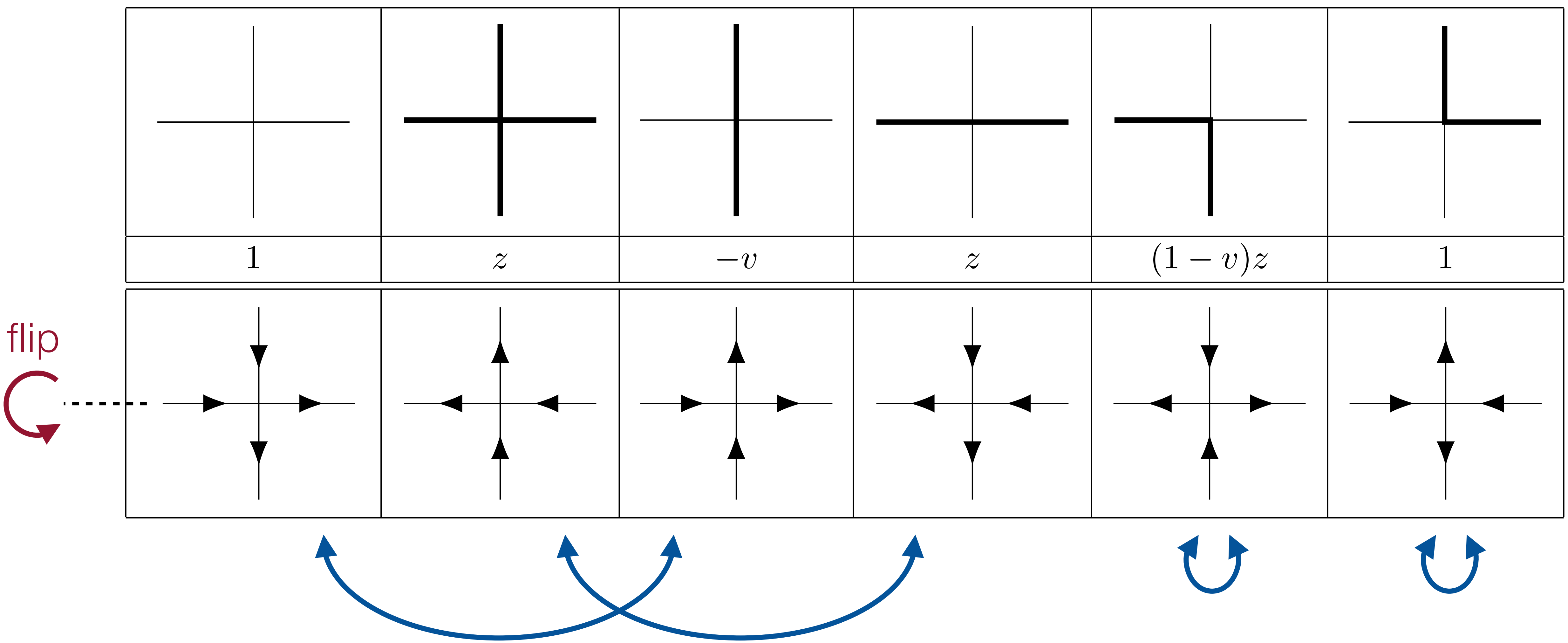
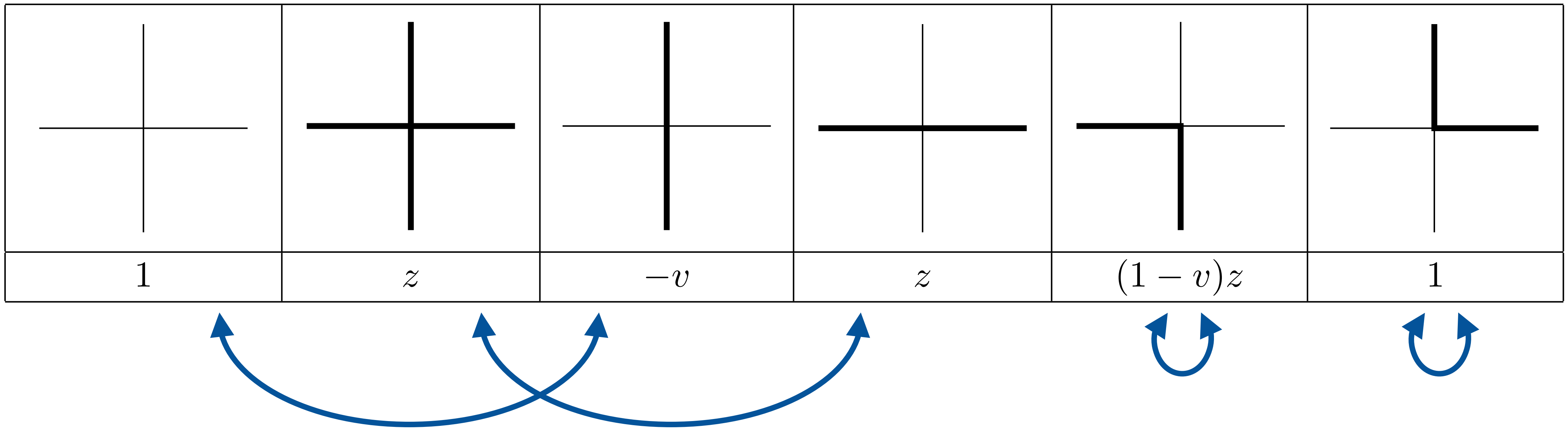


Weights are adjusted for solvability (to satisfy Yang–Baxter equation).

Cauchy identity

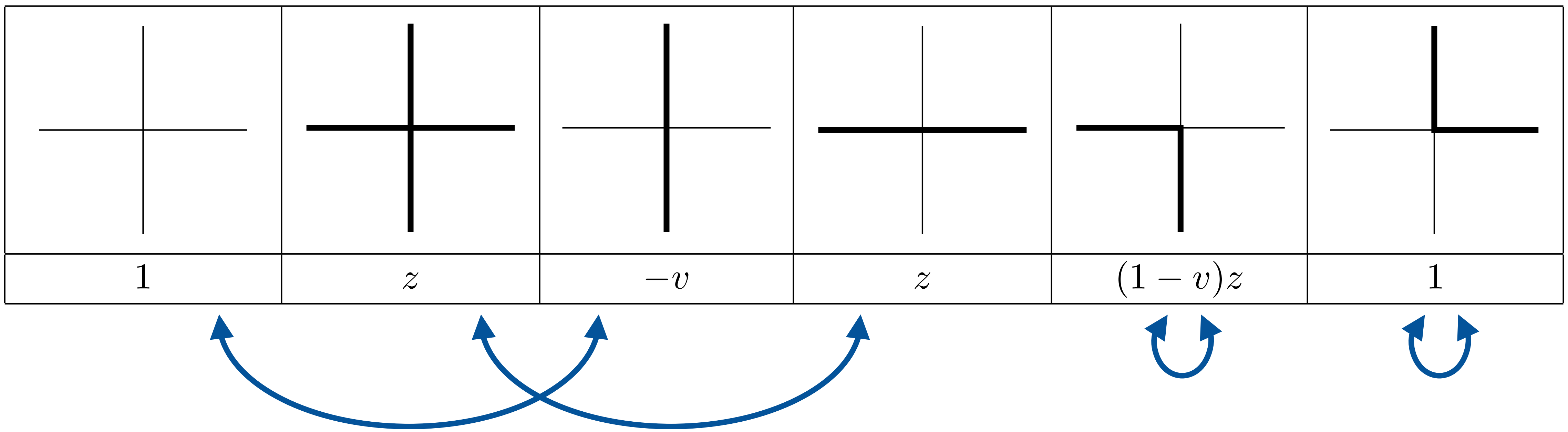


These new weights introduce a slight deformation of the partition function

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^{\rho} \prod_{i<j} (1 - v\frac{z_j}{z_i}) s_{\lambda}(\mathbf{z})$$

[Tokuyama 1988, Hammel–King 2007, Brubaker–Bump–Friedberg 2009]

Cauchy identity



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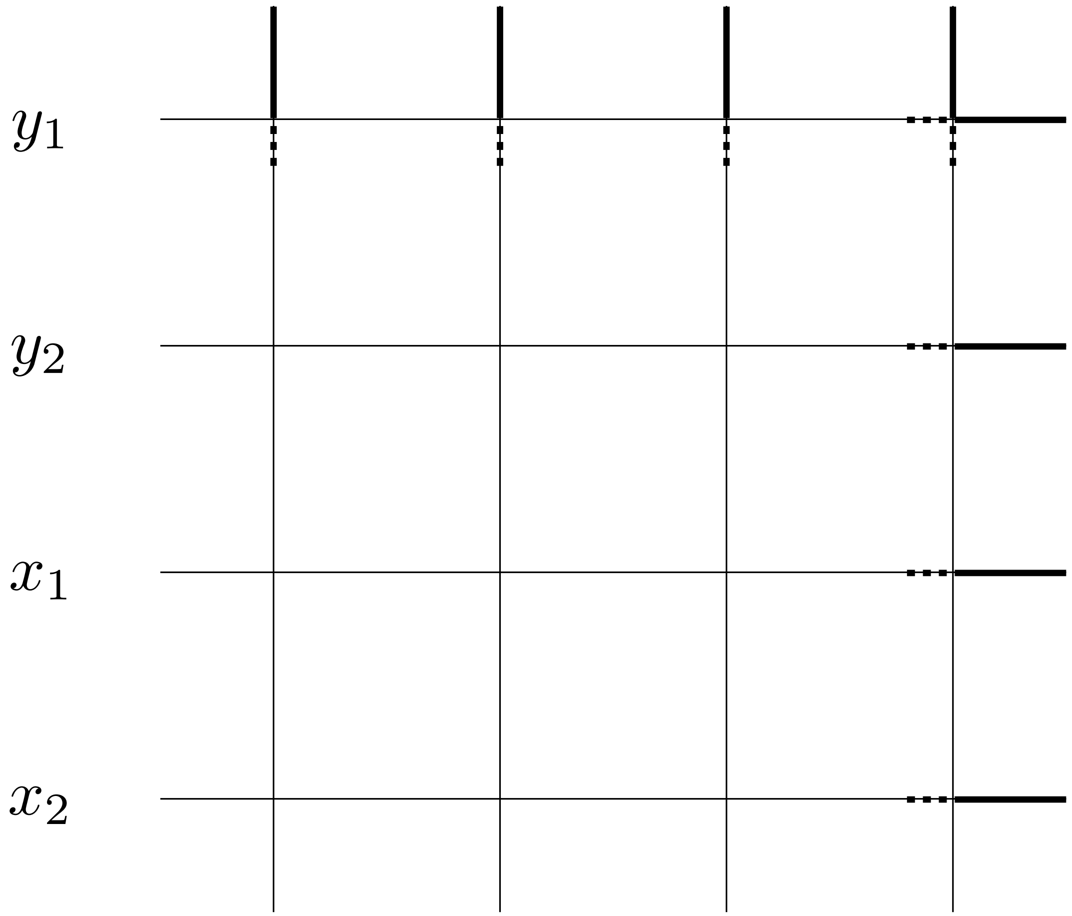
If $v = -1$ then a flip preserves the Boltzmann weight of the state.

Cauchy identity

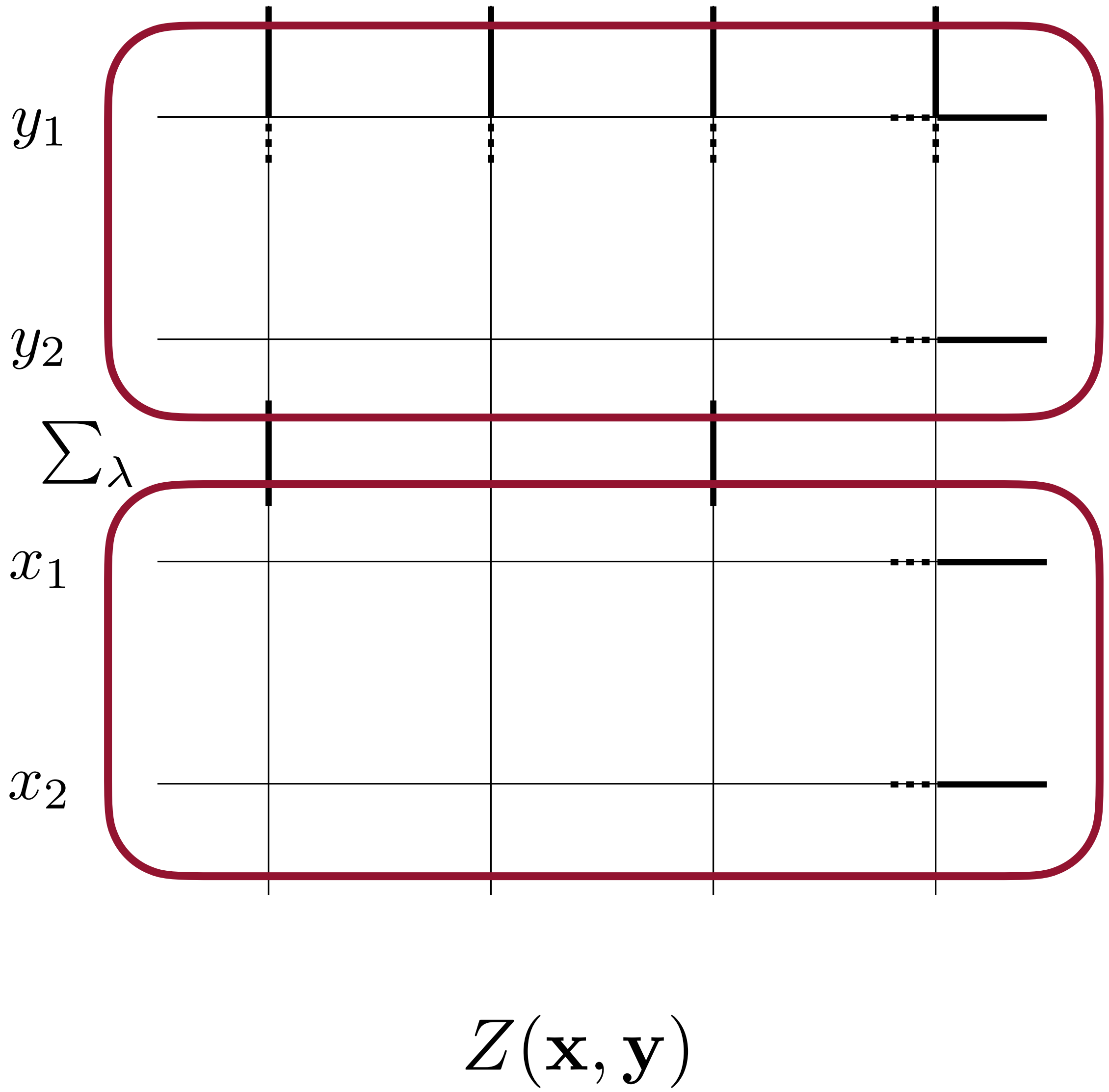


$Z(\mathbf{x}, \mathbf{y})$

The identity is obtained by splitting up a large lattice model into two pieces

[Bump–McNamara–Nakasuji 2014]

Cauchy identity



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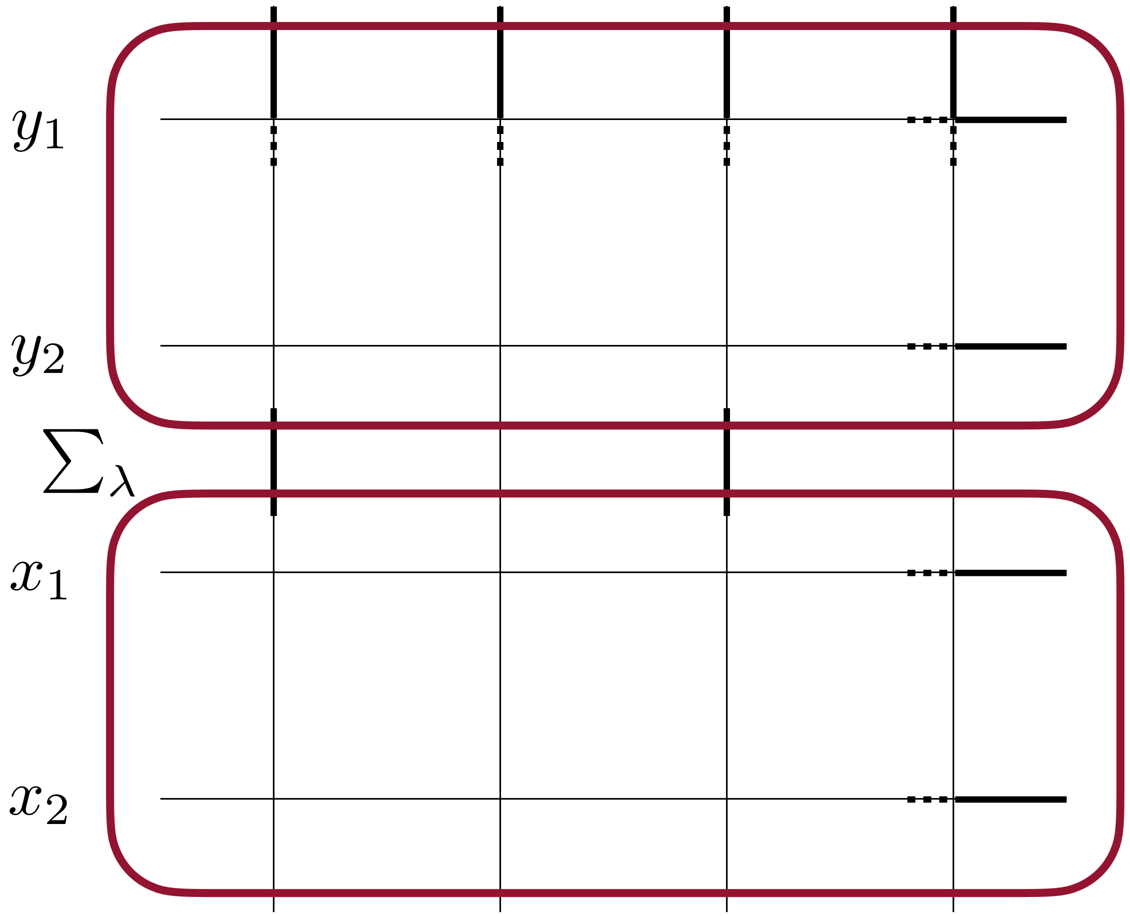
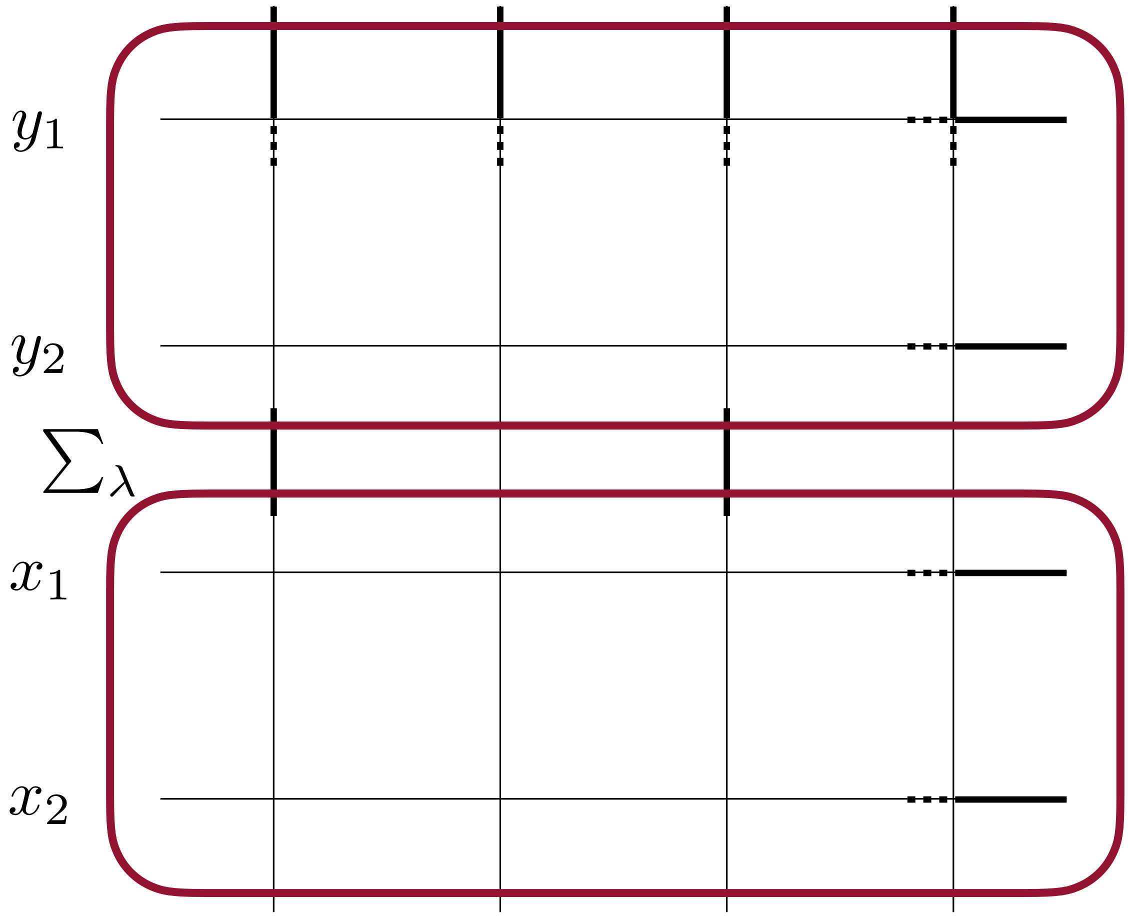


$Z_\lambda^{\text{top half}}(\mathbf{y})$

$Z(\mathbf{x}, \mathbf{y})$

[Bump–McNamara–Nakasuji 2014]

Cauchy identity



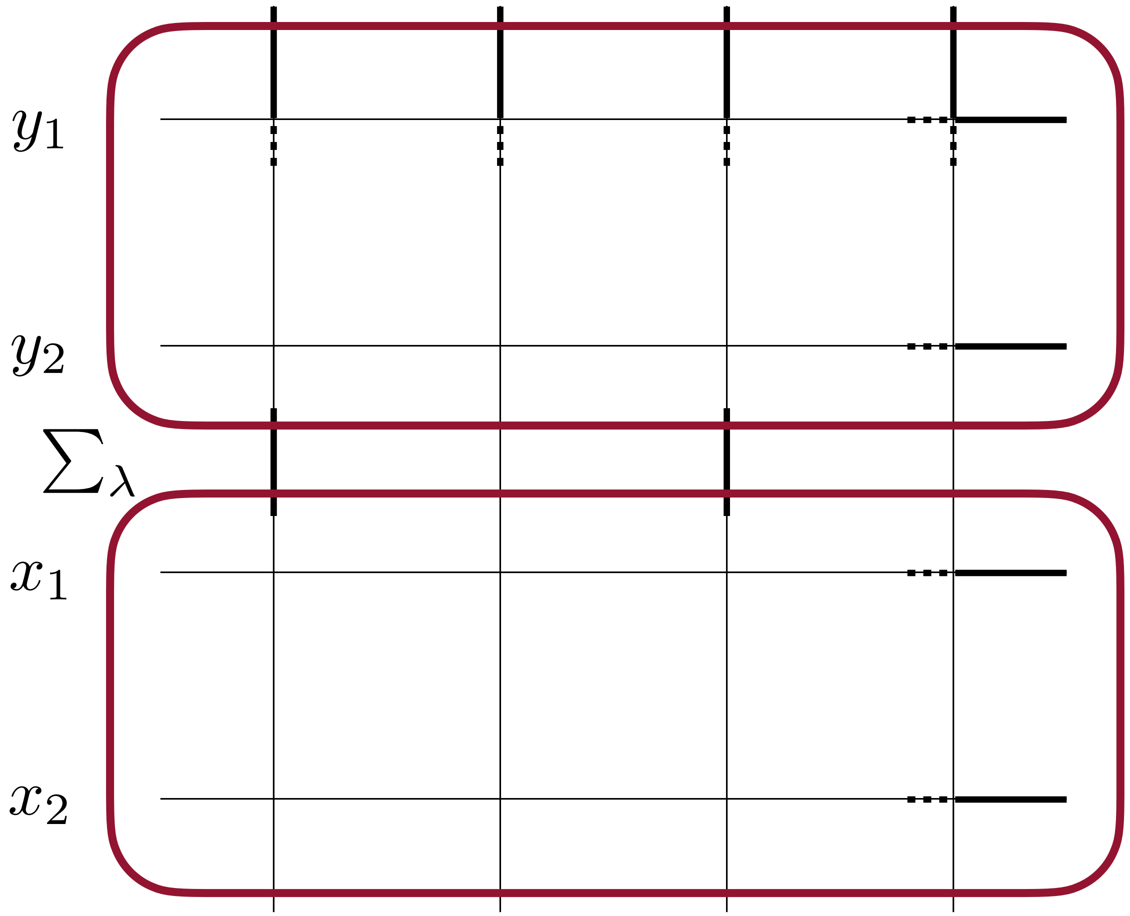
$Z_\lambda^{\text{top half}}(\mathbf{y})$

$Z_\lambda^{\text{bot half}}(\mathbf{x})$

$Z(\mathbf{x}, \mathbf{y})$

[Bump–McNamara–Nakasuji 2014]

Cauchy identity



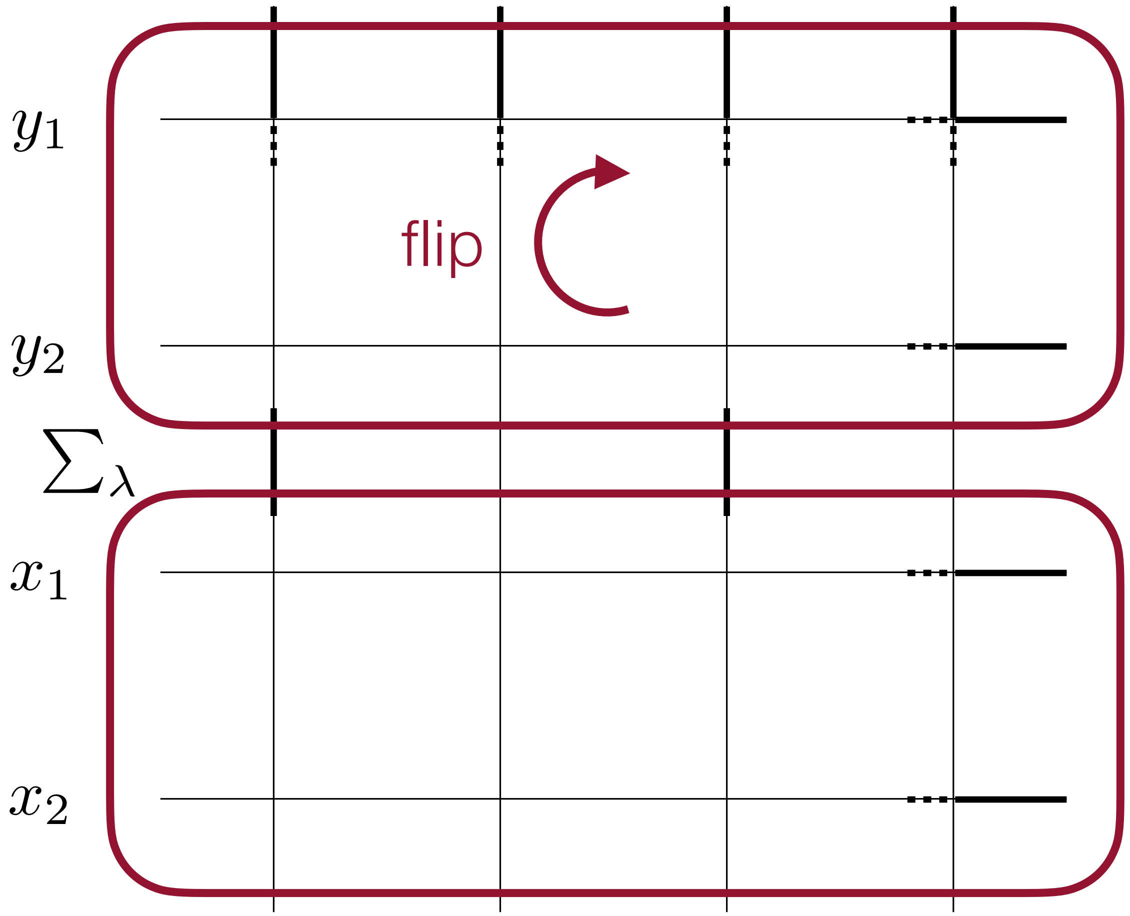
$$Z_\lambda^{\text{top half}}(\mathbf{y})$$

$$Z_\lambda^{\text{bot half}}(\mathbf{x})$$

$$Z(\mathbf{x},\mathbf{y}) = \sum_\lambda Z_\lambda^{\text{bot half}}(\mathbf{x}) \cdot Z_\lambda^{\text{top half}}(\mathbf{y})$$

[Bump–McNamara–Nakasuji 2014]

Cauchy identity



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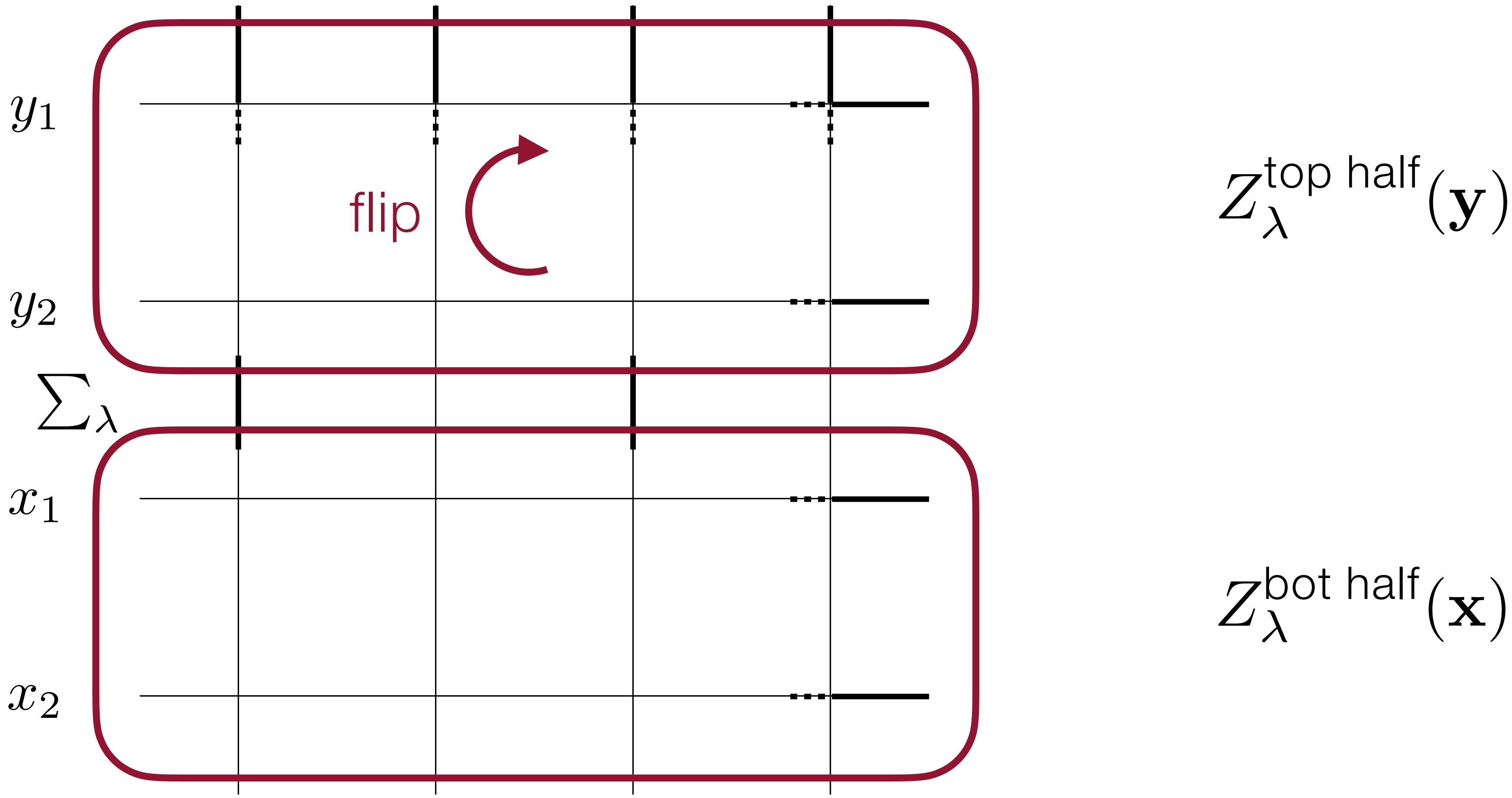

$$Z(\mathbf{x},\mathbf{y}) = \sum_\lambda Z_\lambda^{\text{bot half}}(\mathbf{x}) \cdot Z_\lambda^{\text{top half}}(\mathbf{y})$$

$(v=-1) \Downarrow$ flip

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i+y_j) = \sum_\lambda s_\lambda(\mathbf{x}) \cdot s_{\hat{\lambda}'}(\mathbf{y})$$

[Bump–McNamara–Nakasuji 2014]

Whittaker functions

$$Z(\lambda;\mathbf{z}) = \mathbf{z}^\rho \prod_{i<j}(1 - v\frac{z_j}{z_i})s_\lambda(\mathbf{z})$$

is secretly a Whittaker function

Whittaker functions

$G = \mathrm{GL}_r(\mathbb{Q}_p)$

$$B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \qquad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Character
$\psi : N \to \mathbb{C}^\times$
(generic; standard)

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Whittaker model $\quad \pi \xhookrightarrow{\sim} \mathcal{W}_\psi(\pi) \subset \mathrm{Ind}_N^G(\psi)$

Irreducible representation

$\{f : G \to \mathbb{C} \mid f(ng) = \psi(n) f(g)\}$

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Need to specify representation and embedding:

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Whittaker function $\in \mathcal{W}_\psi(\pi)$

Need to specify representation and embedding:

$$f \in \pi \qquad \mathcal{W}_\psi(f) \;:\; g \longmapsto \int_N f(w_0 n g)\psi(n)^{-1} dn$$

long Weyl group element

The Whittaker model is unique if it exists [Gelfand–Kazhdan 1972, Rodier 1973].

Whittaker functions

Whittaker model $\pi \overset{\sim}{\hookrightarrow} \mathcal{W}_\psi(\pi) \subset \mathrm{Ind}_N^G(\psi)$

Need to specify representation and embedding: $\mathcal{W}_\psi(f) \; : \; g \longmapsto \int_N f(w_0 n g)\psi(n)^{-1} dn$

[Casselman 1980, Casselman–Shalika 1980]

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$$\mathcal{W}_\psi(f_{\mathbf{z}}^\circ)(p^\lambda) = \prod_{i<j}(1 - p^{-1}\tfrac{z_j}{z_i}) s_\lambda(\mathbf{z})$$

[Casselman 1980, Casselman–Shalika 1980]

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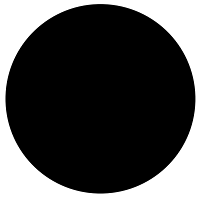
[Casselman 1980, Casselman–Shalika 1980]

Generalizations

Lattice models for other Whittaker functions

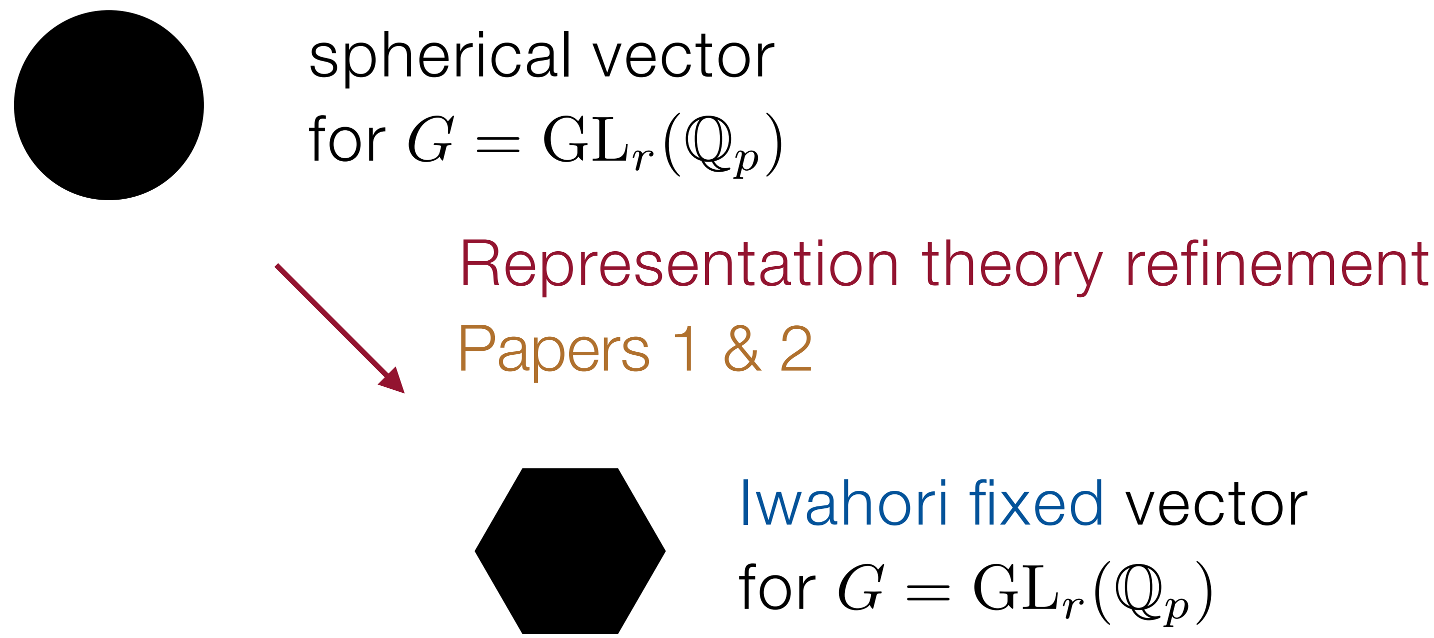
Generalizations



spherical vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Blue terms will be defined in the next slides

Generalizations



Blue terms will be defined in the next slides

Generalizations

spherical vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Number theory generalization
[Brubaker–Bump–Chinta–Friedberg–Gunnells 2012]

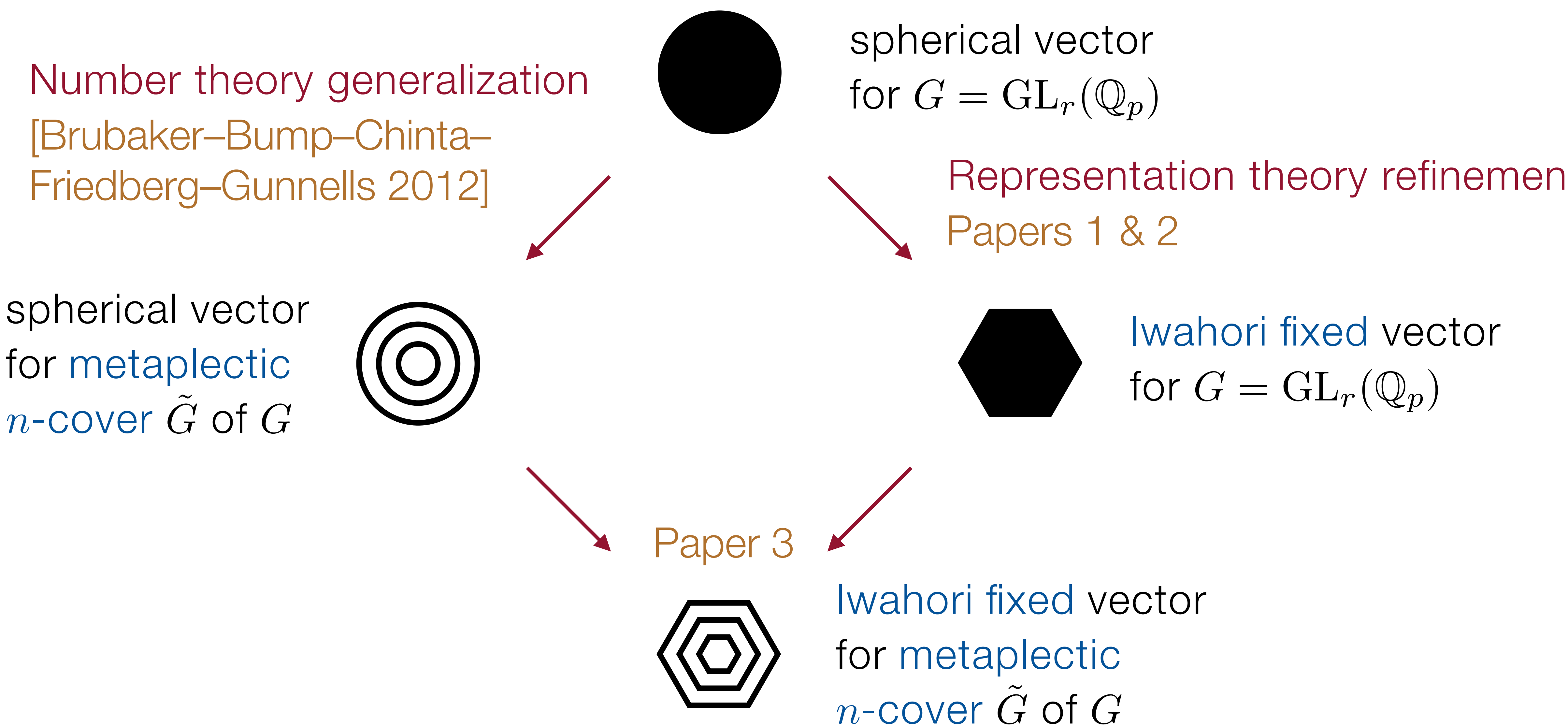
Representation theory refinement
Papers 1 & 2

spherical vector
for metaplectic
n-cover $\tilde{G}$ of G

Iwahori fixed vector
for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Blue terms will be defined in the next slides

Generalizations



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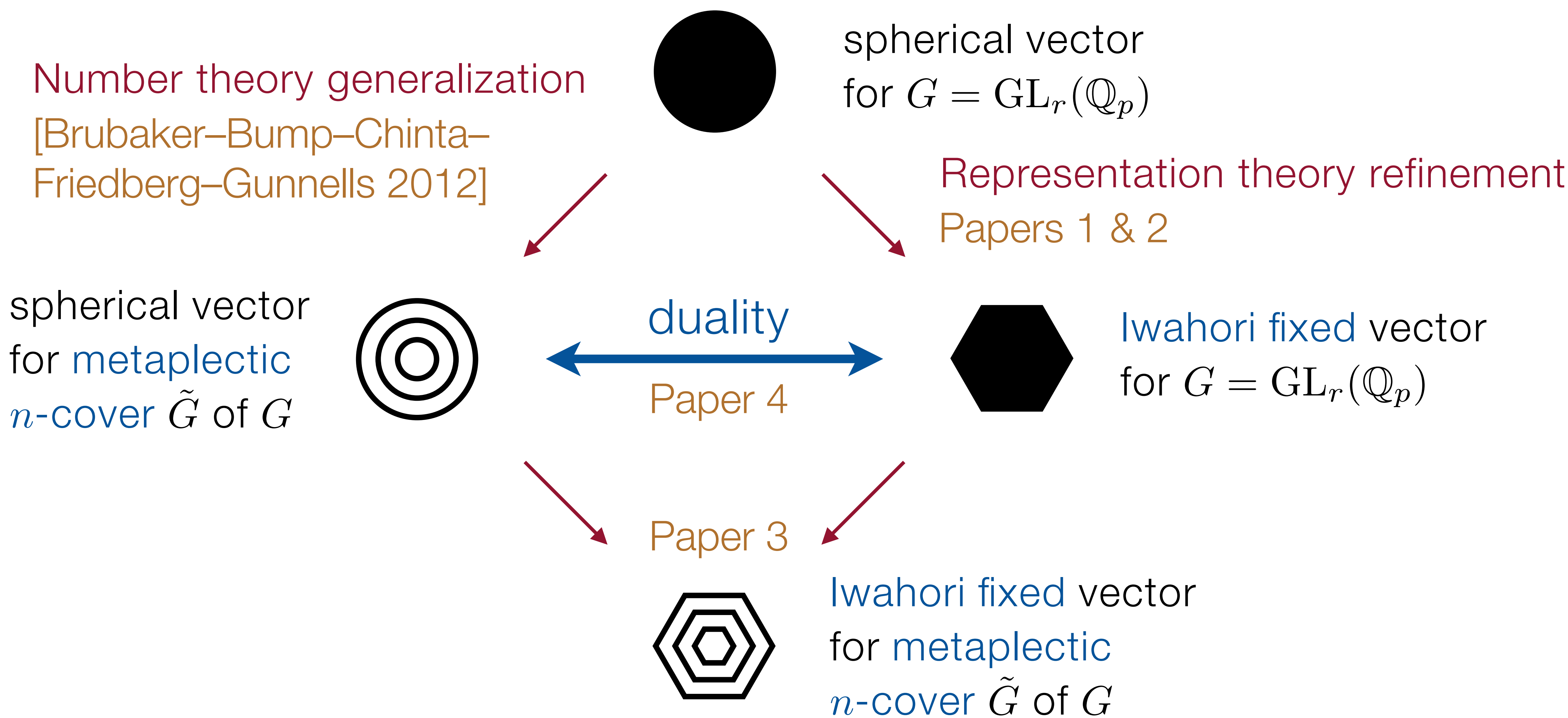
Generalizations



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Generalizations

spherical vector
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Iwahori fixed vector
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Generalizations

spherical vector
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$f^\circ(gk) = f^\circ(g)$ for
$k \in K := \mathrm{GL}_r(\mathbb{Z}_p)$
maximal compact

Iwahori fixed vector
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$G = BK$

$$B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$$

Iwahori fixed vector
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Generalizations

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Iwahori subgroup

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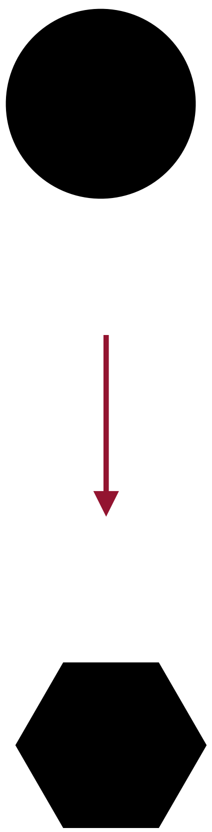
basis enumerated by $W = S_r$

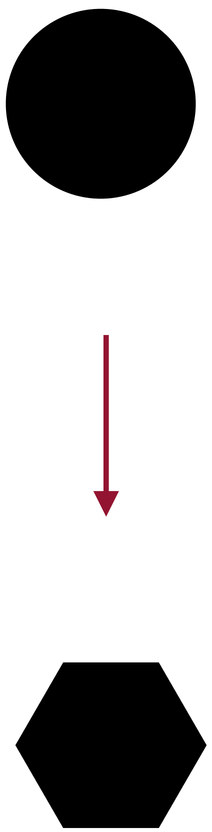
Refinement: $f^\circ_{\mathbf{z}} = \sum_{w \in W} f^{(w)}_{\mathbf{z}}$ each supported only on BwJ

Generalizations



On the lattice model side this refinement corresponds to assigning a different color to each path, making them distinct.

Generalizations



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Schematically (with details to follow):

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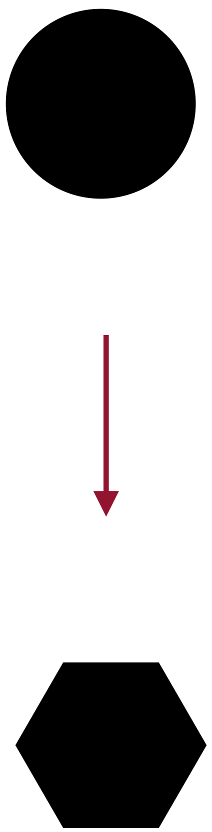


Schematically (with details to follow):

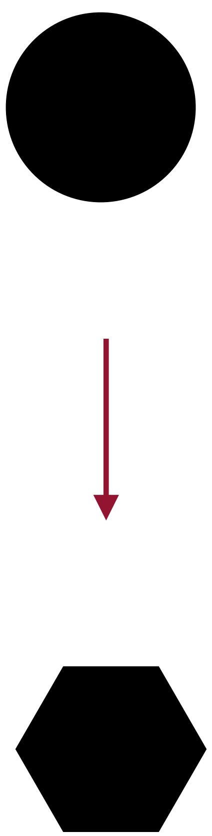
Paper 2 (6-vertex; $v \neq 0$)

spherical Whittaker function

↓ colorize

Iwahori Whittaker functions

Generalizations



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Schematically (with details to follow):

Paper 1 (5-vertex; $v = 0$)	Paper 2 (6-vertex; $v \neq 0$)
Schur polynomial	spherical Whittaker function
↓ colorize	↓ colorize
Demazure atoms	Iwahori Whittaker functions

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Schematically (with details to follow):

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Schur polynomial	spherical Whittaker function
↓ colorize — Duplicate colors: Demazure characters	↓ colorize
Demazure atoms	Iwahori Whittaker functions

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Schematically (with details to follow):

Paper 1 (5-vertex; $v = 0$)

Schur polynomial

colorize

Duplicate colors:
Demazure characters

Demazure atoms

Paper 2 (6-vertex; $v \neq 0$)

spherical Whittaker function

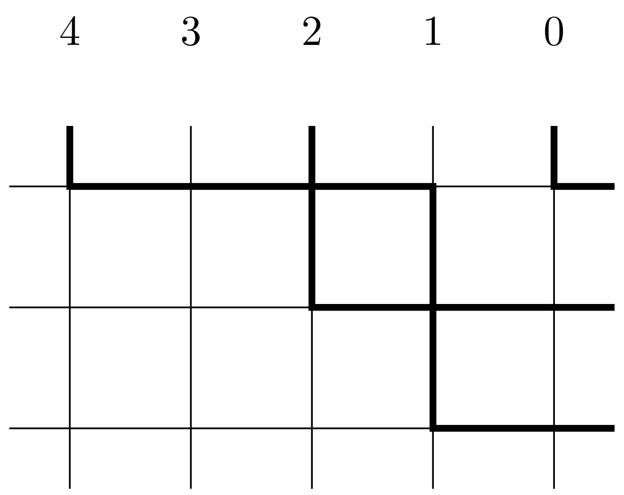
colorize

Duplicate colors:
parahoric Whittaker functions

Iwahori Whittaker functions

Color refinement

Ordered palette of r colors: R $>$ B $>$ G



[Papers 1 & 2] Concept based on [Borodin–Wheeler 2018]

Color refinement

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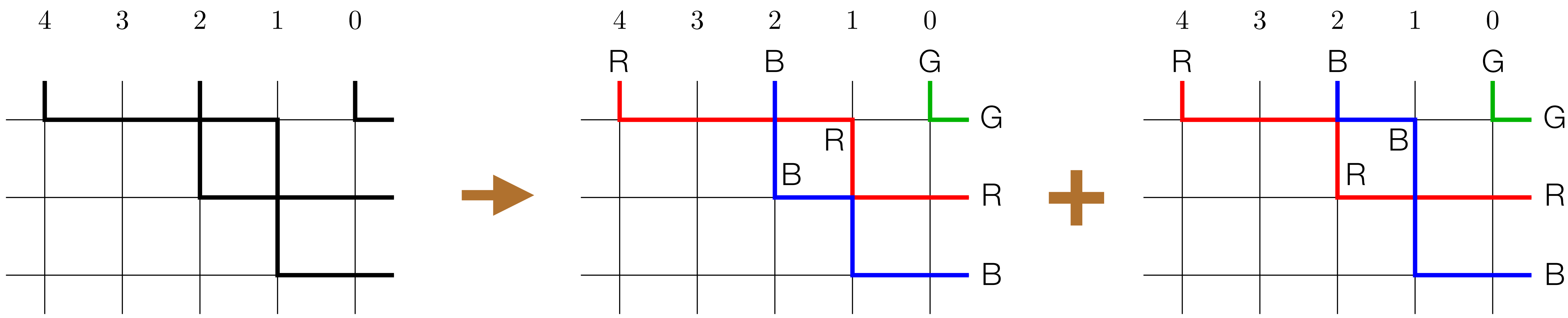

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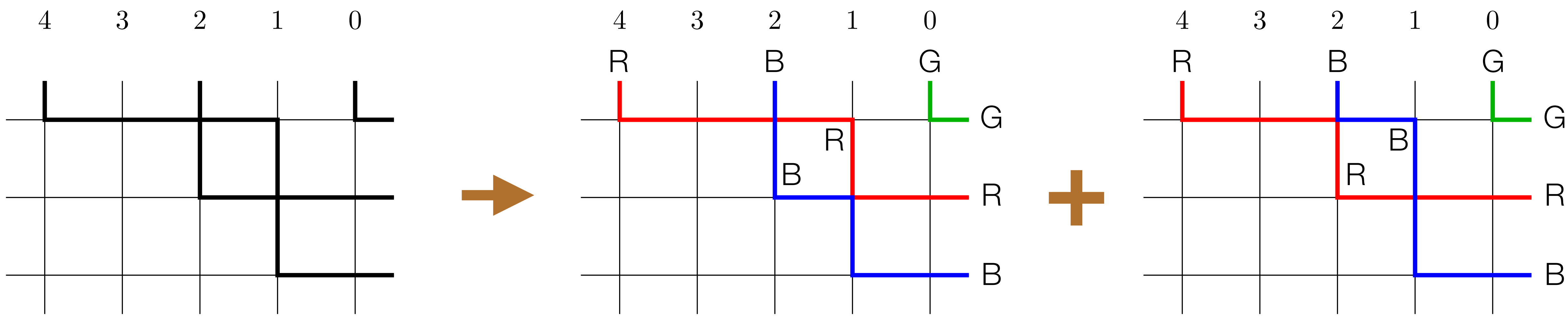

New right boundary data: permutation $w \in S_r$ of (R, B, G)

[Papers 1 & 2] Concept based on [Borodin–Wheeler 2018]

Color refinement

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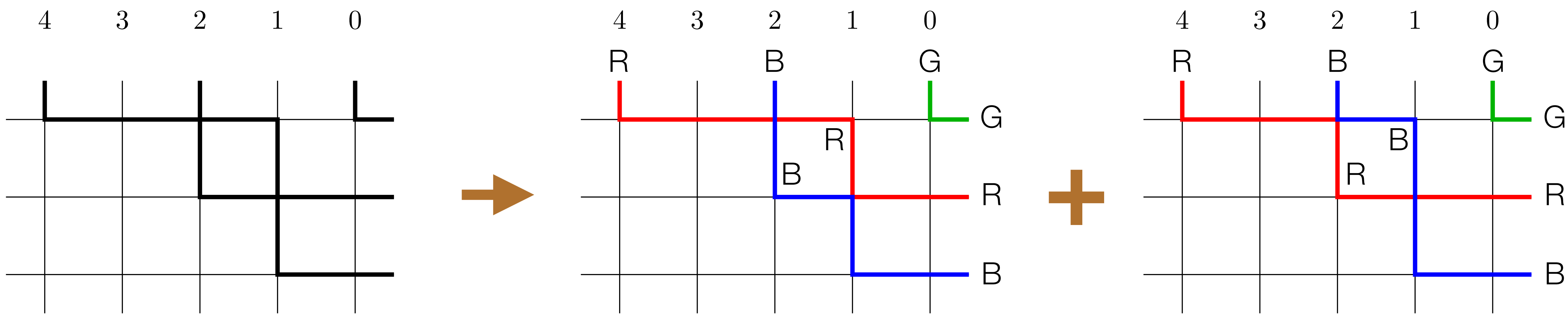


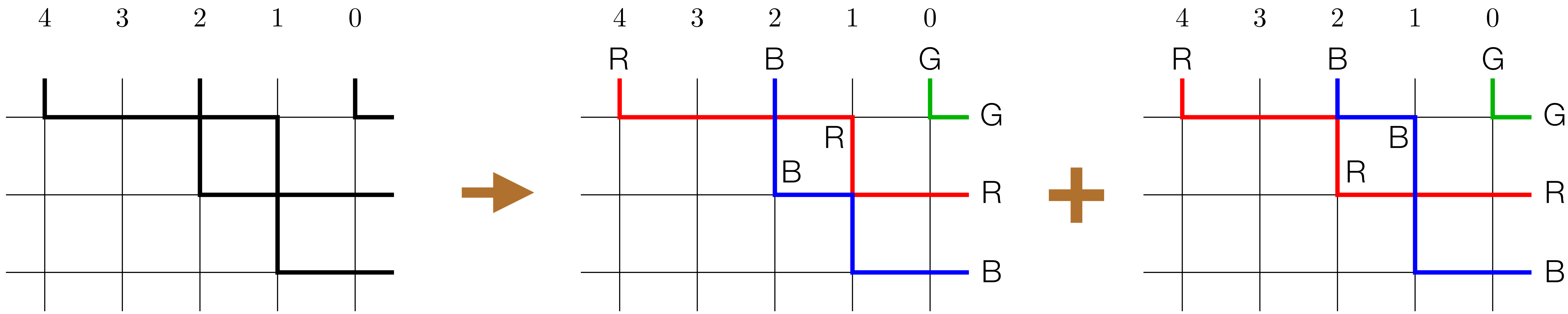
New right boundary data: permutation $w \in S_r$ of (R, B, G)

Have constructed vertex configuration weights such that the partition function is refined to:

uncolored $$Z(\lambda; \mathbf{z}) = \sum_{w \in S_r} Z(\lambda, w; \mathbf{z})$$ colored

[Papers 1 & 2] Concept based on [Borodin–Wheeler 2018]

Color refinement



uncolored $\quad Z(\lambda;\mathbf{z}) = \sum_{w\in S_r} Z(\lambda,w;\mathbf{z})$ $\quad$ colored

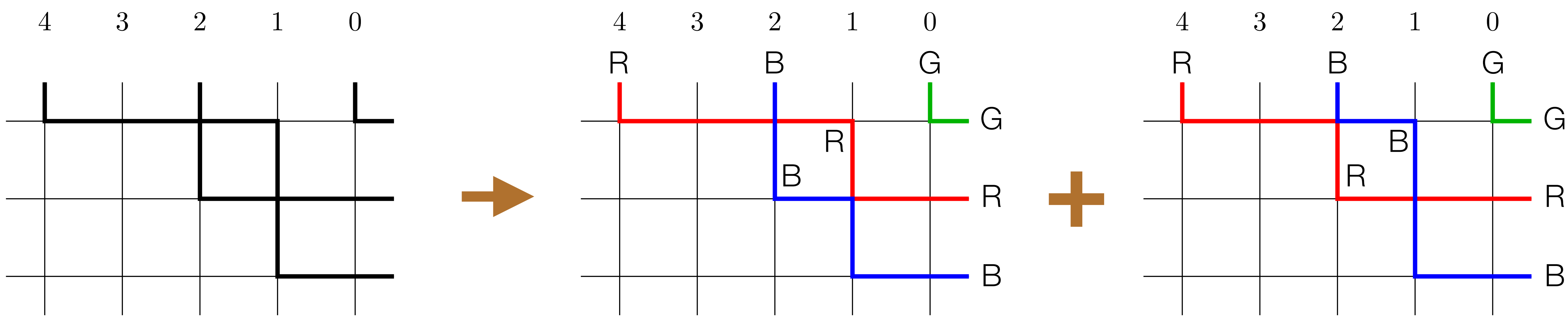
In more detail:

Paper 1 $\quad$ (5-vertex; $v=0$)

Theorem:

$Z(\lambda,w;\mathbf{z})_{v=0}$ = Demazure atom

Color refinement



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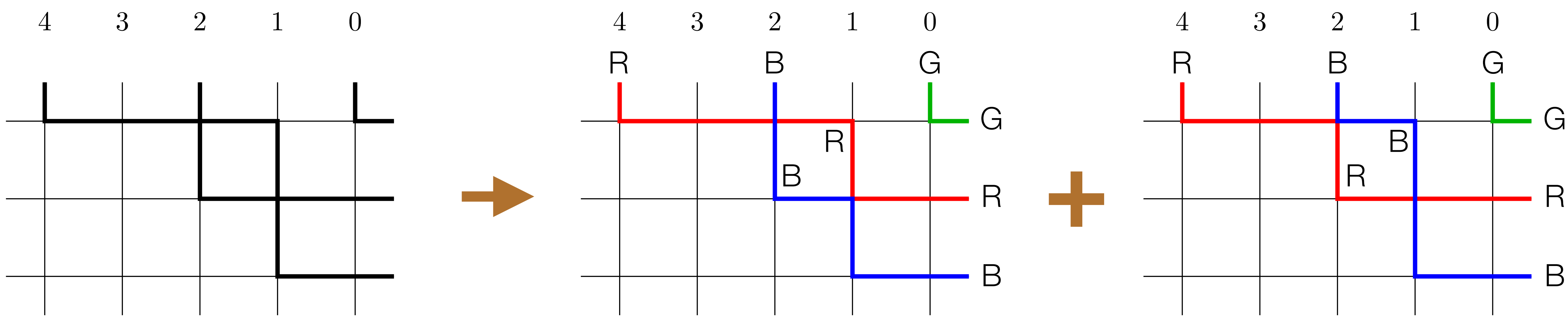
$\sum_{w \in S_r}$ ⟶ Schur polynomial

Color refinement



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$Z(\lambda, w; \mathbf{z})_{v=0} =$ Demazure atom

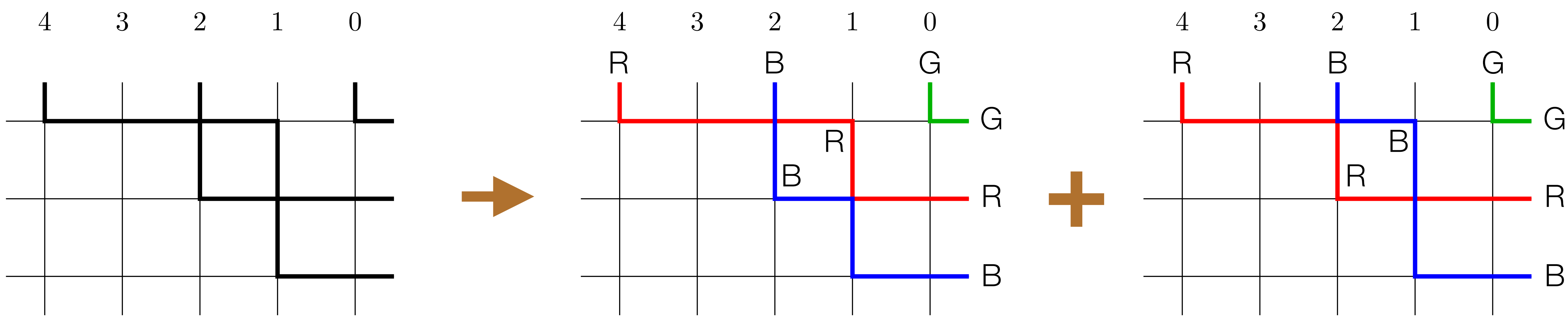
$\sum_{w \in S_r} \longrightarrow$ Schur polynomial

Paper 2 (6-vertex; $v \neq 0$)

Theorem:

$Z(\lambda, w; \mathbf{z})_{v=p^{-1}} =$ Iwahori Whittaker function $\mathcal{W}_\psi(f_{\mathbf{z}}^{(w)})(p^\lambda)$

Color refinement



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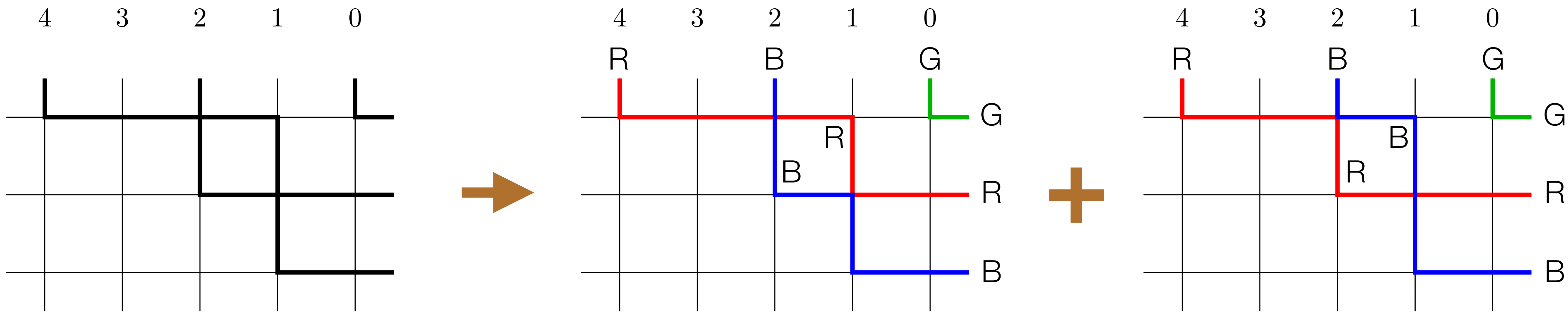
$\sum_{w\in S_r}$ ⟶ Spherical Whittaker function

Color refinement



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In more detail:

Bijection of data

states $\longleftrightarrow$ crystal Demazure atoms

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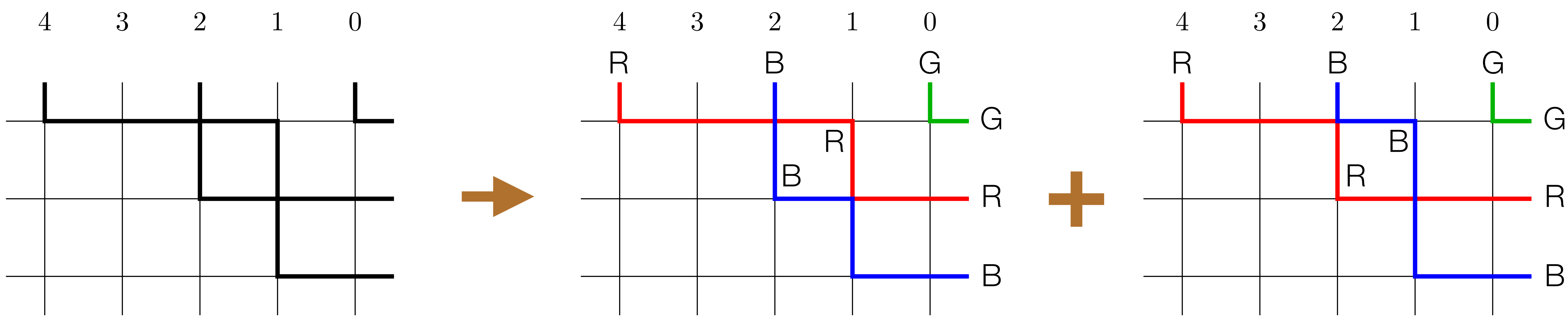
Color refinement



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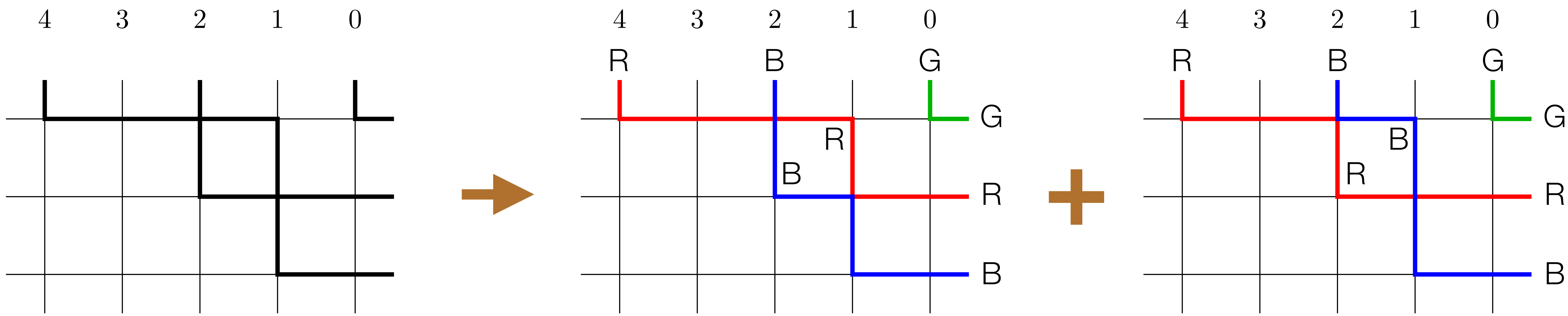
Bijection of data

states $\longleftrightarrow$ crystal Demazure atoms

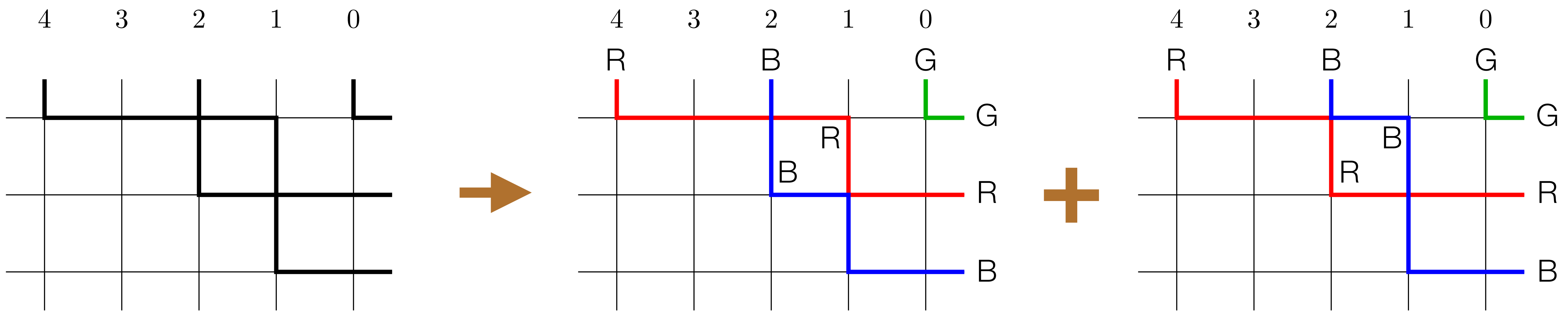
boundary data $\longleftrightarrow$ Whittaker data

Color refinement



Extra structure ⟶ Partition function from recurrence relations

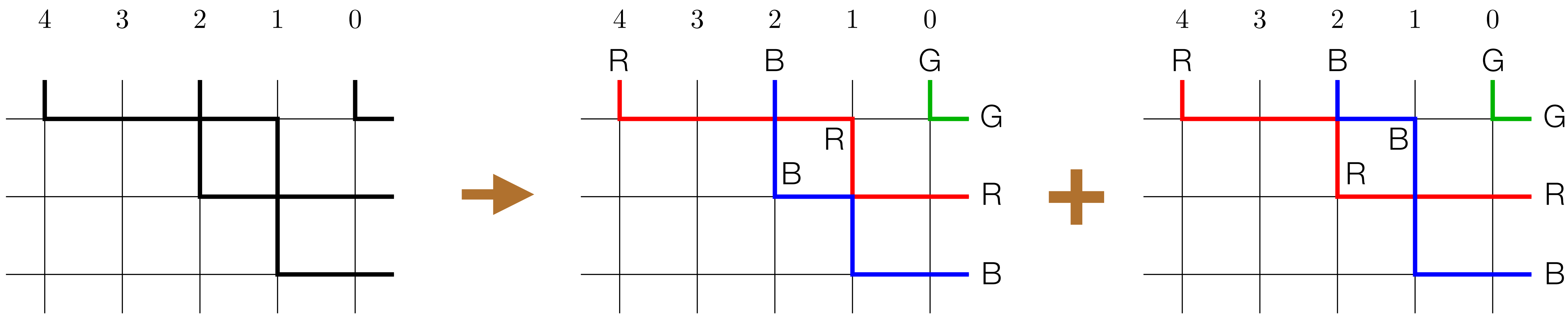
Color refinement



Extra structure ⟶ Partition function from recurrence relations

Solvability via Yang–Baxter equations from underlying quantum group

Color refinement

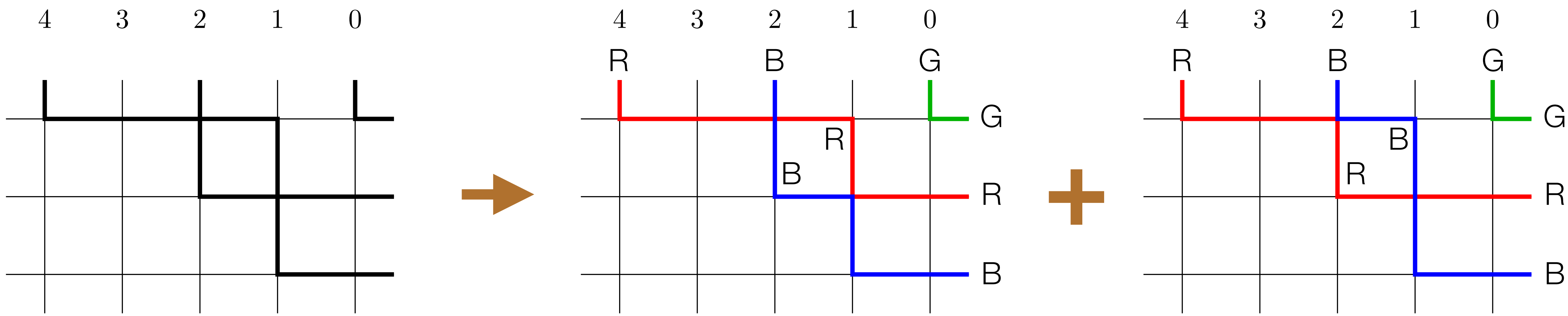


Extra structure ⟶ Partition function from recurrence relations

Solvability via Yang–Baxter equations from underlying quantum group

Theorem: [Papers 1, 2] $Z(\lambda, s_{i_1} \cdots s_{i_r}; \mathbf{z}) = T_{i_1} \cdots T_{i_r} \mathbf{z}^{\lambda+\rho}$

Color refinement



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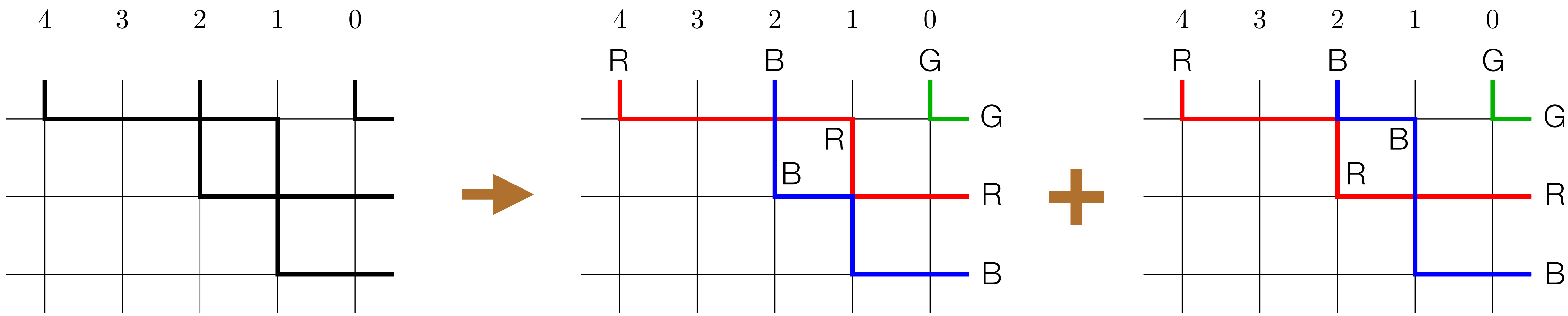
Divided difference Demazure operators

$$T_i f(\mathbf{z}) = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} f(s_i \mathbf{z}) + \frac{v-1}{1-\mathbf{z}^{\alpha_i}} f(\mathbf{z})$$

Color refinement



Extra structure ⟶ Partition function from recurrence relations

Solvability via Yang–Baxter equations from underlying quantum group

Theorem: [Papers 1, 2] $Z(\lambda, s_{i_1} \cdots s_{i_r}; \mathbf{z}) = T_{i_1} \cdots T_{i_r} \mathbf{z}^{\lambda+\rho}$

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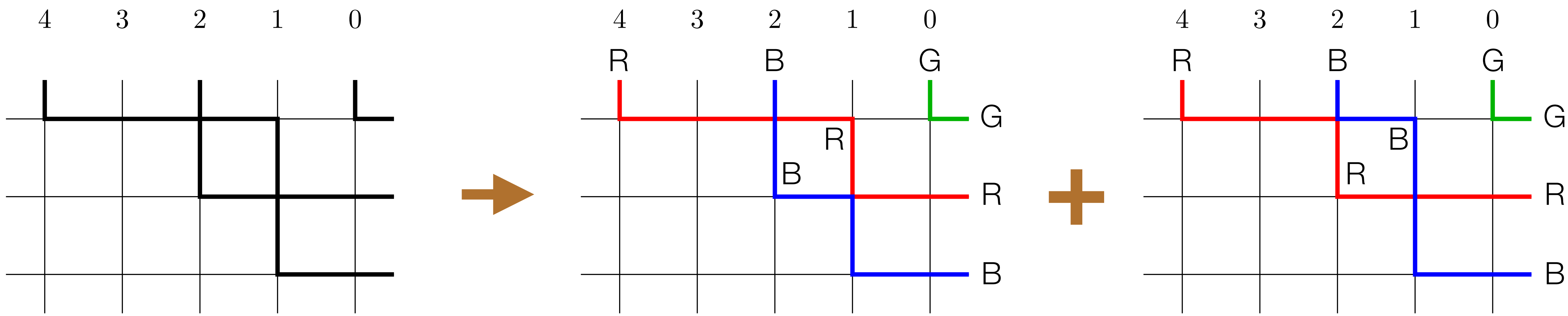

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and 3 (metaplectic)

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Metaplectic groups

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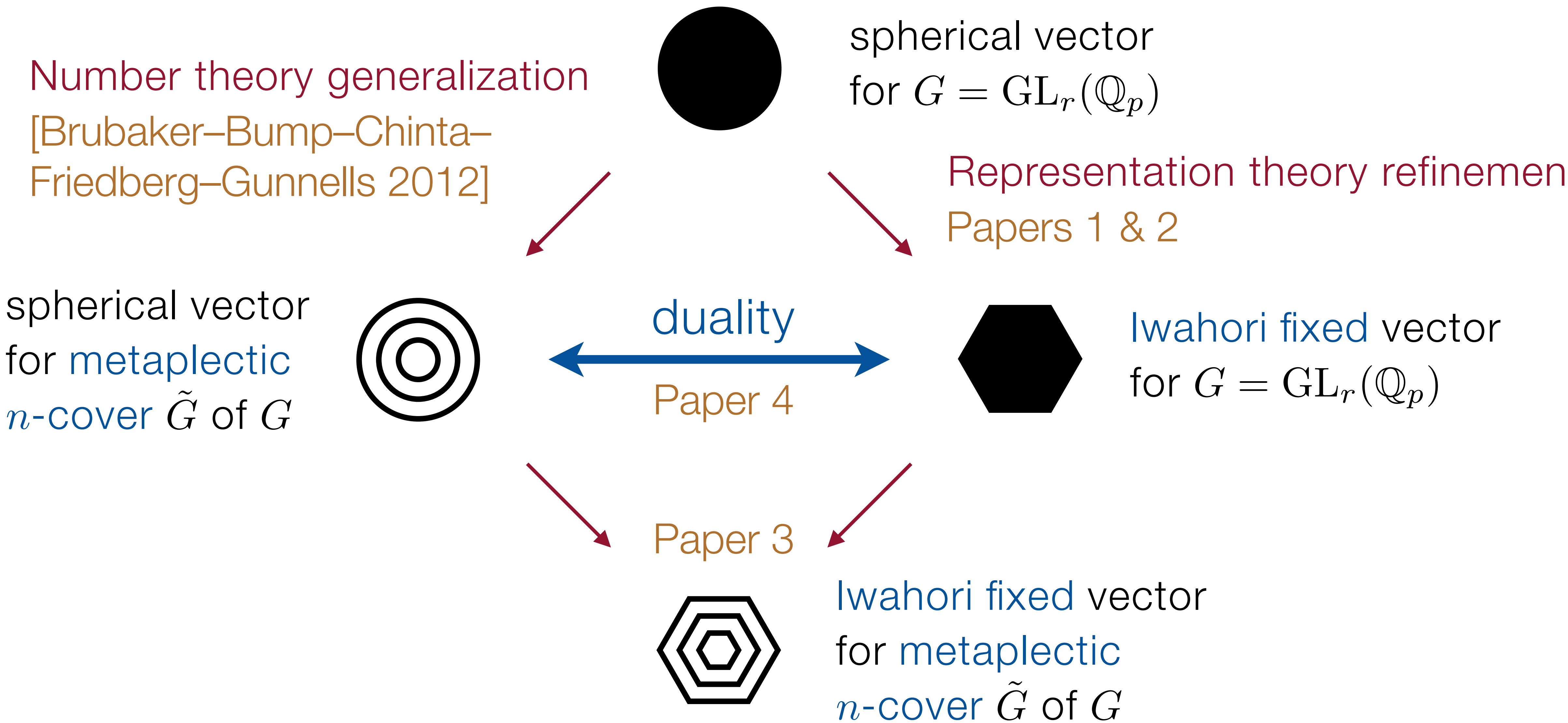
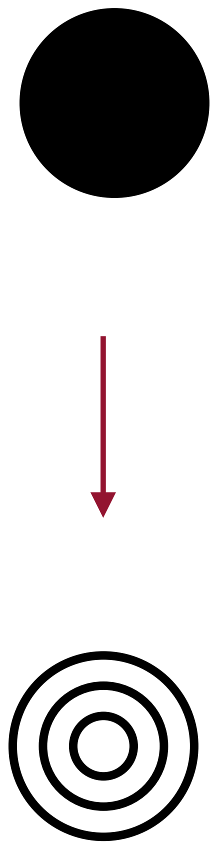


Blue terms will be defined in the next slides

Metaplectic Whittaker functions

The metaplectic n-cover $\tilde{G}$ of G is a central extension:

$$1 \longrightarrow \langle e^{2\pi i/n} \rangle \longrightarrow \tilde{G} \xrightarrow{\text{proj}} G \longrightarrow 1$$



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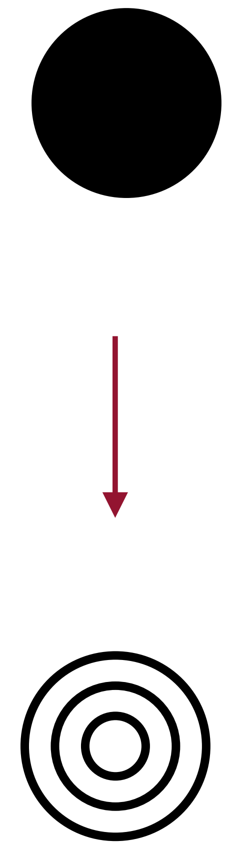

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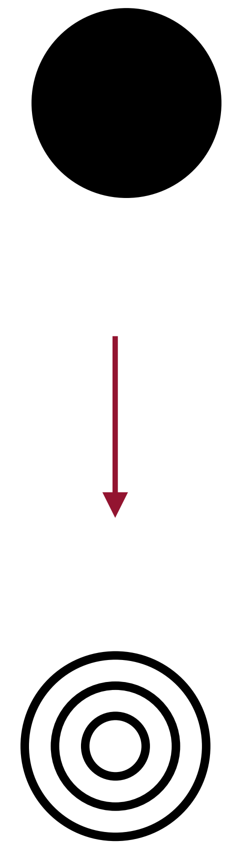

The principal series representation $\pi_{\mathbf{z}}$ with $\mathbf{z} \in (\mathbb{C}^\times)^r$ is constructed similarly, but is now vector-valued of dimension n^r.

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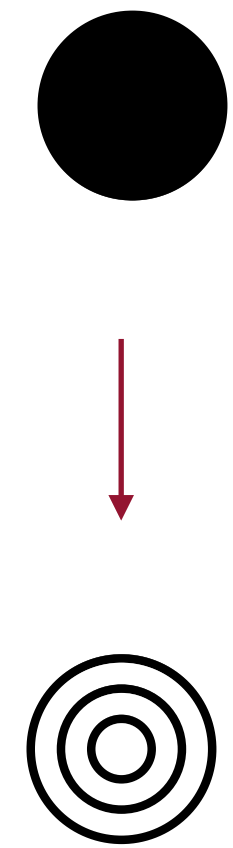

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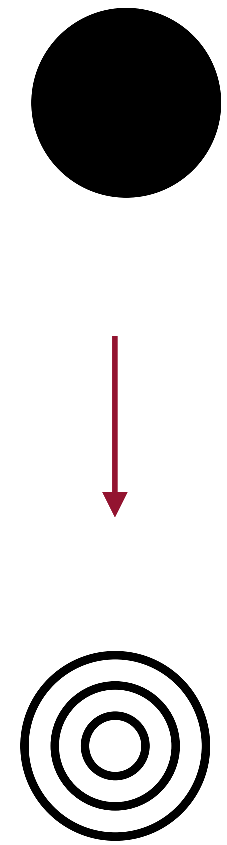

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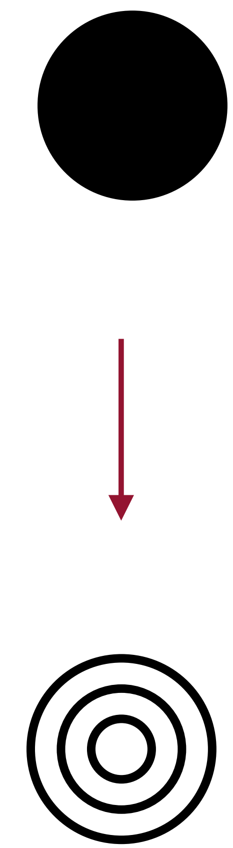

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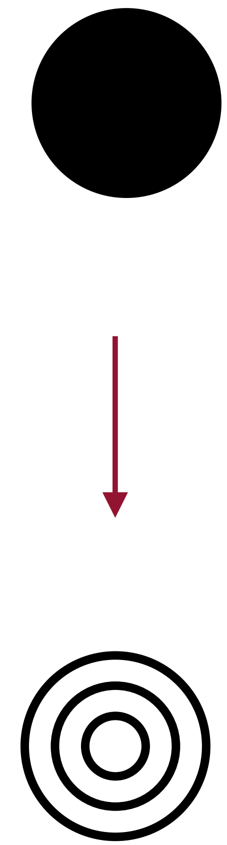

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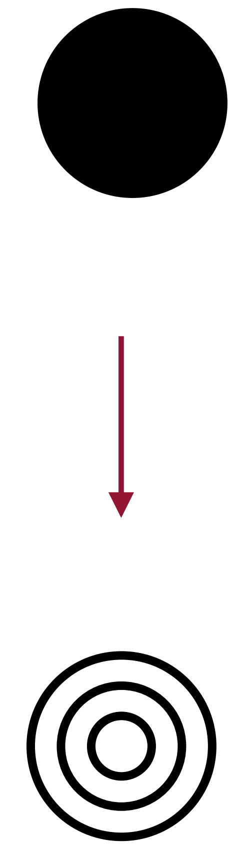

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Often, (e.g. [Chinta–Gunnells–Puskás 2017, Patnaik–Puskás 2017, McNamara 2016, Sahi–Stokman–Venkateswaran 2021]) the σ-average is considered.

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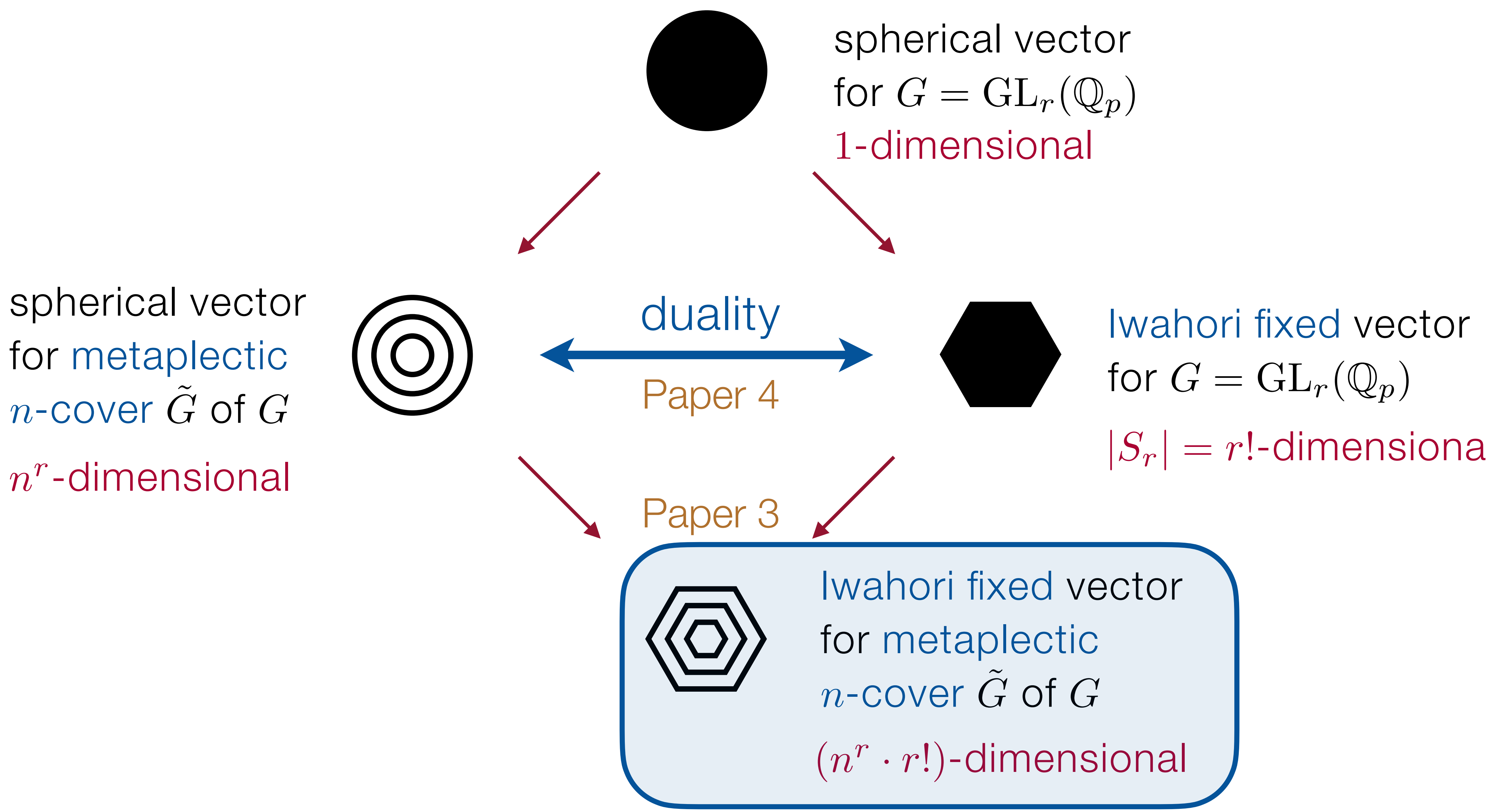
Used for constructing Weyl group multiple Dirichlet series which have, together with their applications to L-functions, been studied extensively in:

Kubota 1969, Bump–Friedberg–Hoffstein 1996, Diaconu-Goldfeld-Hoffstein 2003, Friedberg–Hoffstein–Lieman 2003, Brubaker–Bump–Chinta–Friedberg–Hoffstein 2006, Brubaker–Bump 2006, Chinta–Gunnells 2009, Brubaker–Bump–Friedberg 2011, ...

Metaplectic groups



Includes the others as subcases

Metaplectic Iwahori lattice model

Two sets of r paths assigned colors from two different palettes:

- r colors distinct south-east moving paths
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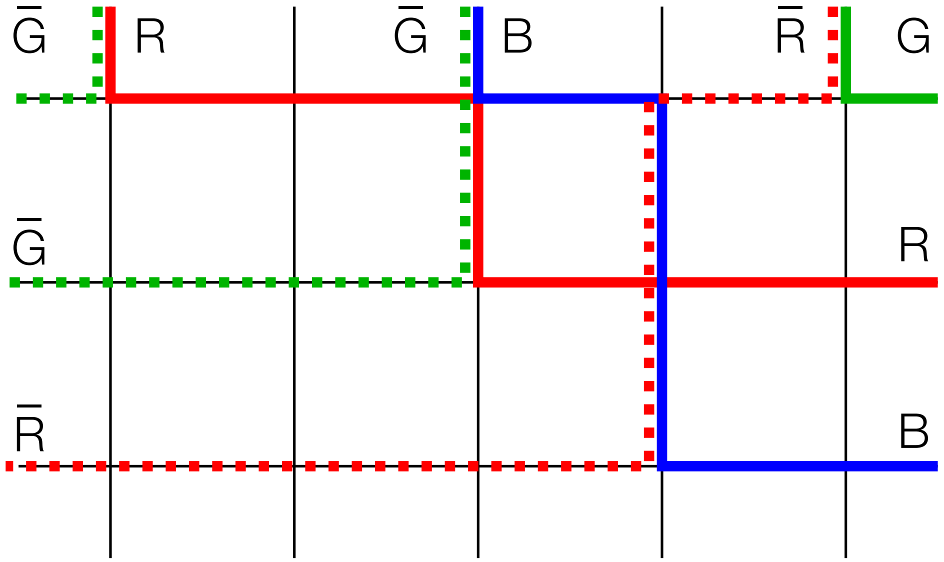

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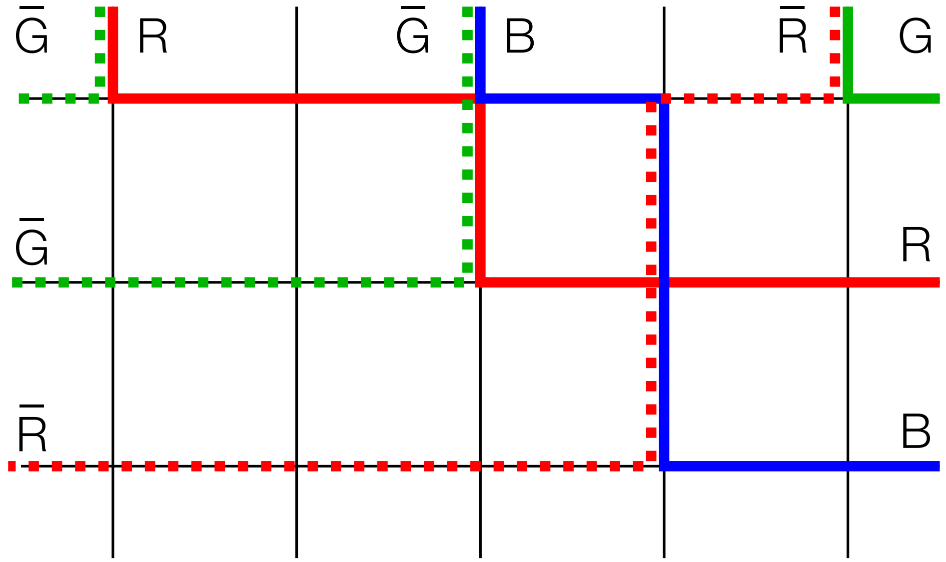

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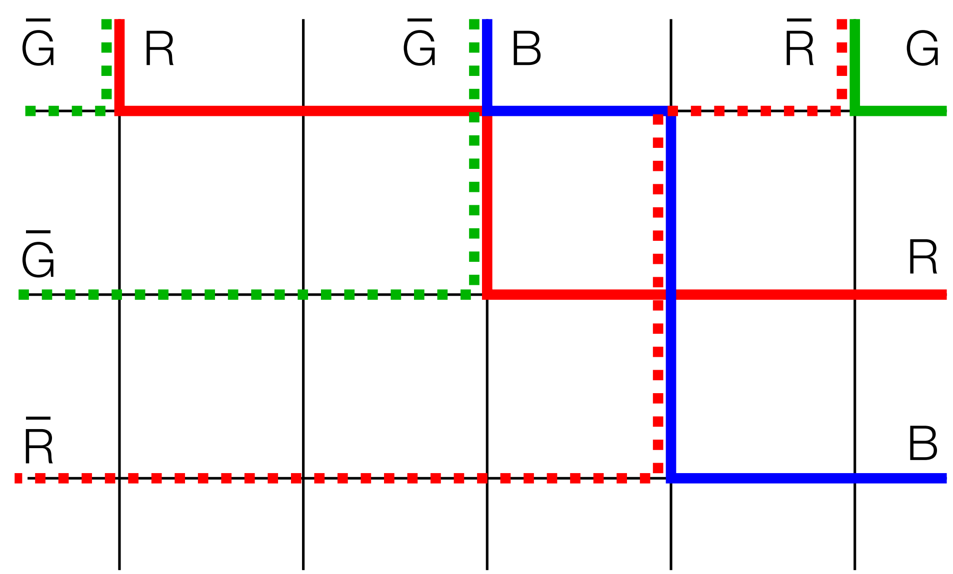

Iwahori basis

$w \in S_r$

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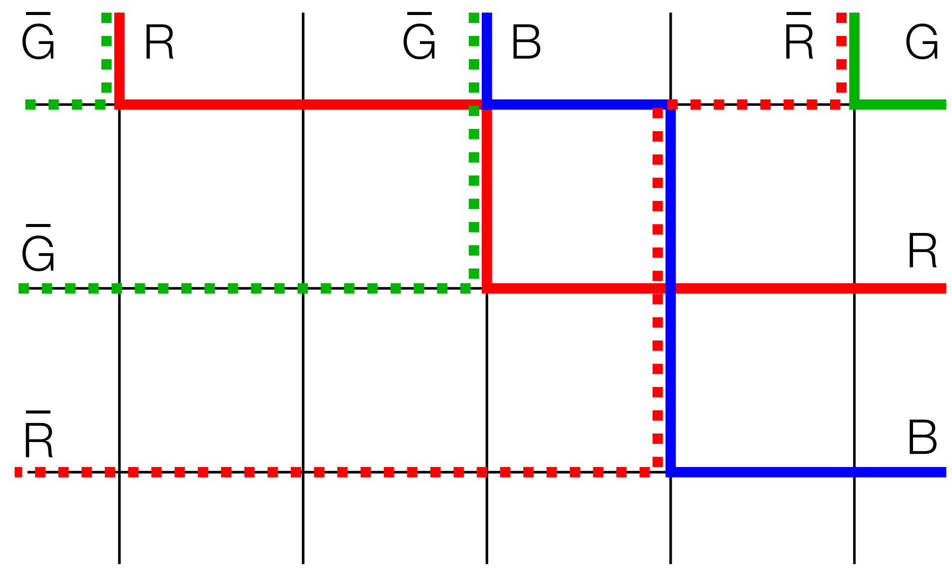

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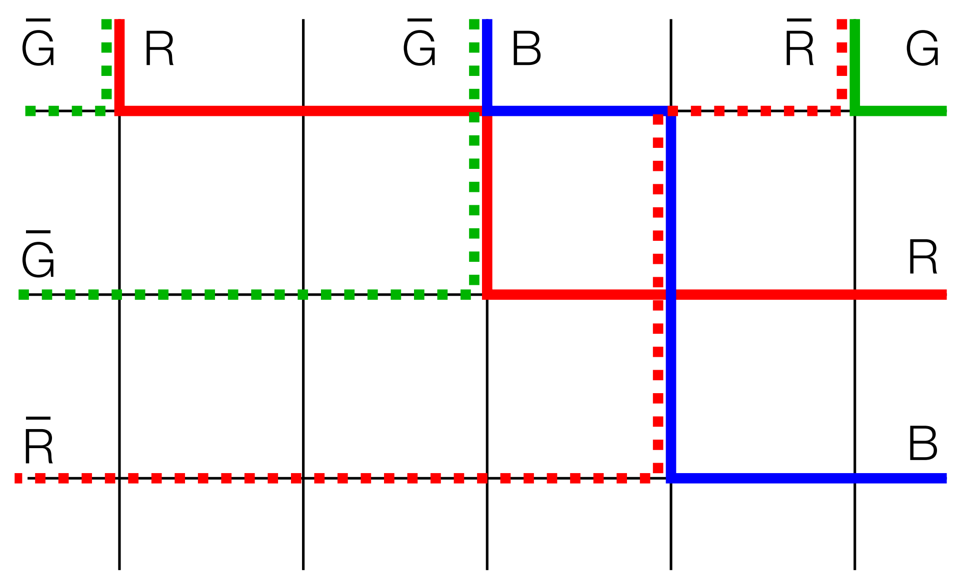

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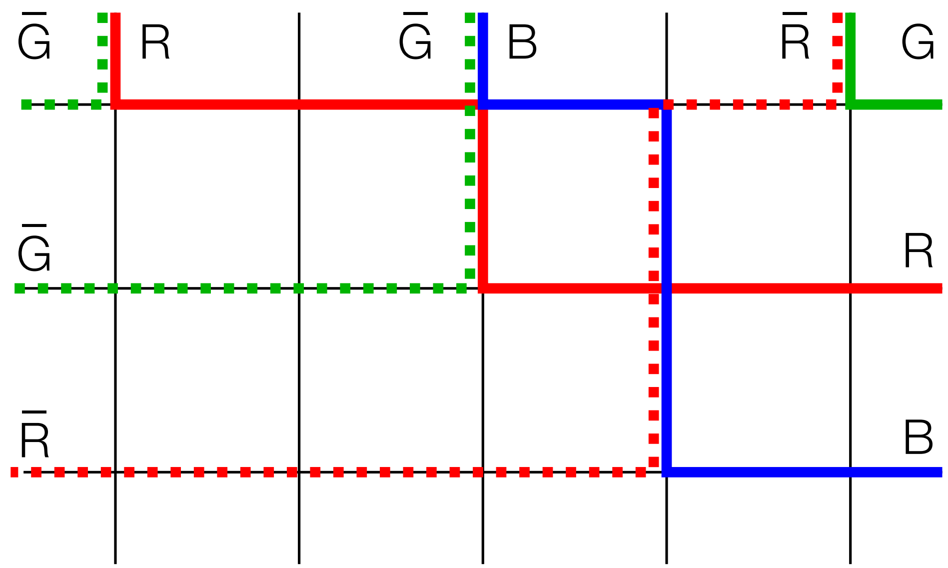

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$Z(\lambda, w, \sigma; \mathbf{z})$ = metaplectic Iwahori Whittaker function $\mathcal{W}_\psi^\sigma(f_{\mathbf{z}}^{(w)})(p^\lambda)$

Bijection: boundary data $\longleftrightarrow$ Whittaker function data

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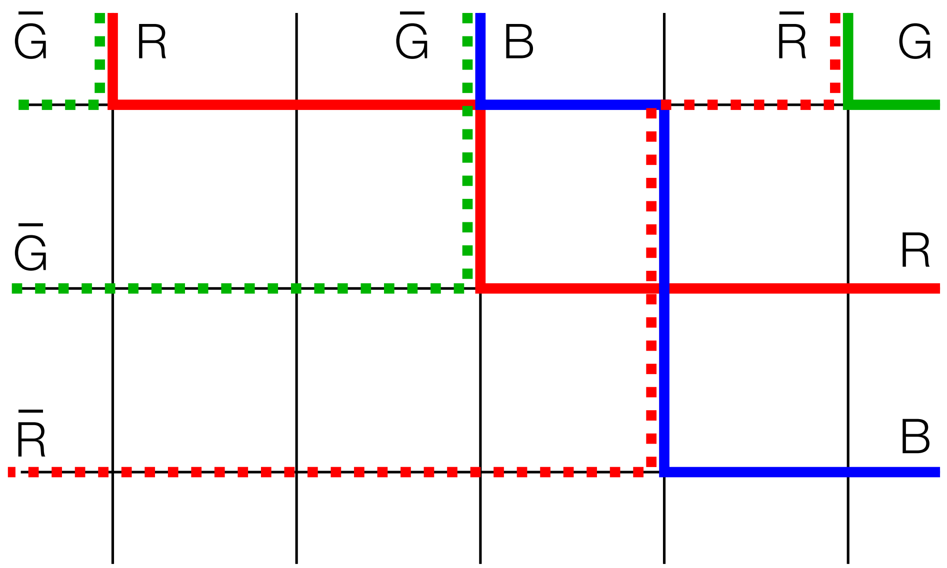

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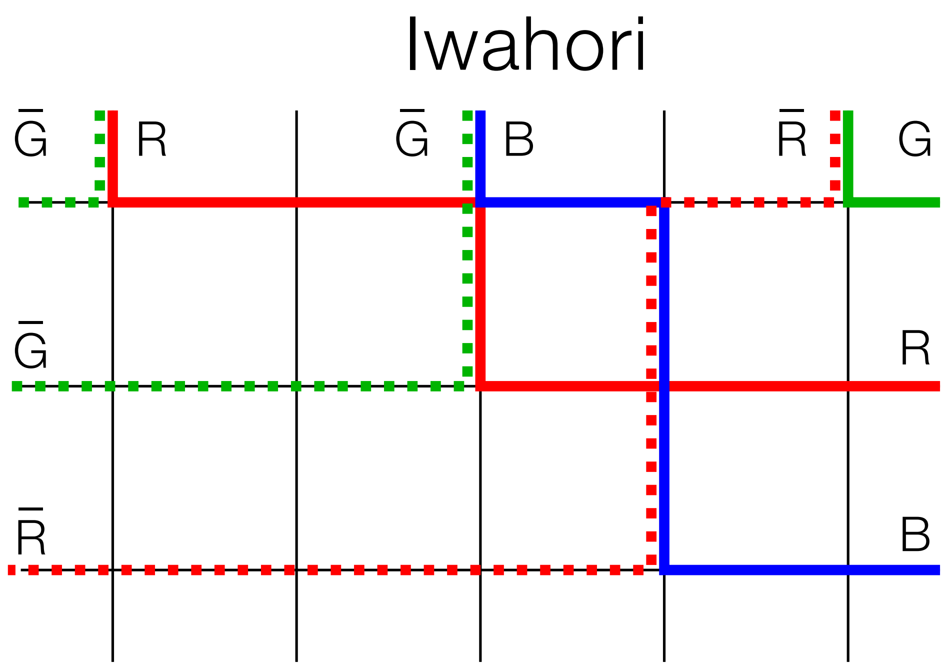

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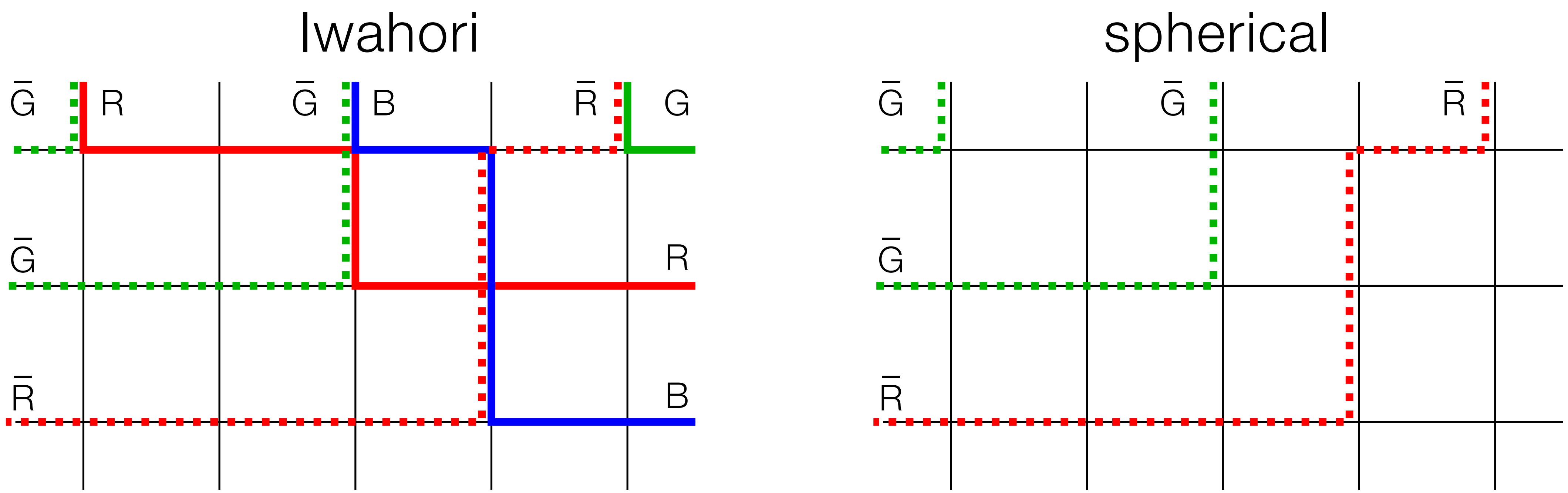

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Iwahori–metaplectic duality

[Paper 4]

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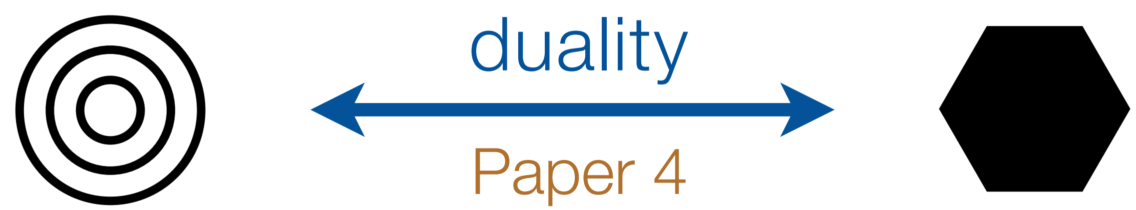
spherical vector for metaplectic n-cover $\tilde{G}$ of G



Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

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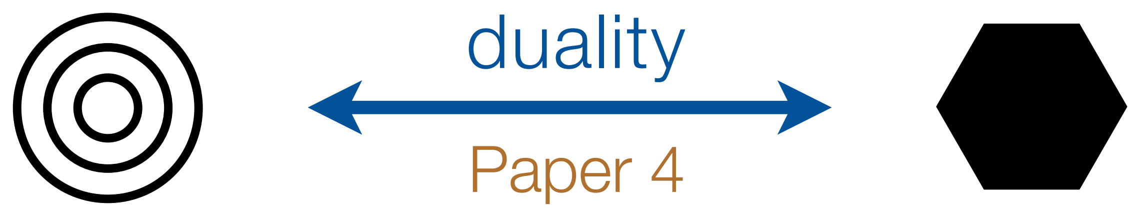

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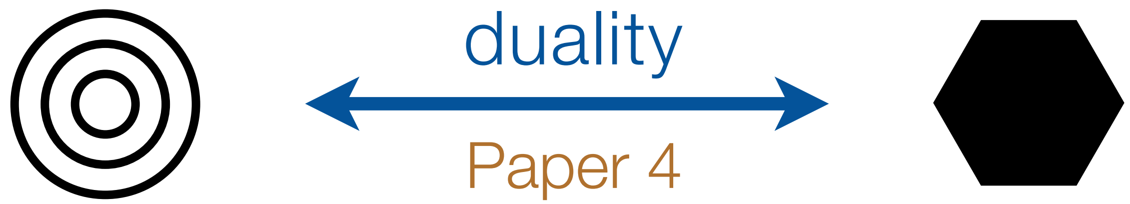

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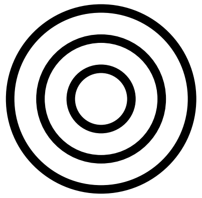

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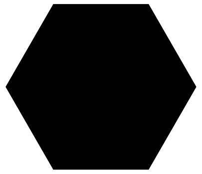

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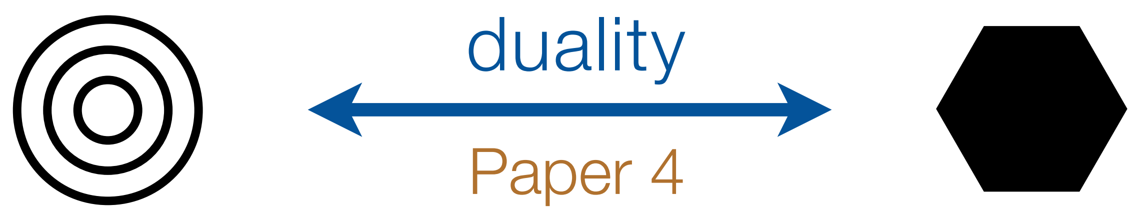

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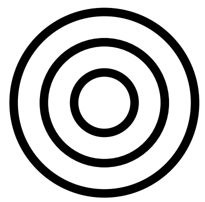

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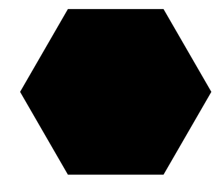

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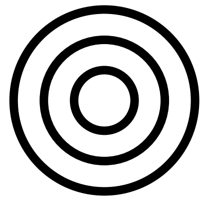
Conjecture: in general $\tilde{\phi}^\circ_\sigma(\mathbf{z}; g) \approx$ (non-metaplectic parahoric Whittaker function) with more complicated insertions of Gauss sums.

[Paper 4]

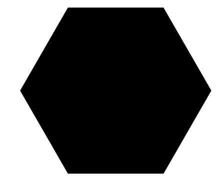
Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G



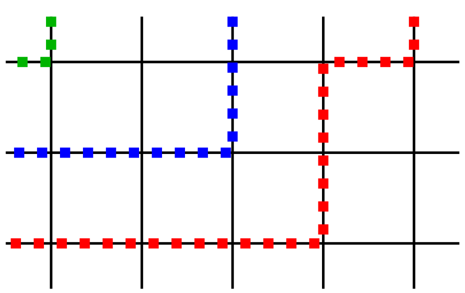
duality

Paper 4



Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

Idea:



metaplectic spherical

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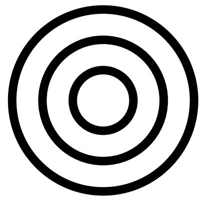
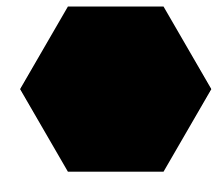


duality

Paper 4



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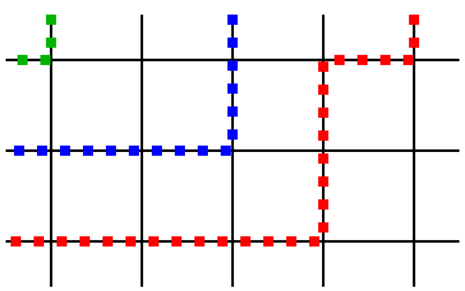
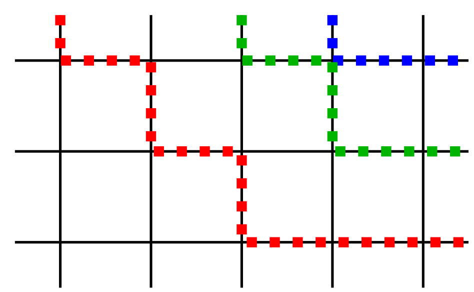


metaplectic spherical

Γ-Δ correspondence

equality for partition functions (not for individual states)

[Brubaker–Bump–Friedberg 2011, Brubaker–Buciumas–Bump 2019]



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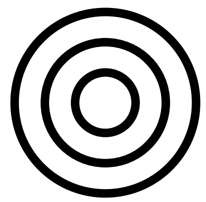
Iwahori–metaplectic duality

spherical vector for metaplectic n-cover $\tilde{G}$ of G 

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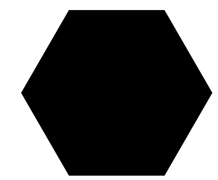
Paper 4

 Iwahori fixed vector for $G = \mathrm{GL}_r(\mathbb{Q}_p)$

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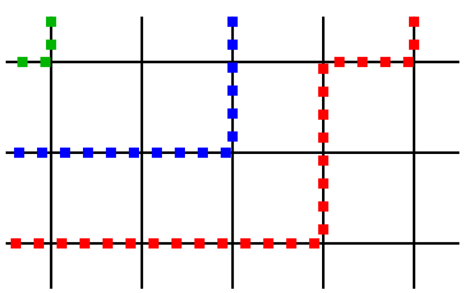

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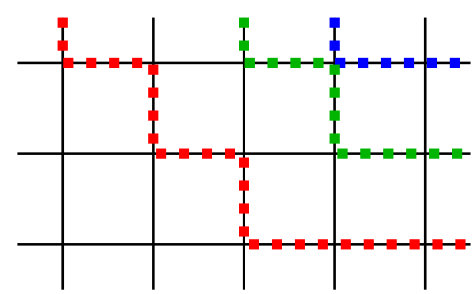
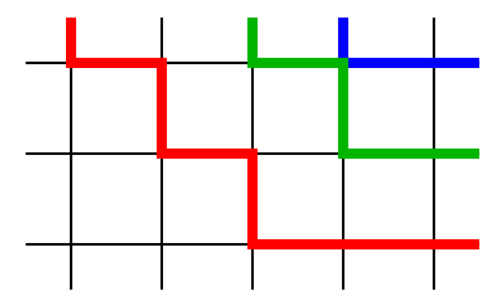


Drinfeld twist

equality for states (changes Boltzmann weights and partition function)



non-metaplectic Iwahori

[Paper 4]

Iwahori–metaplectic duality

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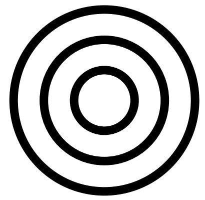
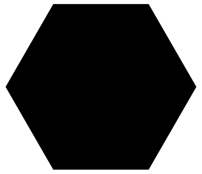


duality

Paper 4



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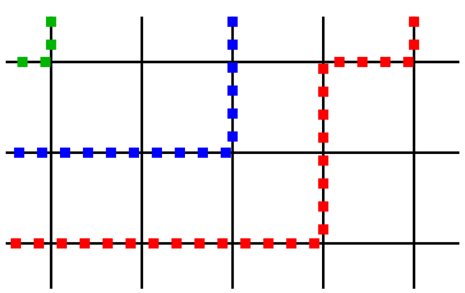

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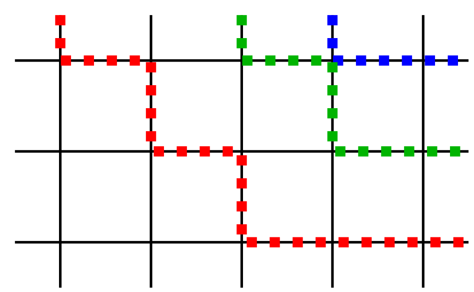

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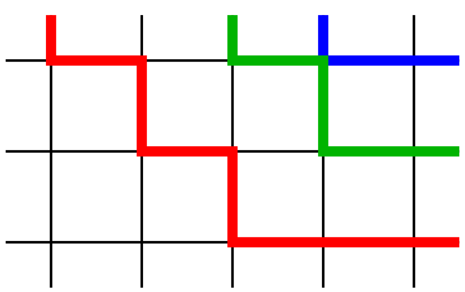


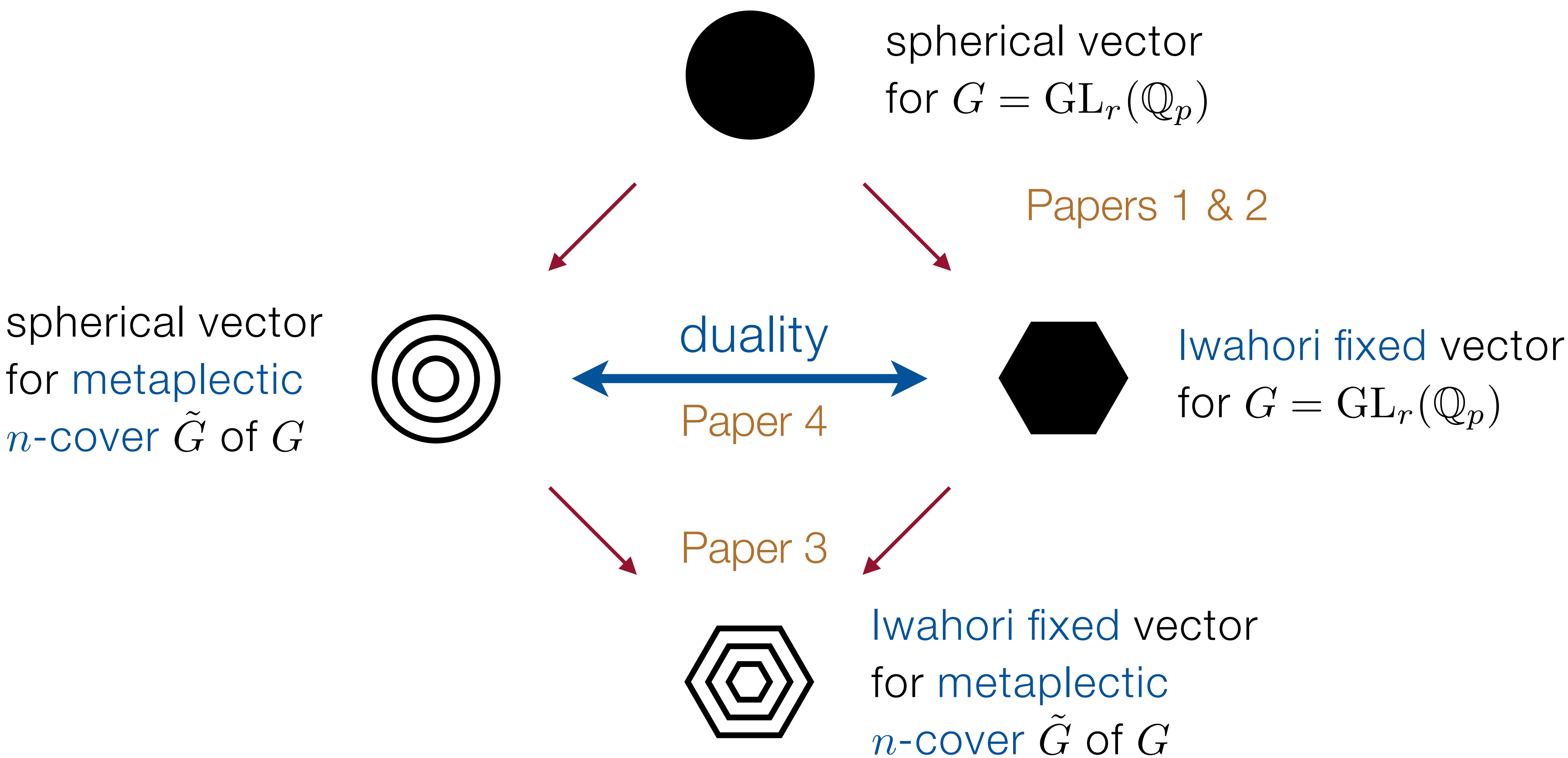
non-metaplectic Iwahori

There is also a representation theoretical version using Demazure operators

[Paper 4]

Summary

partition functions $\stackrel{\sim}{\longleftrightarrow}$ Whittaker functions



Other results

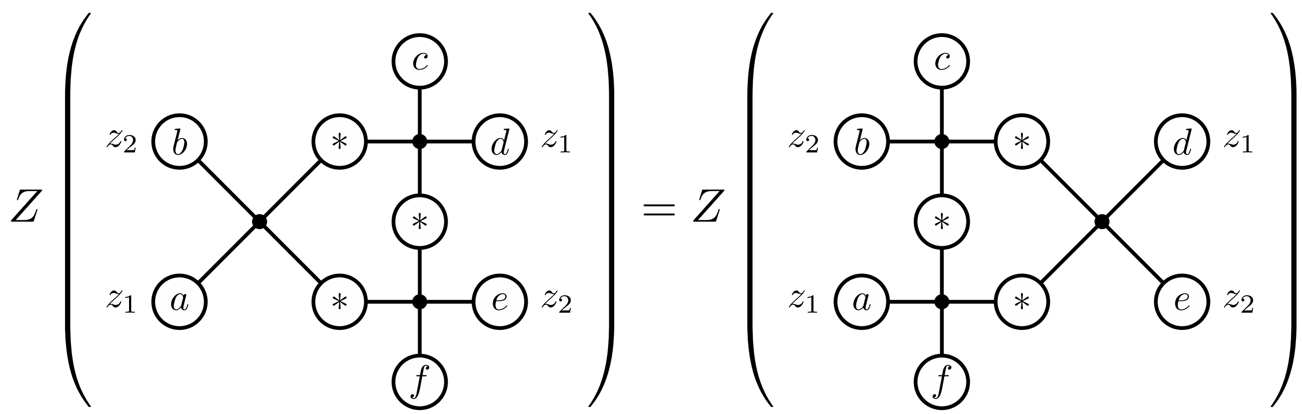
- Cauchy identities for metaplectic spherical Whittaker functions [Paper 4]
- Partition functions as quantum Fock space vertex operators [Paper 0 & 4]
- *All* values of (metaplectic) Iwahori Whittaker functions (previously only on torus arguments) and showed that any σ-component can easily be extracted from the σ-average) [Papers 2,3]
- Parahoric Whittaker functions are Macdonald polynomials with prescribed symmetry [Paper 2]
- Algorithms for computing Lascoux-Schützenberger keys (of interest in combinatorics) [Paper 1]
- Constructing solvable lattice models via fusion of vertices [Papers 2 & 3]

Thank you!

Slides are available at

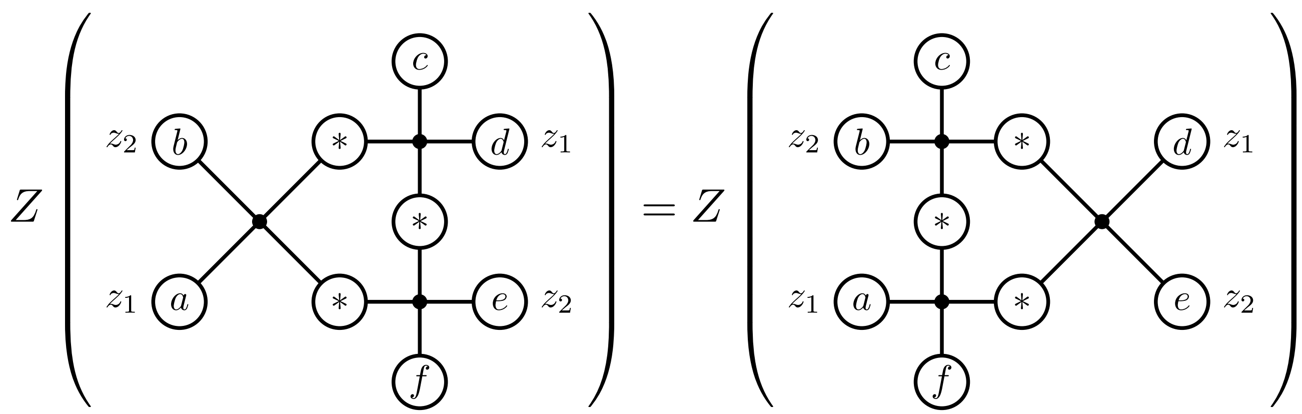
https://hgustafsson.se

Yang-Baxter equations



Train argument

Yang-Baxter equations



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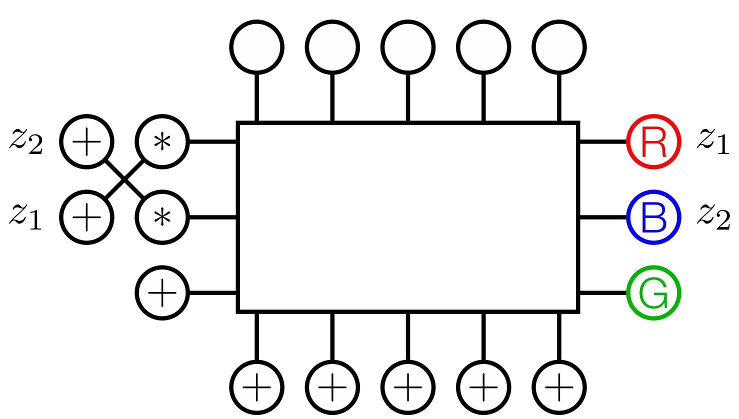

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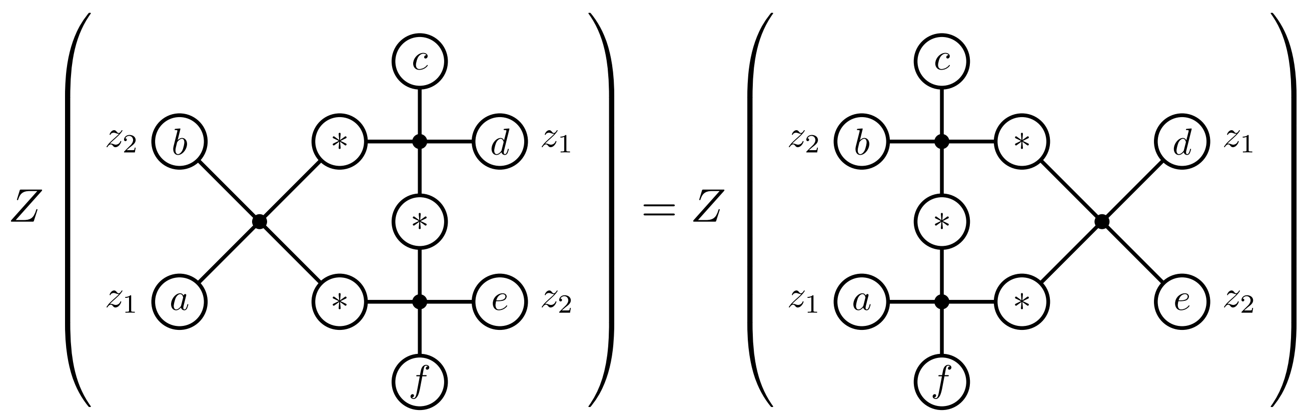

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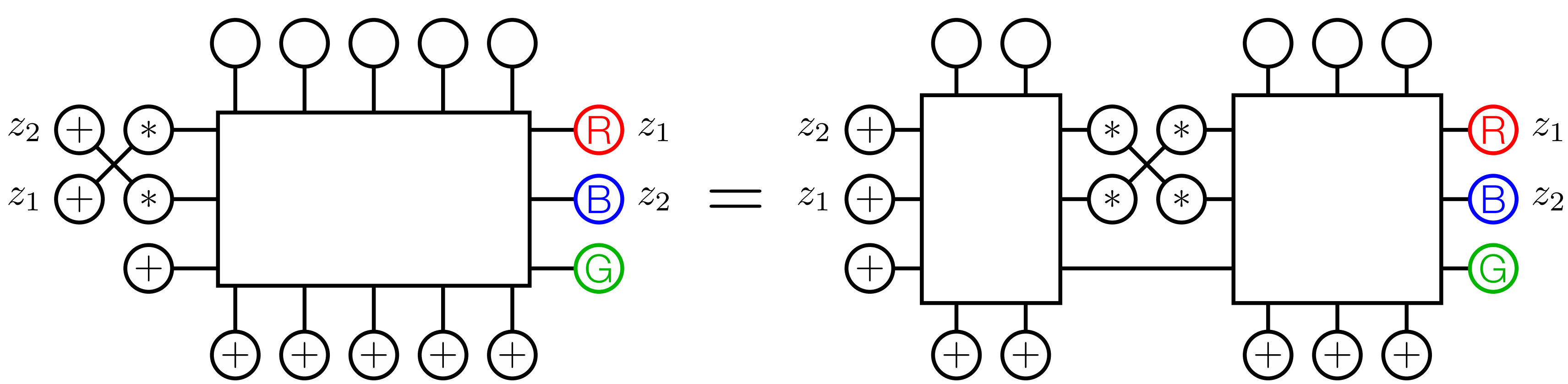

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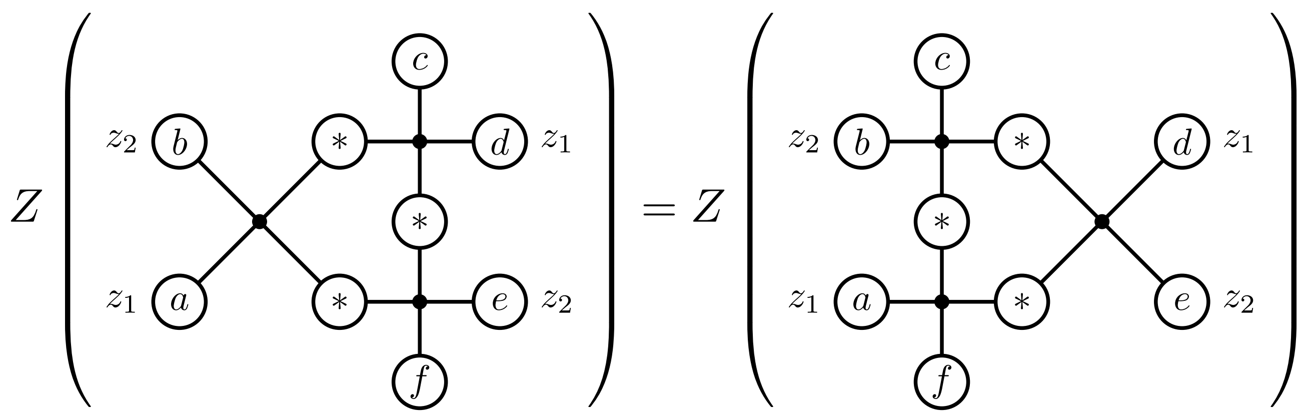

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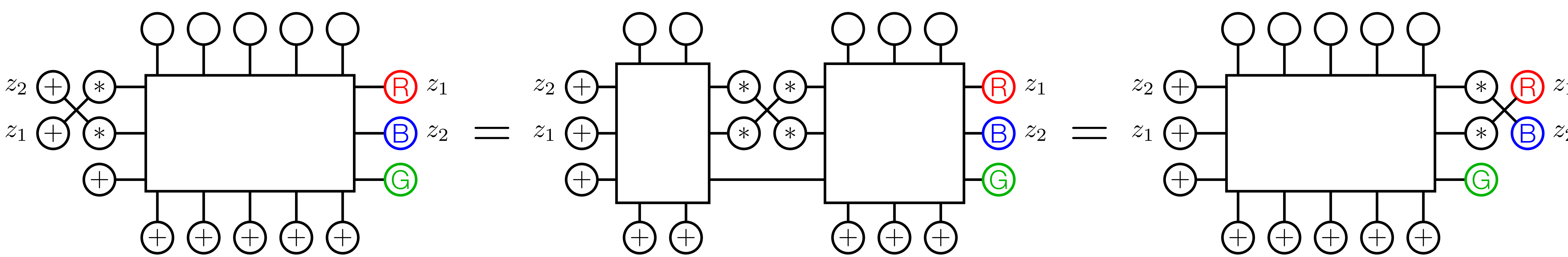

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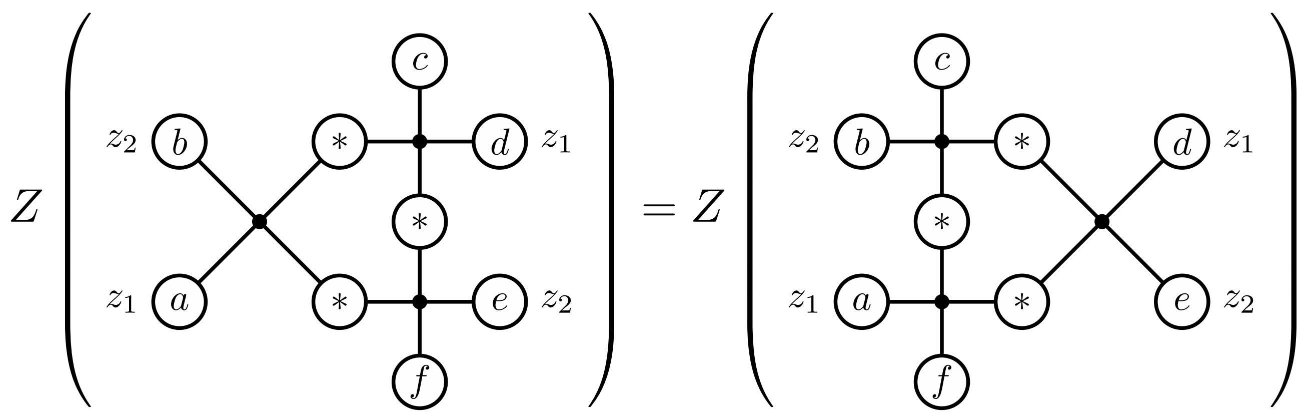

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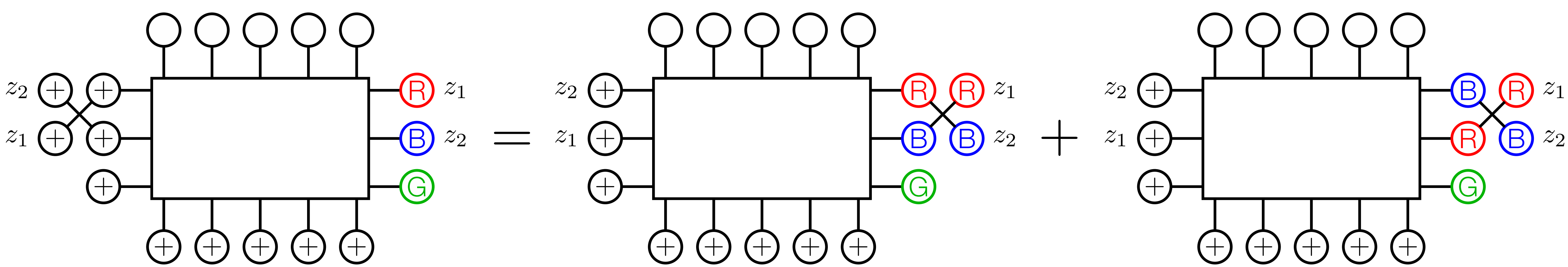

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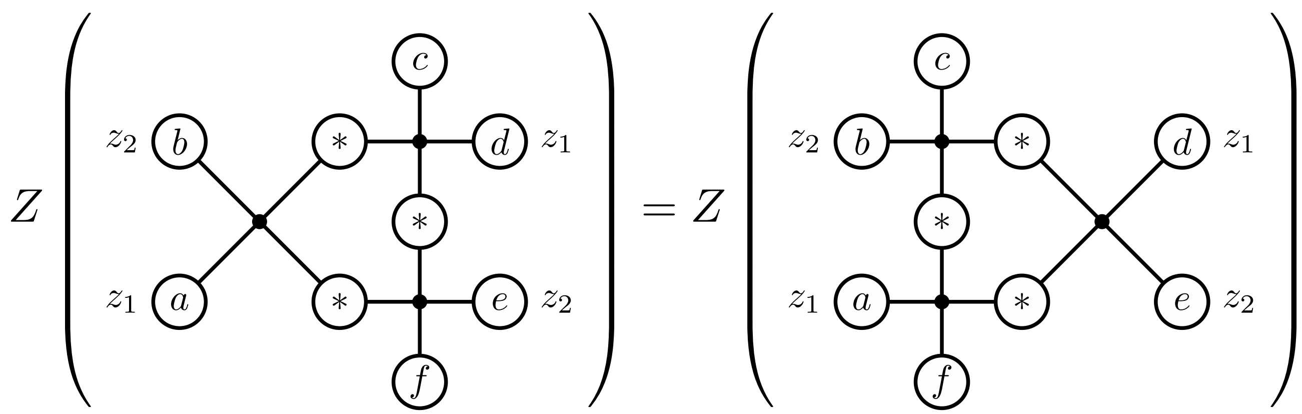

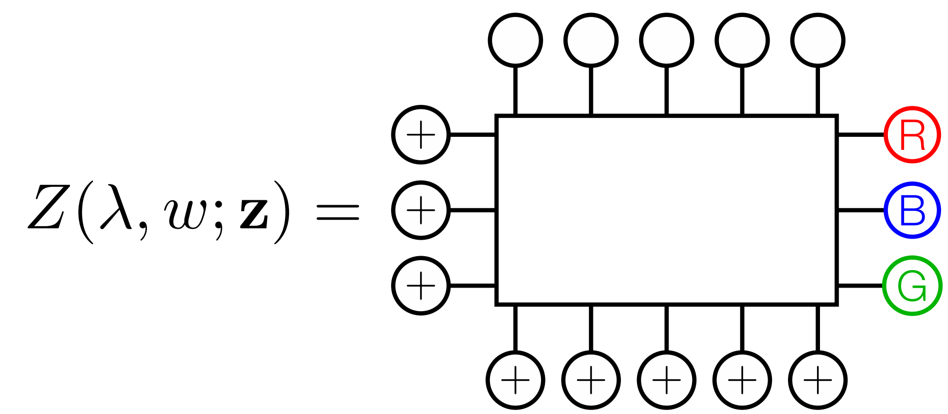

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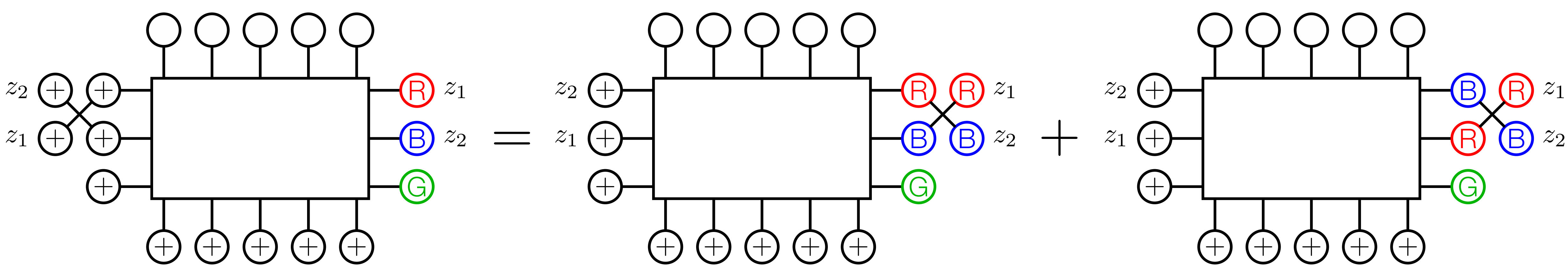

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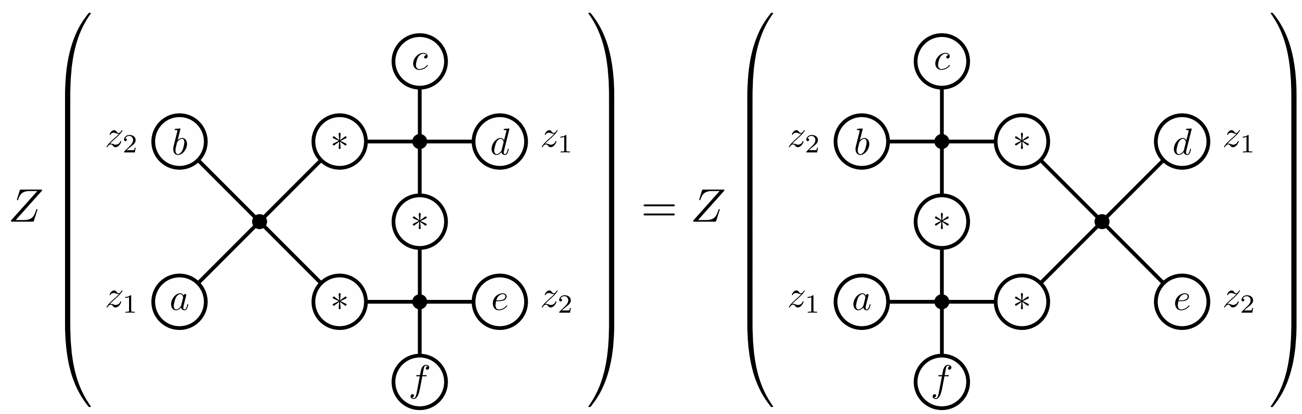
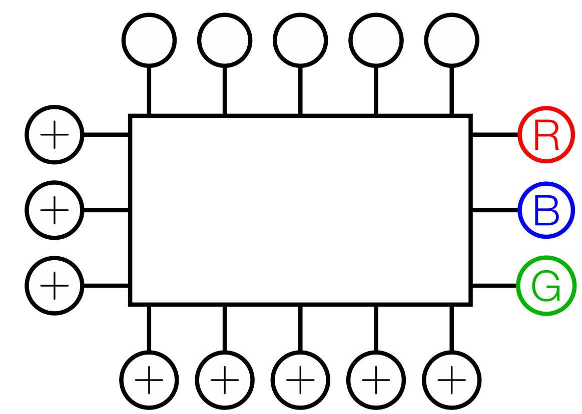


$Z(\lambda, w; \mathbf{z}) =$



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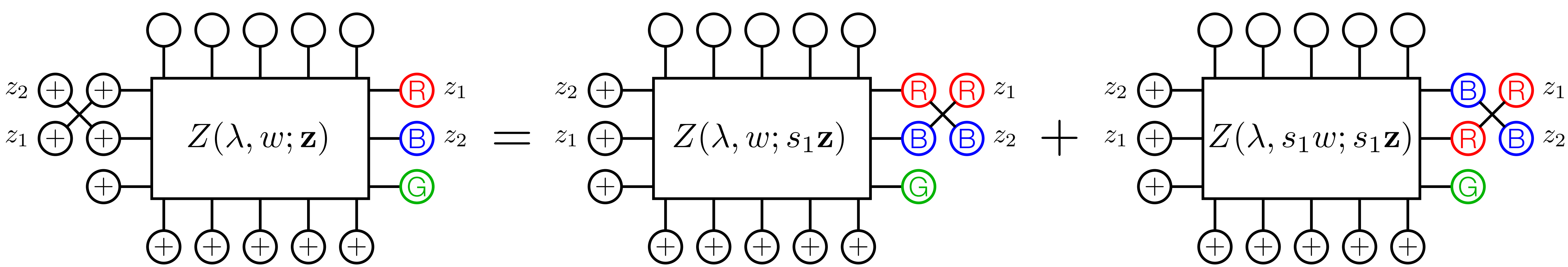
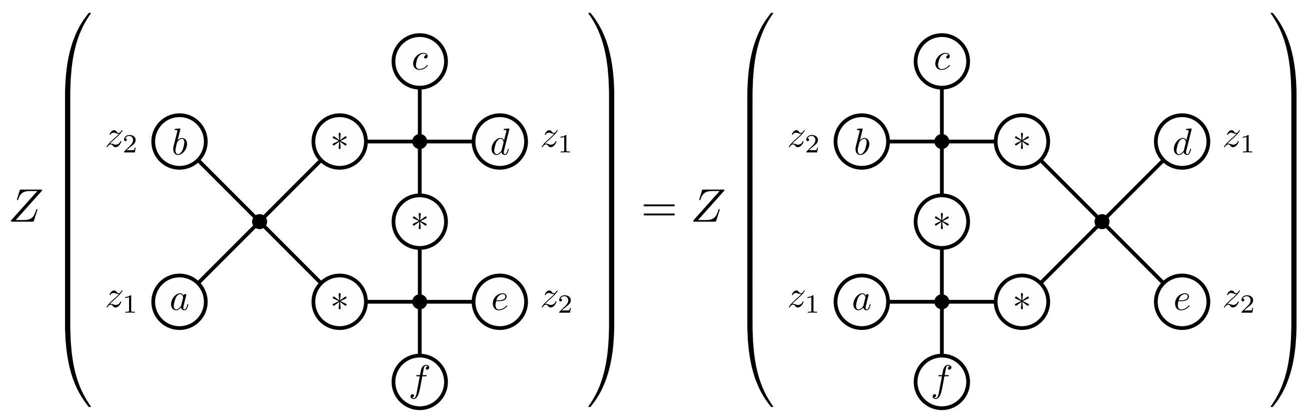
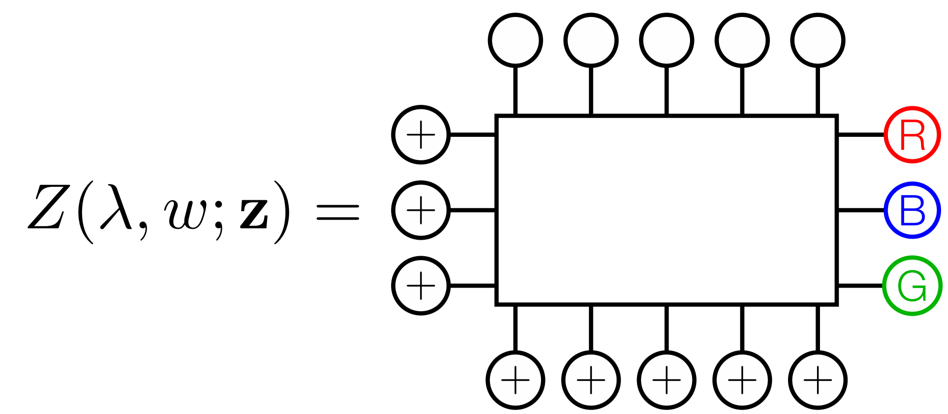


Yang-Baxter equations





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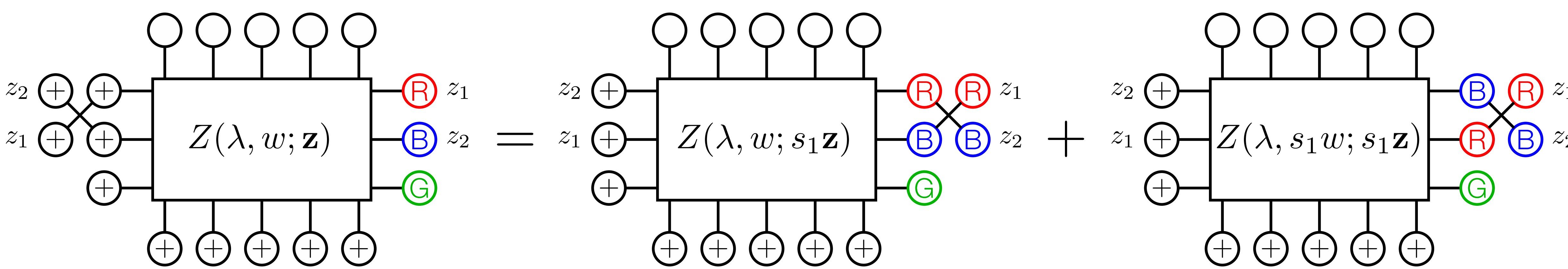
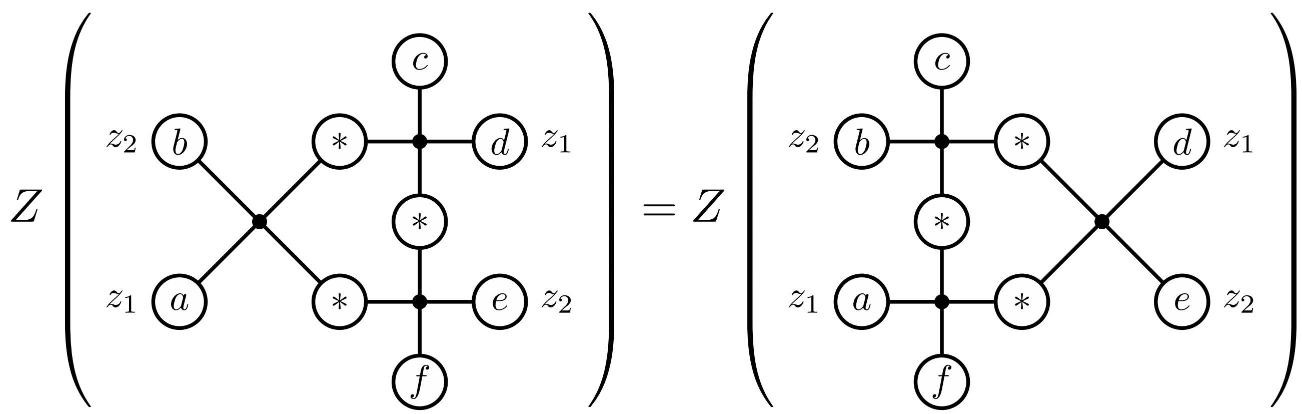
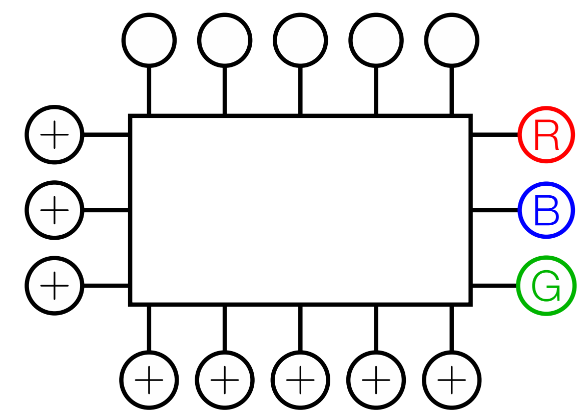


We can solve for $Z(\lambda, s_1 w; s_1 \mathbf{z})$ in terms of $Z(\lambda, w; s_1 \mathbf{z})$ and $Z(\lambda, w; \mathbf{z})$

Yang-Baxter equations



$Z(\lambda, w; \mathbf{z}) =$ 

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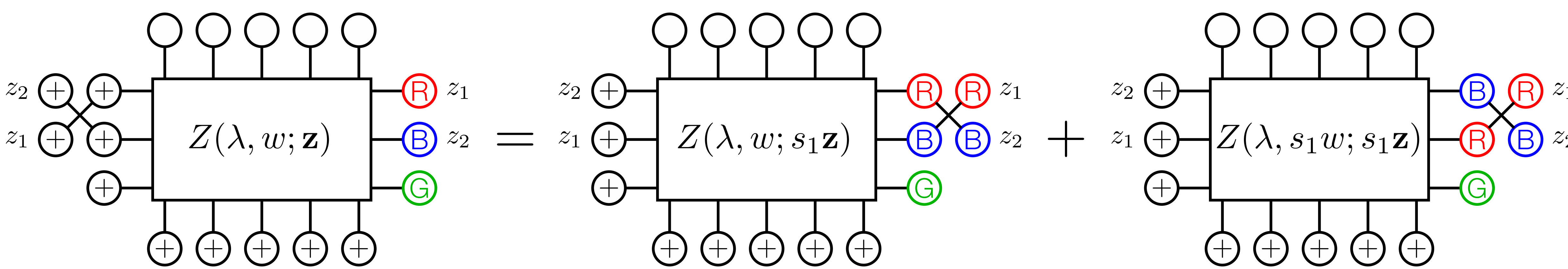

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Recursion relation in terms of Demazure (divided difference) operators

Quantum groups

Solutions to the Yang-Baxter equations arise from quantum groups. These are q-deformations of universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$.

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The module for the vertical edges is not known for any of the lattice models in this talk. The weights had to be constructed, and the Yang-Baxter equations had to be checked by hand.

Special polynomials

Table 2. Relations between different Whittaker functions and associated special polynomials.

Whittaker function		Special polynomial
Spherical Whittaker function $\sum_{w\in W}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Schur polynomial $\prod_{\alpha\in\Delta^+}(1-v\mathbf{z}^{-\alpha})s_\lambda(\mathbf{z})$
Li's Whittaker function $\sum_{w\in W}(-v)^{-\ell(w)}\phi_w(\mathbf{z};\varpi^{-\lambda})$	$=$	Hall-Littlewood polynomial $\mathbf{z}^{-\rho}P_{\lambda+\rho}(\mathbf{z},v^{-1})$
Iwahori Whittaker function $\phi_{w_1}(\mathbf{z};\varpi^{-\lambda})$	$=$	Non-symmetric Macdonald polynomial $(-v)^{\ell(w)}\mathbf{z}^{-\rho}w_0E_{w_0w(\lambda+\rho)}(\mathbf{z};\infty,v)$
Parahoric Whittaker function $\psi_1^{\mathbf{J}}(\mathbf{z};\varpi^{-\lambda})$	$=$	Macdonald polynomial with prescribed symmetry $\mathbf{z}^{-\rho}S_{\lambda+\rho}^{(\emptyset,\mathbf{J})}(\mathbf{z};0,v^{-1})a_{\lambda+\rho}^{(\emptyset,\mathbf{J})}$

References for each row is found in Table 1 in the same paper: arXiv:1906.04140