– BASIC NOTIONS SEMINAR IN NUMBER THEORY –

MULTIPLE DIRICHLET SERIES, METAPLECTIC WHITTAKER FUNCTIONS AND LATTICE MODELS

HENRIK P. A. GUSTAFSSON

JULY 21, 2020

Overview

As the title suggests, the focus of this talk is the connections between the following topics:

Multiple Dirichlet series $\longleftrightarrow$ Metaplectic Whittaker functions $\longleftrightarrow$ Lattice models

My research is focused on the latter two, but since this is a number theory seminar I will start from multiple Dirichlet series and work my way to the right.

Metaplectic here refers to Whittaker functions on a so called metaplectic cover of a reductive group and lattice models refers to statistical mechanics models on a two-dimensional lattice.

1. Multiple Dirichlet series

We are familiar with the usual Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_n, s \in \mathbb{C}$. Take for example the series for the Riemann zeta function $\zeta(s)$ with all $a_n = 1$ which converges for $\mathrm{Re}(s) > 1$. Properly normalized as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ it satisfies the functional equation $\xi(s) = \xi(1-s)$ which allows us to obtain the values of the meromorphic continuation beyond the domain of convergence.

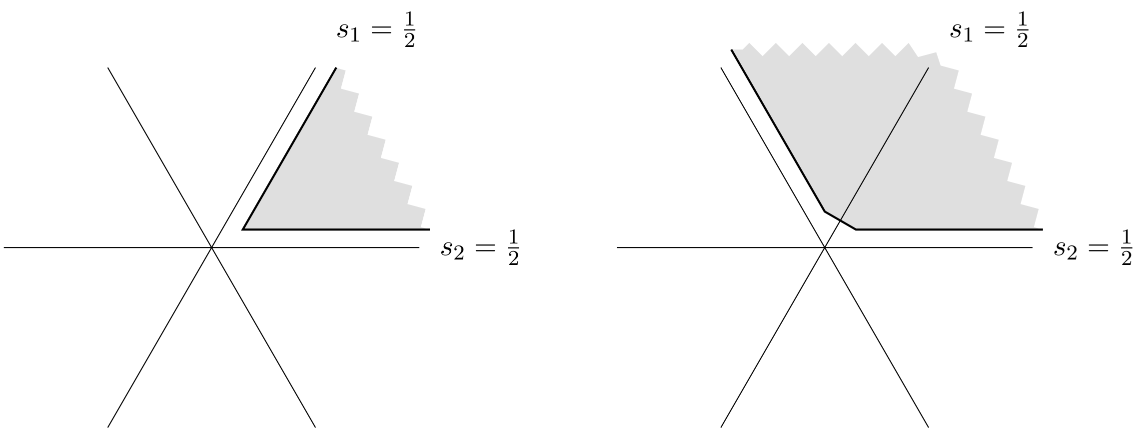
A multiple Dirichlet series is a series in multiple variables $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ and can be written as

$$Z(\mathbf{s}) = \sum_{n_1,\ldots,n_r} A_{n_1,\ldots,n_r} n_1^{-s_1} \cdots n_r^{-s_r} \tag{1.1}$$

I have left the summation range unspecified since we will want to sum over different rings of integers. Early examples arise from considering usual Dirichlet series and taking the coefficients a_n to be Dirichlet series themselves. Multiple Dirichlet series can be used to study L-functions: for example their mean values and moments [Goldfeld-Hoffstein-85, Diaconu-Goldfeld-Hoffstein-03], and nonvanishing properties via Rankin-Selberg integrals [Friedberg-Hoffstein-95, Chinta-Friedberg-Hoffstein-06].

The multiple Dirichlet series will have some domain of convergence and we want functional equations for meromorphic continuation. One type of multiple Dirichlet series that we will be interested in are so call *Weyl group multiple Dirichlet series* which have a group of functional equations isomorphic to the Weyl group of a root system Φ of rank r.

For example, if Φ is of type A_2 we may have the following gray area as domain of convergence where we have projected to the real parts of s_1 and s_2. The functional equations would then act as reflections in these lines transforming the domain of convergence similar to how the fundamental Weyl chamber is mapped over the whole weight space by Weyl transformations. We have tilted the s_1 axis such that the functional equations become reflections in these lines.



Using a variation of [Bochner-38] we can obtain a meromorphic continuation for the complex hull of the combined domains.

The Weyl group multiple Dirichlet series have another defining property concerning the multiplicativity of its coefficients, but to discuss that we first need some setup. We will then specify the remaining coefficients to fully define a Weyl group multiple Dirichlet series.

Let k be an algebraic number field which contains the group μ_n of n-th roots of unity which we embed in $\mathbb{C}$. For convenience we will actually also assume that $\mu_{2n} \subset k$ so that -1 is an n-th power in k. Fix a finite set of places S containing all the archimedean places as well as those ramified over $\mathbb{Q}$, and large enough so that the ring of S-integers $\mathcal{O}_S = \{x \in k : |x|_\nu \leqslant 1 \text{ for all } \nu \notin S\}$ is a principal ideal domain.

The following data

- reduced root system Φ of rank r
- $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$
- twisting parameter $\mathbf{m} \in (\mathcal{O}_S)^r$
- Ψ in a finite dimensional vector space of functions $(k_S^\times)^r \to \mathbb{C}$ with certain transformation properties (will not be important here)

specifies a Weyl group multiple Dirichlet series of order n by

$$Z(\mathbf{s}, \mathbf{m}, \Psi) = \sum_{\mathbf{c}} \frac{H(\mathbf{c}; \mathbf{m})\Psi(\mathbf{c})}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}} \tag{1.2}$$

where $\mathbf{c} = (c_1, \ldots, c_r)$ and each c_i ranges over $(\mathcal{O}_S \setminus \{0\})/\mathcal{O}_S^\times$ and $|c_i| := |\mathcal{O}_S/c_i\mathcal{O}_S|$. The coefficients H are left to be defined.

The twisted multiplicativity mentioned above is stated for the coefficients H. Let $\mathbf{c} = (c_1, \ldots, c_r)$ and $\mathbf{c}' = (c'_1, \ldots, c'_r)$. Then, if $\gcd(c_1 \cdots c_r, c'_1 \cdots c'_r) = 1$,

$$H(c_1c'_1, \ldots, c_rc'_r; \mathbf{m}) = \varepsilon_{\mathbf{c},\mathbf{c}'} H(\mathbf{c}; \mathbf{m}) H(\mathbf{c}'; \mathbf{m}) \tag{1.3}$$

where $\varepsilon_{\mathbf{c},\mathbf{c}'} \in \mu_n$ is a product of n-th power residue symbols. We also have a similar property for $\mathbf{m}$. This means that we only have to define H for $\mathbf{c} = (p^{k_1}, \ldots, p^{k_r})$ and $\mathbf{m} = (p^{l_1}, \ldots, p^{l_r})$ for primes p of $\mathcal{O}_S$. We create a generating series called the p-part of the multiple Dirichlet series

$$\sum_{k_i=0}^{\infty} H(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) |p|^{-2(k_1s_1+\cdots+k_rs_r)}. \tag{1.4}$$

There are several ways of defining these p-parts, and thus, several ways of defining the Weyl group multiple Dirichlet series. These developed in parallel and at first it was not clear if they would agree.

- Weyl group action average on the field of rational functions [Chinta-Gunnells-10]
- Sums over crystal bases or Gelfand–Tsetlin patterns (type A) [Brubaker-Bump-Friedberg-11a]
- Metaplectic p-adic Whittaker functions [BBF-11a, Chinta-Offen-13, McNamara-11, McNamara-16]
- Partition functions of lattice models (type A and C) [Brubaker-Bump-Chinta-12, Ivanov-12].

In fact there were even two ways to define the multiple Dirichlet series using Gelfand–Tsetlin patterns and a major part of the book [Brubaker-Bump-Friedberg-11b] is devoted to combinatorially prove their equivalence. Later, by rewriting the statement in terms of two corresponding lattice models, their equivalence becomes drastically simplified using a tool in statistical mechanics called the Yang–Baxter equation.

With the above H-coefficients the Weyl group multiple Dirichlet series Z has a meromorphic continuation to $\mathbf{s} \in \mathbb{C}^r$ [Chinta-Gunnells-10].

It was shown in [Brubaker-Bump-Friedberg-11a] that global Whittaker coefficients of Borel Eisenstein series on metaplectic covers of GL_{r+1} are actually full, type A, multiple Dirichlet series in r variables. These Whittaker coefficients satisfy the twisted multiplicativity and are not Eulerian.

At the end of this talk I will discuss how to write down the H-coefficients using statistical models, which is one example of a metaplectic Whittaker function that they can be used to describe. First let us discuss these metaplectic Whittaker functions.

2. Metaplectic Whittaker functions

For convenience we will restrict to $\mathbf{G} = \mathrm{GL}_{r+1}$. Let F be a non-archimedean **local** field with ring of integers $\mathcal{O}$ and a choice of uniformizer p, and let $q = |\mathcal{O}/p\mathcal{O}|$. We assume that F contains μ_{2n}.

Let $G = \mathbf{G}(F)$. Its n-fold metaplectic cover $\widetilde{G}$ is defined by the central extension

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \xrightarrow{\mathrm{pr}} G \longrightarrow 1 \tag{2.1}$$

Fix a section $s : G \to \widetilde{G}$. As a set $\widetilde{G} \cong G \times \mu_n$, but the multiplication is determined by a choice of map $\sigma : G \times G \to \mu_n$ by $s(g_1)s(g_2) = \sigma(g_1, g_2)s(g_1g_2)$ which is a 2-cocycle defining a class in $H^2(G, \mu_n)$. (Associativity $\implies$ closed, Choice of s $\implies$ up to exact element).

We will use the explicit cocycle described in [Kazhdan-Patterson-84] based on earlier work of [Matsumoto-69, Kubota-69]. One can also use the framework of [Brylinski-Deligne-01]. Denote the maximal torus of diagonal elements in G by T. For a weight $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda \cong \mathbb{Z}^r$ and $x \in F^\times$ let $x^\lambda = \mathrm{diag}(x^{\lambda_1}, \dots, x^{\lambda_r}) \in T$. The cocycle is chosen such that

$$[s(x^\lambda), s(y^\mu)] = (x, y)^{\langle \lambda, \mu \rangle} \tag{2.2}$$

where $(\cdot, \cdot) : F^\times \times F^\times \to \mu_n$ is the n-th power Hilbert symbol and $\langle \cdot, \cdot \rangle$ is the usual inner product on the weight space. This means that the preimage $\widetilde{T} := \mathrm{pr}^{-1}(T) \subset \widetilde{G}$ is not abelian.

We want to consider Whittaker functions for an unramified principal series representation of $\widetilde{G}$, but let us first remind ourselves of the corresponding non-metaplectic definitions for G.

For the non-metaplectic case, T is abelian and the irreducible representations are isomorphic to characters $\chi : T \to \mathbb{C}^\times$. Let $B \supset T$ be the Borel subgroup of G of upper triangular matrices and trivially extend χ as a function on B. Then

$$I(\chi) := \mathrm{Ind}_B^G(\chi) = \{ f : G \to \mathbb{C} \mid f(bg) = \chi(b) f(g) \text{ for } b \in B, g \in G \} \tag{2.3}$$

is a principal series representation of G. We also restrict to spherical vectors, that is f which are right invariant under the maximal compact subgroup $K = \mathrm{GL}_{r+1}(\mathcal{O})$. These are unique up to a constant factor and we chose such a function ϕ_K with convenient normalization.

A Whittaker functional on a representation (π, V) of G is a linear functional $W : V \to \mathbb{C}$ for which

$$W(\pi(n)\phi) = \psi(n) W(\phi) \text{ for all } n \in N, \phi \in V\,. \tag{2.4}$$

where ψ is a character on non-degenerate character on N. We may take for example the integral

$$W(\phi) = \int_N \phi(n)\psi(n)^{-1}\, dn \tag{2.5}$$

A Whittaker function can then be expressed as a function $G \to \mathbb{C}$, $g \mapsto W(\pi(g)\phi)$ for $\phi \in V$. For the non-metaplectic case, there is a unique spherical Whittaker function $W(\pi(g)\phi_K)$ [Gelfand-Kazhdan-71, Rodier-72]. This implies that global spherical Whittaker coefficients are Eulerian with each factor proportional to the corresponding unique local spherical Whittaker function.

This is not the case for metaplectic $\widetilde{G}$. The group $\widetilde{T} = \mathrm{pr}^{-1}(T)$ is non-abelian so it takes some work to describe its irreducible representations. We may identify the maximal compact K with its preimage in $\widetilde{G}$. Let H be the centralizer of $\widetilde{T} \cap K$ in $\widetilde{T}$, which is abelian. We have that $\widetilde{T}/H$ is parametrized by cosets $\varpi^\lambda H$ with $\lambda \in \Lambda / n\Lambda$. Let χ a character on H trivial on $\widetilde{T} \cap K$ such that $\chi(\zeta g) = \zeta \chi(g)$ for $\zeta \in \mu_n \subset \mathbb{C}^\times$. Every irreducible unramified representation of $\widetilde{T}$ is isomorphic to $i(\chi) := \mathrm{Ind}_H^{\widetilde{T}}(\chi)$ for some χ [Savin-04].

The module V_χ for $i(\chi)$ is of dimension $|\widetilde{T}/H| = |\Lambda/n\Lambda| = n^r$. We make a further trivial extension to $\widetilde{B}$ and induction to $\widetilde{G}$ as above to obtain the principal series $I(\chi)$, but now these are not $\mathbb{C}$-valued functions, but rather V_χ-valued. Thus, while there is still a unique spherical vector ϕ_K in $I(\chi)$, it is vector-valued and to obtain a $\mathbb{C}$-valued Whittaker function we must pair it with an element of the dual vector space. In summary, we get a n^r-dimensional space of Whittaker functions.

This explains why global metaplectic Whittaker coefficients are, in general, not Eulerian. Instead we get a twisted multiplicativity, which carries over to the H-coefficients in the multiple Dirichlet series.

Indeed, with a suitable chosen functional on V_χ, we get that the Whittaker function evaluated at $g = p^\lambda$ equals the p-part (1.4) with the following dictionary:

- χ is given by Langlands parameters s_i
- λ correspond to the l's giving the twist $\mathbf{m}$

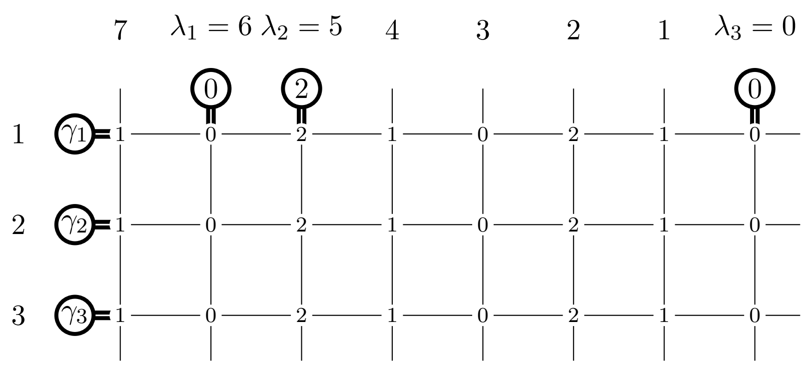
The identification of the p-parts of the Weyl group multiple Dirichlet series as certain metaplectic Whittaker functions was shown by [McNamara-11, Brubaker-Bump-Chinta-Friedberg-Gunnels-12, Chinta-Offen-13] for type A and by [McNamara-16] in general.

I still have to specify how to easily compute these p-parts, or in fact any other metaplectic Whittaker function, which I will do using lattice models.

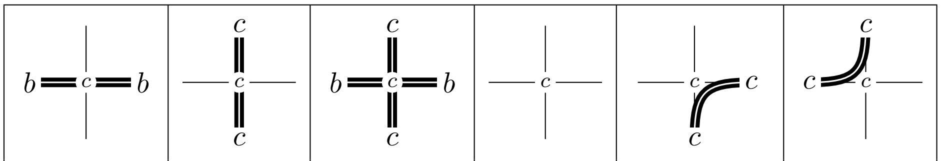
3. Lattice models

The following is my variation of the model described in [Brubaker-Buciumas-Bump-16].

We start with a grid like below with corresponding labels of the r rows and sufficiently many columns, and we give each vertex a label by the column number modulo n. We fix some boundary data as follows. We mark r top boundary positions according to λ and attribute them with a so called charge according to the vertex label. This λ will specify the argument of the Whittaker function by $g = p^\lambda$. On the left boundary we mark all r positions with some charges $\gamma_i \in \mathbb{Z}/n\mathbb{Z}$. These will specify one of the n^r conveniently chosen basis elements in V_χ^*.

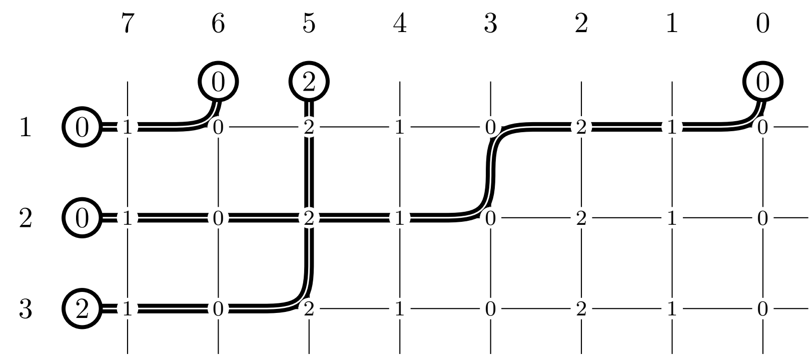


A state of this lattice model with these boundary conditions is then an assignment of vertex configurations according to the following table to the remaining vertices forming different lines between the boundary starting points. Each state is weighted by the product of the vertex configuration weights which contain the variable s_i at row i. The Whittaker function is then computed by the partition function which is the sum of these weights over all possible states given the fixed boundary.



1	s_i	$g(b-c)$	s_i	$(1-q^{-1})s_i$	1

Here $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ is an n-th order Gauss sum, which is a certain sum over elements in μ_n. The following is one of the possible states for the above boundary configuration.



An advantage of rewriting the Whittaker function as the partition function of lattice model is that one can take advantage of the many tools in statistical mechanics such as the Yang–Baxter equation. This gives a further connection to the field of quantum groups which was formalized by [Drinfeld-87]. The Yang-Baxter equation for the above model is related to the particular quantum group $U_q(\widehat{\mathfrak{gl}}(n|1))$. As mentioned above, using a Yang–Baxter equation [Brubaker-Bump-Chinta-12] was able to show the equivalence of the two Gelfand–Tsetlin descriptions in a very elegant way.

Together with Brubaker, Buciumas and Bump, I am currently working on metaplectic Whittaker functions which are not spherical, but rather invariant under a smaller Iwahori group, which is the subgroup of K of lower triangular matrices mod p. On the representation theory side we evaluate these using Demazure–Lusztig type operators and define a corresponding lattice model where we also have colored lines going from the top boundary to the right boundary and the order of the colors on the right boundary determine an Iwahori fixed vector enumerated by Weyl words.

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540
Department of Mathematics, Rutgers University, Piscataway, NJ 08854
Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, SE-412 96 Gothenburg, Sweden
URL: http://hgustafsson.se