Whittaker functions and lattice models

Henrik Gustafsson

Member, School of Mathematics
Institute for Advanced Study

IAS Postdoc short talks

September 29, 2020

https://hgustafsson.se

Rutgers University | University of Gothenburg and Chalmers | Swedish Research Council

Two projects

1. Fourier coefficients of automorphic forms in small automorphic representations of reductive groups.

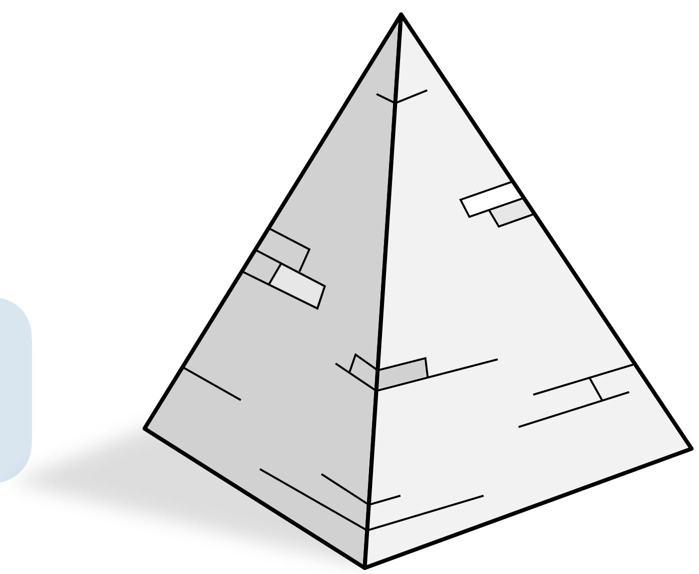
   Joint with: Dmitry Gourevitch, Axel Kleinschmidt,
   Daniel Persson, Siddhartha Sahi

   (See last year's postdoc short talk)

Two projects

2. Joint with: Ben Brubaker, Valentin Buciumas, Daniel Bump

Lattice models



Whittaker functions

Quantum groups

Special polynomials

Schur polynomials

Schur polynomials

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ be a partition and define the Schur polynomial $s_\lambda : \mathbb{C}^r \to \mathbb{C}$ by [Cauchy 1815, Jacobi 1841]

$$s_\lambda(z_1, \ldots, z_r) := \frac{\det(z_i^{(\lambda+\rho)_j})}{\det(z_i^{\rho_j})}$$

$\rho = (r-1, r-2, \ldots, 0)$

Form a linear basis for the space of symmetric polynomials

Characters of irreducible representations of $\mathrm{GL}_r(\mathbb{C})$

$$s_\lambda(z_1, \ldots, z_r) = \frac{\sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_i z_i^{(\lambda+\rho)_{\sigma_i}}}{\sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_i z_i^{\rho_{\sigma_i}}}$$

Weyl character formula

Important for when we will discuss Whittaker functions

Schur polynomials

Combinatorial formula using semistandard Young tableaux T of shape λ [Littlewood 1937]

$$\mathbf{z} = (z_1, \ldots, z_r) \;\in \mathbb{C}^r$$

$$\mathrm{wt}(T) = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$$

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

$$\mathbf{z}^\mu = \prod_{i=1}^{r} z_i^{\mu_i} \qquad \mathrm{wt}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = (1, 2, 1)$$

Stepping stone towards lattice model

states $\mathfrak{s} \longleftrightarrow$ semistandard Young–tableaux T

shape

$$\lambda \left\{\begin{array}{ccccc} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array}\right\} \longleftrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Gelfand–Tsetlin pattern

Schur polynomials

Combinatorial formula using semistandard Young tableaux T of shape λ [Littlewood 1937]

$$\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$$

$$\mathrm{wt}(T) = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$$

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

$$\mathbf{z}^\mu = \prod_{i=1}^{r} z_i^{\mu_i} \qquad \mathrm{wt}\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right) = (1, 2, 1)$$

Stepping stone towards lattice model

states $\mathfrak{s} \longleftrightarrow$ semistandard Young–tableaux T

shape

$$\left\{ \begin{array}{ccccc} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \right\} \longleftrightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

Gelfand–Tsetlin pattern

Schur polynomials

Combinatorial formula using semistandard Young tableaux T of shape λ [Littlewood 1937]

$$\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$$

$$\mathrm{wt}(T) = (\mu_1, \dots, \mu_r) \in \mathbb{Z}^r$$

$$s_\lambda(\mathbf{z}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{z}^{\mathrm{wt}(T)}$$

$$\mathbf{z}^\mu = \prod_{i=1}^{r} z_i^{\mu_i} \qquad \mathrm{wt}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\right) = (1,2,1)$$

Stepping stone towards lattice model

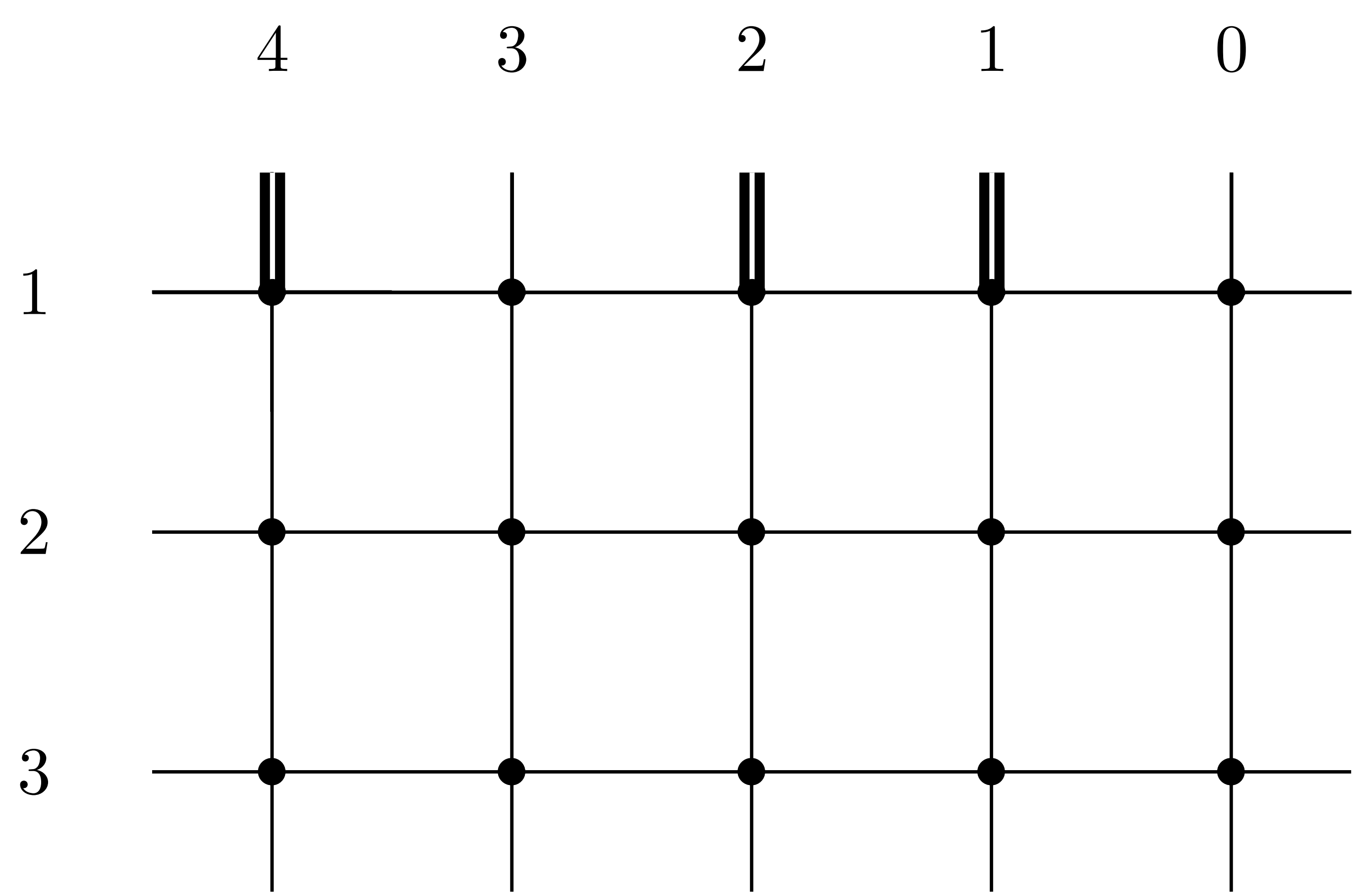
states $\mathfrak{s} \longleftrightarrow$ semistandard Young–tableaux T

shape

$$\mathfrak{s} \longleftrightarrow \left\{\begin{matrix} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 1 & & \end{matrix}\right\} = \left\{\begin{matrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{matrix}\right\} + \left\{\begin{matrix} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{matrix}\right\} \longleftrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

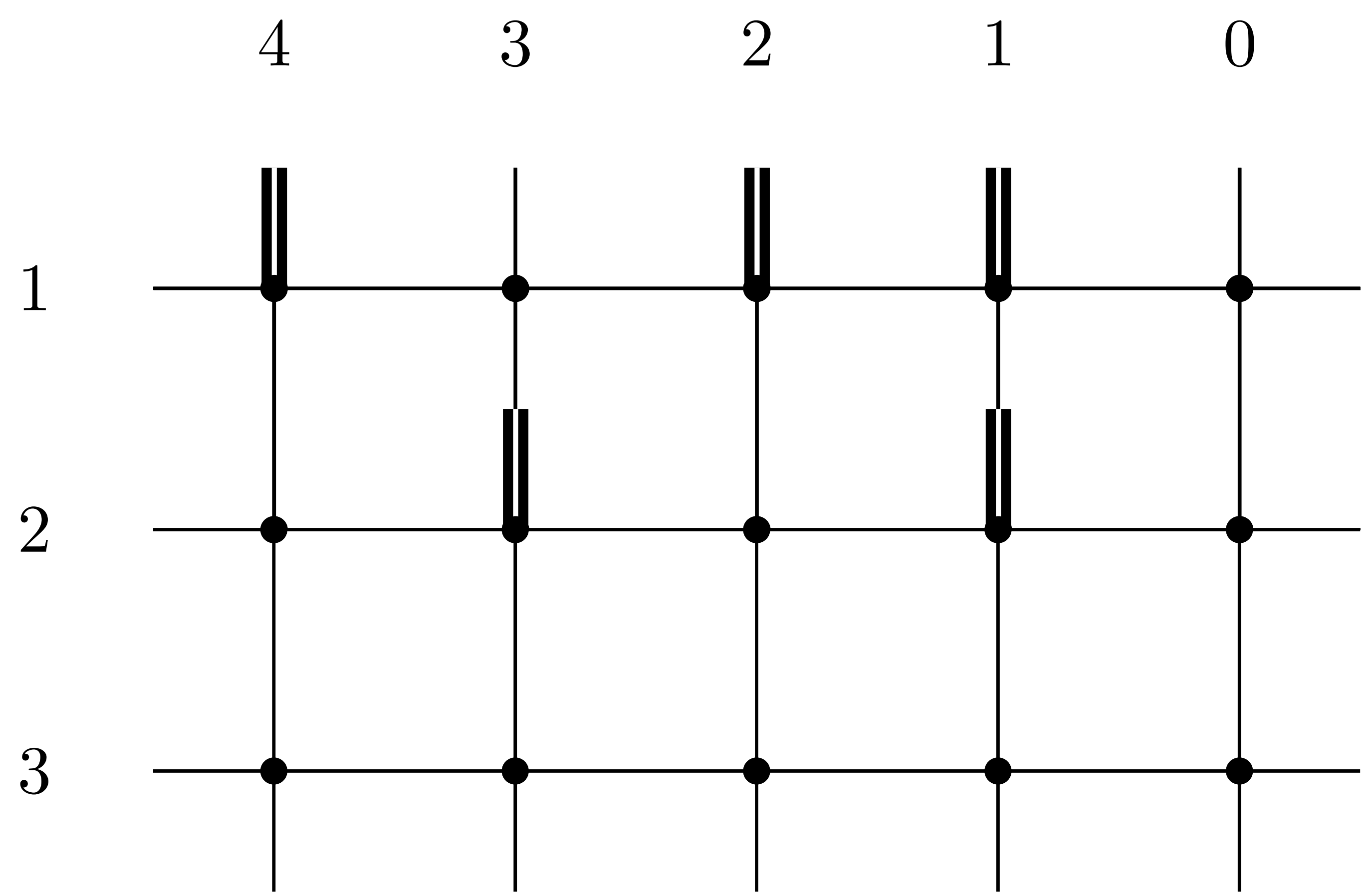
Gelfand–Tsetlin pattern

Lattice model



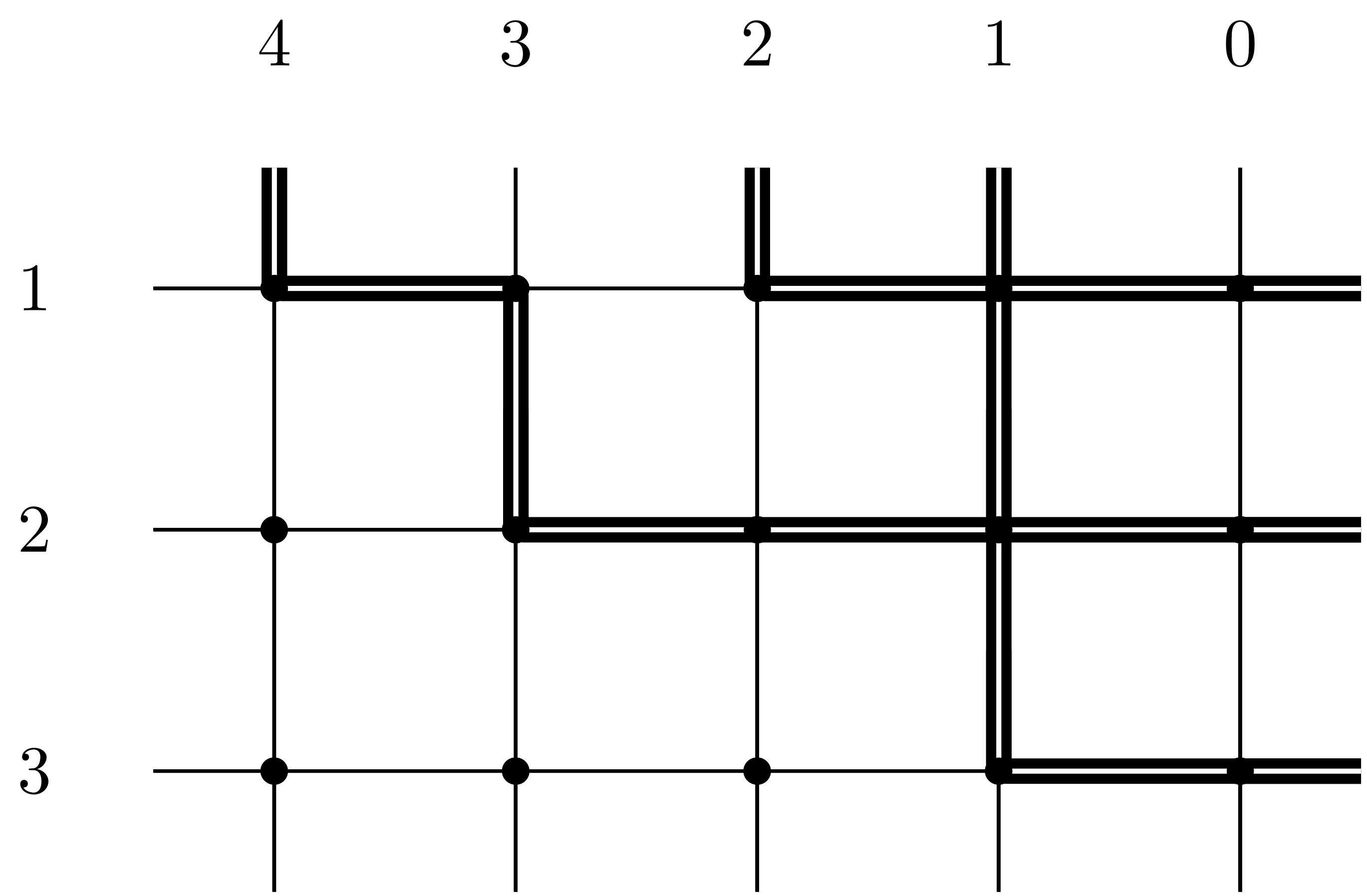
$$\mathfrak{s} \longleftrightarrow \left\{ \begin{matrix} \overbrace{4 \quad & & 2 & & \quad 1}^{\lambda+\rho} \\ & 3 & & 1 & \\ & & 1 & & \end{matrix} \right\} = \left\{ \begin{matrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{matrix} \right\} + \left\{ \begin{matrix} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{matrix} \right\} \longleftrightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

Lattice model



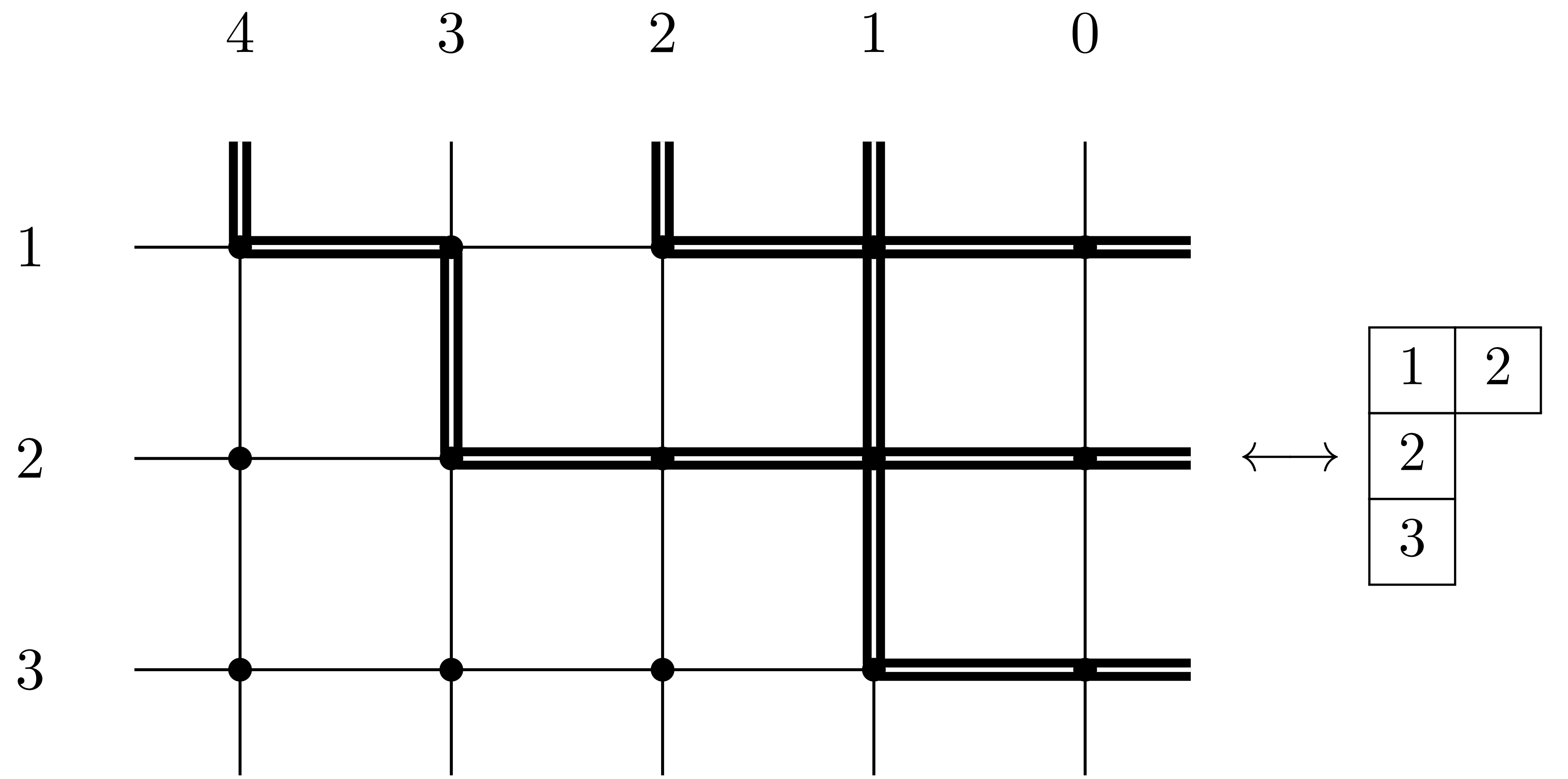
$$\mathfrak{s} \longleftrightarrow \left\{\begin{array}{ccccc} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 1 & & \end{array}\right\} = \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\} + \left\{\begin{array}{ccccc} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array}\right\} \longleftrightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Lattice model



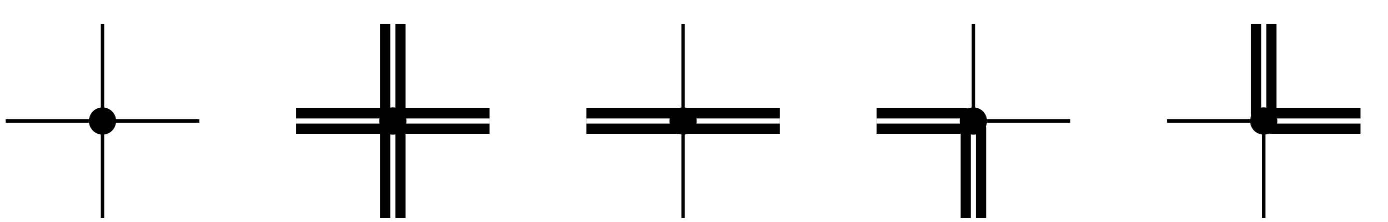
$$\mathfrak{s} \longleftrightarrow \left\{\begin{matrix} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 1 & & \end{matrix}\right\} = \left\{\begin{matrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{matrix}\right\} + \left\{\begin{matrix} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{matrix}\right\} \longleftrightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

Lattice model

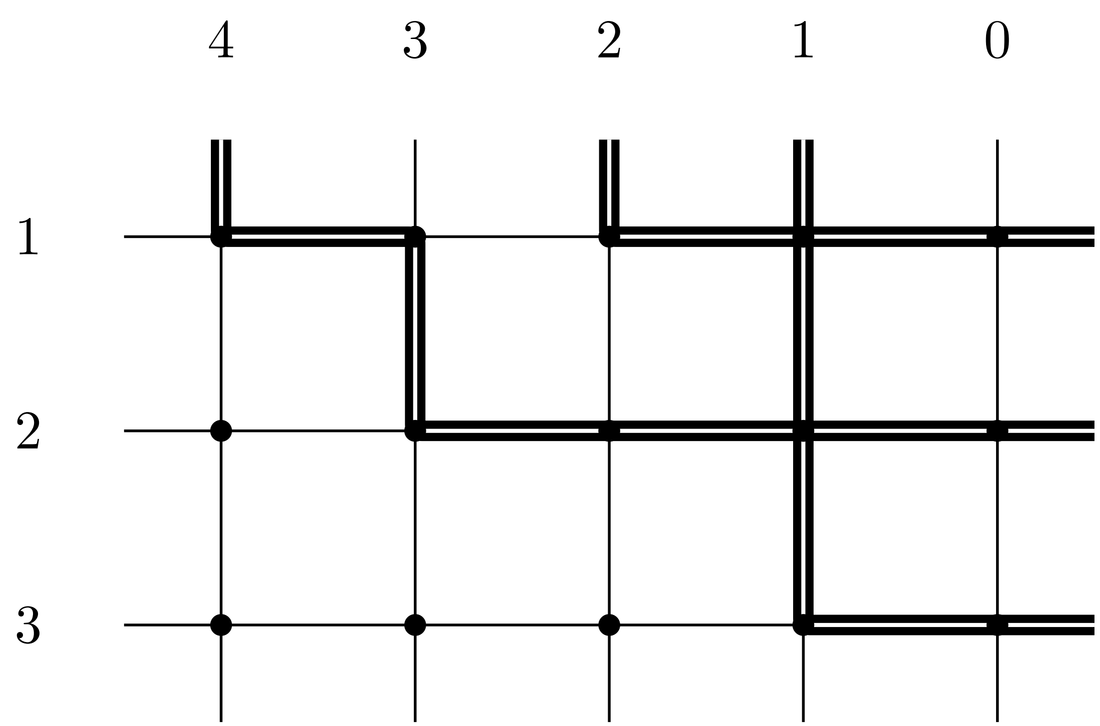


Five-vertex model

Weight $\beta(v)$ for vertex v at row i



Lattice model



$$\beta(\mathfrak{s}) = \prod_{\text{vertex } v} \beta(v)$$

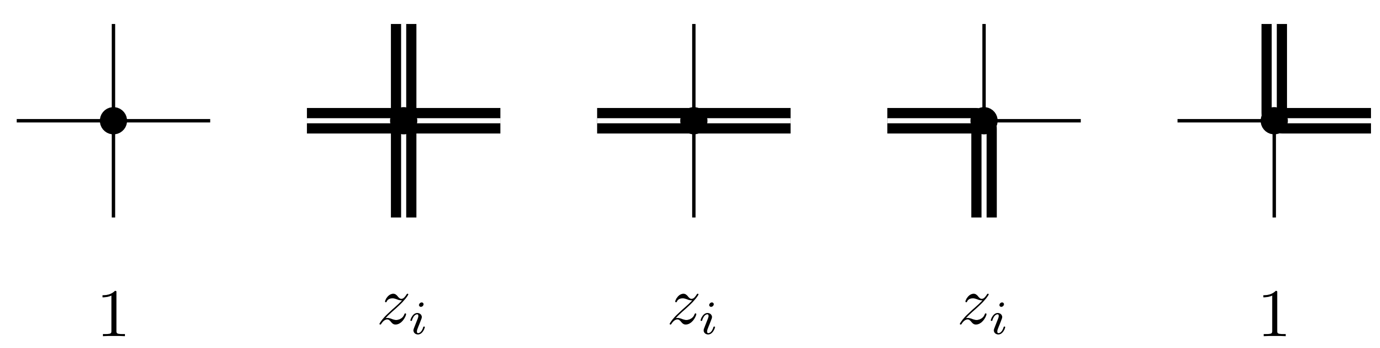
Partition function

$$Z_\lambda(\mathbf{z}) = \sum_{\substack{\text{states } \mathfrak{s} \text{ with} \\ \text{top lines at } \lambda+\rho}} \beta(\mathfrak{s})$$

$\mathrm{SSYT}(\lambda)$

Five-vertex model

Weight $\beta(v)$ for vertex v at row i



Whittaker function

$s_\lambda(\mathbf{z})$ is a Whittaker function in disguise.

Let $G = \mathrm{GL}_r(\mathbb{Q}_p)$ and N the subgroup of lower triangular matrices with unit diagonal. (more generally: any non-archimedean local field)

A Whittaker model for a repr. (π, V) of G is an embedding

$$\pi \hookrightarrow \mathrm{Ind}_N^G \psi = \{\underbrace{\phi}_{\text{Whittaker function}} : G \to V \mid \phi(ng) = \underbrace{\psi}_{\text{Non-degenerate character on } N}(n)\phi(g)\} \qquad n \in N, g \in G$$

We take $(\pi, V = \mathbb{C})$ to be an unramified principal series representation from a character defined by

$$p^{-\lambda} := \begin{pmatrix} p^{-\lambda_1} & & & \\ & p^{-\lambda_2} & & \\ & & \ddots & \\ & & & p^{-\lambda_r} \end{pmatrix} \mapsto \mathbf{z}^\lambda \qquad \begin{aligned} \mathbf{z} &= (z_1, \ldots, z_r) \in \mathbb{C}^r \\ \lambda &= (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \end{aligned}$$

Whittaker function

right-invariant under maximal compact subgroup $\mathrm{GL}_r(\mathbb{Z}_p)$

Unique spherical Whittaker function ϕ. Determined by the values

$$\phi_{\mathbf{z}}(p^{-\lambda}) =$$

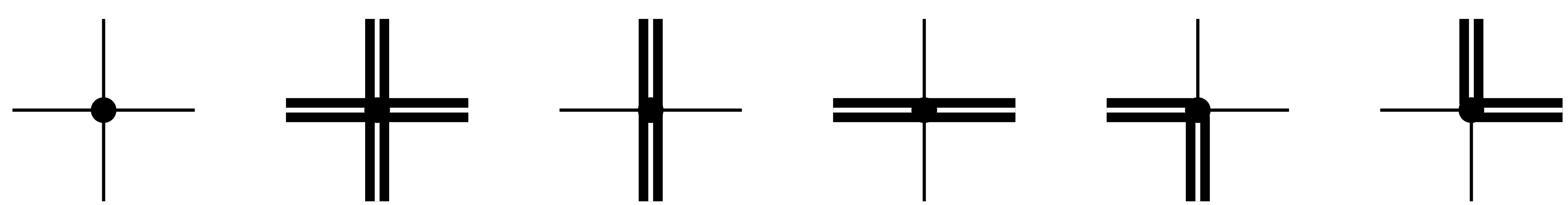
Casselman–Shalika formula

Character for Langlands dual group $\hat{G}(\mathbb{C}) = \mathrm{GL}_r(\mathbb{C})$, i.e. Schur polynomial

The extra factor, a deformed Weyl denominator, can be captured by introducing another vertex configuration

Six-vertex model

$$Z_\lambda(\mathbf{z}) = \mathbf{z}^\rho \phi_{\mathbf{z}}(p^{-\lambda})$$



1	z_i	$-\frac{1}{p}$	z_i	$(1-\frac{1}{p})z_i$	1

[Hamel–King 07, Brubaker–Bump–Friedberg 09, Tokuyama 88]

Motivation

Why relate Whittaker functions or special polynomials to lattice models?

Use toolset from lattice models to study original objects

- Functional equations
  (e.g. for analytic continuation of multiple Dirichlet series)
  [Brubaker–Bump–Friedberg 09, 11, Brubaker–Buciumas–Bump–Gray 19]

- Cauchy-type identities
  [Wheeler–Zinn-Justin 15, Bump–McNamara–Nakauji 14]

- Other combinatorial properties
  Recent preprints: Borodin–Wheeler, Buciumas–Scrimshaw,
  Brubaker–Frechette–Hardt–Tibor–Weber

Iwahori Whittaker function

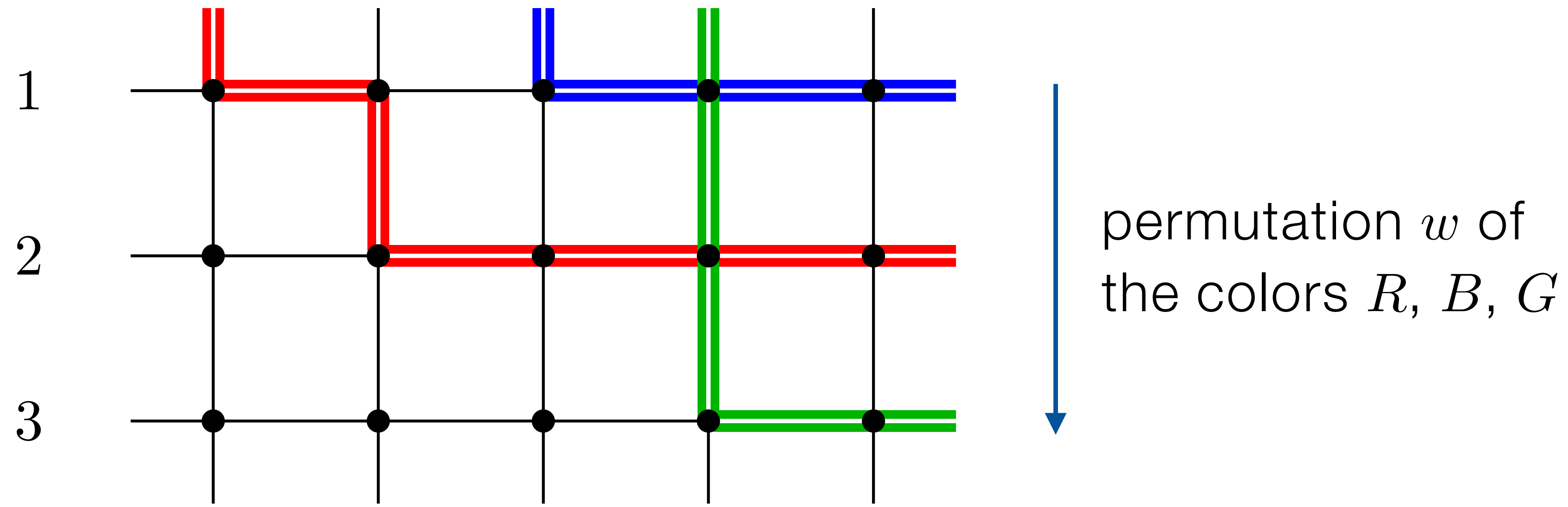
Relax: spherical $\longrightarrow$ Iwahori fixed

right-invariant under subgroup of $\mathrm{GL}_r(\mathbb{Z}_p)$ lower triangular $\bmod p$

$\in W$

Iwahori Whittaker functions enumerated by Weyl group: $\phi_{\mathbf{z}}^{w}$

Computed using lattice model with colors



Theorem [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]

partition function $= \mathbf{z}^{\rho}\phi_{\mathbf{z}}^{w}$

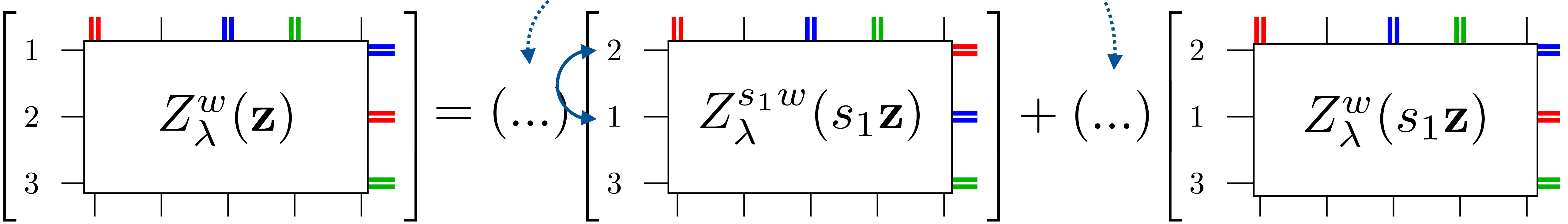
Iwahori Whittaker function

Proof: ~~Young–tableau description.~~ Recursion relations

$$\phi_{\mathbf{z}}^{s_i w} = \mathfrak{T}_i \phi_{\mathbf{z}}^{w} \qquad \ell(s_i w) > \ell(w) \qquad \mathfrak{T}_i f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1} - p^{-1} \frac{f(\mathbf{z}) - \mathbf{z}^{-\alpha_i} f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}$$

Similar to the Demazure–Lusztig operators

Equivalent to the Yang–Baxter equation quantum group $U_q(\widehat{\mathfrak{gl}}(r|1))$



After comparing base case: $\quad Z_\lambda^w(\mathbf{z}) = \mathbf{z}^\rho \phi_{\mathbf{z}}^w(p^{-\lambda})$

[Brubaker–Buciumas–Bump–HG arXiv:1906.04140]

Strategy

Whittaker function = partition function of lattice model

or special polynomial

Recursion relations $\longleftrightarrow$ Yang–Baxter equations

Applied in a series of papers:

- Iwahori and parahoric Whittaker functions

  [Brubaker–Buciumas–Bump–HG arXiv:1906.04140]

- Demazure atoms and Lascoux-Schützenberger keys

  [Brubaker–Buciumas–Bump–HG arXiv:1902.01795. To appear in JCTA]

- Metaplectic Iwahori Whittaker functions

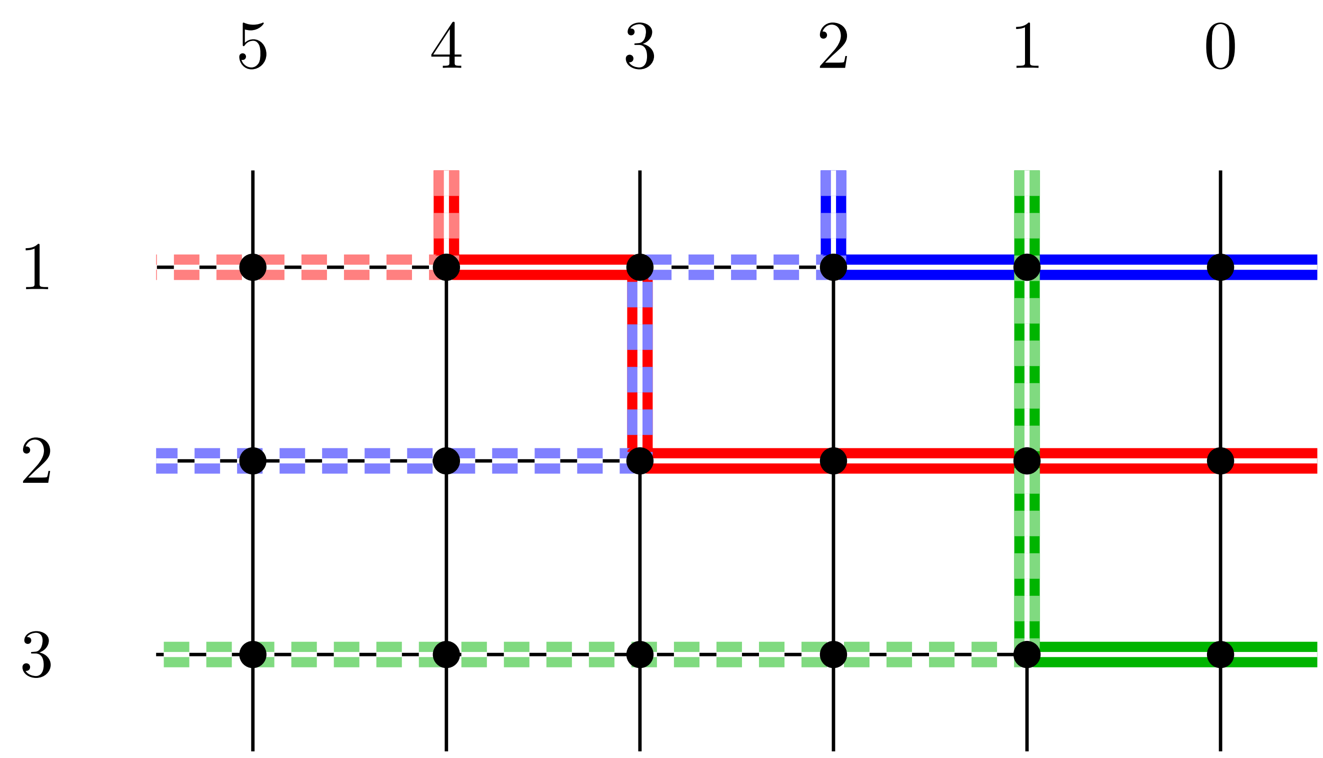
  [Brubaker–Buciumas–Bump–HG in progress]

Along the way: results independent of lattice model relation

Current work

Iwahori Whittaker functions for metaplectic n-covers of GL_r

Spherical Whittaker function identified with p-parts of multiple Dirichlet series



Quantum group $U_q\big(\widehat{\mathfrak{gl}}(r|n)\big)$

Open problem: quantum group module for *vertical* edges

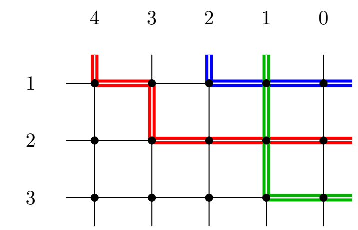
$\Longrightarrow$ vertex weights from first principles

Thank you!

Slides available at

https://hgustafsson.se

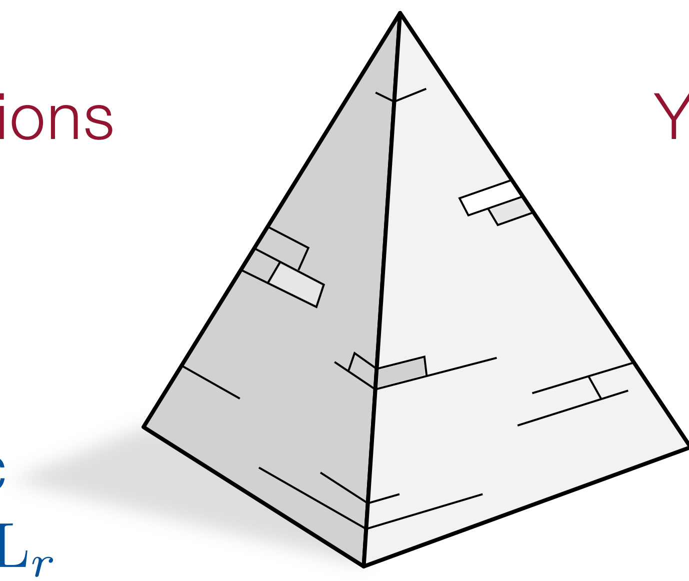
Summary



Lattice models

Partition functions

Yang–Baxter equations



Whittaker functions

spherical, Iwahori, parahoric

n-fold metaplectic cover of GL_r

Quantum groups

$U_q(\widehat{\mathfrak{gl}}(r|n))$

Special polynomials

Schur polynomials

nonsymmetric Macdonald polynomials

p-parts of multiple Dirichlet series