Instantons and Automorphic Representations

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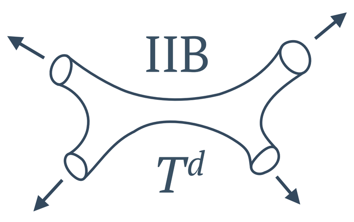
Automorphic forms in string theory



Consider Type IIB string theory compactified on a torus and, specifically, the four graviton corrections to the effective action.

These corrections can schematically be expressed as

$$S = \int R + f_1 \mathcal{R}^4 + f_2 \partial^4 \mathcal{R}^4 + f_3 \partial^6 \mathcal{R}^4 + \ldots$$ [Green-Schwarz, ...]

where $\mathcal{R}^4$ is a standard contraction of four Riemann tensors.

D	$G(\mathbb{R})$
10	$SL(2,\mathbb{R})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$
7	$SL(5,\mathbb{R})$
6	$Spin(5,5,\mathbb{R})$
5	$E_6(\mathbb{R})$
4	$E_7(\mathbb{R})$
3	$E_8(\mathbb{R})$

[Cremmer-Julia, Hull-Townsend]

The coefficients f_1, f_2 and f_3 are functions on the moduli space $G(\mathbb{R})/K$, invariant under the U-duality group $G(\mathbb{Z})$. This means that they are so called automorphic forms.

These automorphic forms, which capture both the perturbative and non-perturbative corrections to the effective action, are known explicitly and have been studied extensively in the literature.

[Ginzburg-Rallis-Soudry, Green-Gutperle, Green-Miller-Vanhove, Pioline]

Supersymmetry and automorphic representations

Besides U-duality, the effective action is also constrained by supersymmetry which allows for only a few non-vanishing contributions. These constraints can also be realized as differential equations for the automorphic forms above.

In the language of automorphic forms, these differential equations imply that the coefficients should be attached to small automorphic representations.

Small with respect to functional dimension

$f_1 \in$ minimal representation, $f_2 \in$ next-to-minimal representation, ...
½ BPS ¼ BPS

[Ginzburg-Rallis-Soudry, Green-Sethi, Green-Miller-Vanhove, Pioline, Bossard-Kleinschmidt]

Extracting physical information with Fourier coefficients

Let us first consider the ten-dimensional case where $G(\mathbb{R}) = SL(2,\mathbb{R})$ and the moduli space is parametrized by the axio-dilaton $\tau = \chi + ie^{-\phi} \in SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

Since f_1 is $SL(2,\mathbb{Z})$-invariant it is periodic in χ and can be Fourier expanded as follows

$$f_1(\tau) = Ag_s^{-\frac{3}{2}} + Bg_s^{\frac{1}{2}} + \sum_N{}' \mu(N)\, e^{2\pi(iN\chi - |N|g_s^{-1})} \left(1 + \mathcal{O}(g_s)\right)$$

Instanton charge; Perturbative corrections; Instanton measure

[Green-Gutperle]

The first two terms belong to the zero-mode and are perturbative in the string coupling constant. The remaining Fourier modes, characterized by an instanton charge N, are non-perturbative. From this expansion we can obtain the instanton measure counting the number of states for a given charge.

Using the full expression for f_1 one finds that, in this case, the instanton measure can be expressed as a divisor sum $\mu(N) = \sum_{\substack{m|N \\ m>0}} m^{-2}$.

This can be interpreted as the number of ways the instanton charge can be factorized as a wrapping number and units of momenta around a circle in the compactified T-dual picture.

[Green-Gutperle]

Fourier coefficients for larger groups

If we want to obtain similar information about instantons in lower dimensions we need to compute Fourier coefficients for a larger group G. Let U be a unitary subgroup of G and ψ a multiplicative character on $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$. Such a character is given by a set of instanton charges and generalizes the mode $\exp(2\pi i N\chi)$ from the ten-dimensional case.

The corresponding Fourier coefficient of an automorphic form f is then defined as

$$F_\psi[f] = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(ug)\, \psi(u)\, du$$

These, however, are difficult to compute and are only known in certain cases for f_1, f_2 and f_3. For example, when $U = N$ is maximal, the Fourier coefficients, called Whittaker vectors, can, in part, be computed using the Casselman-Shalika formula.

arXiv:1412.5625
[HG, Axel Kleinschmidt & Daniel Persson]

Our results

In our paper we present a method for computing these Fourier coefficients for automorphic forms in small automorphic representations by organizing the instanton charges into nilpotent orbits called character variety orbits.

Using theorems from Matumoto and Mœglin-Waldspurger, we show that, by regrouping the Fourier expansion into terms corresponding to the same nilpotent orbit, we can treat the contributions representation by representation

$$f = f_{\text{trivial-rep}} + f_{\text{½-BPS}} + f_{\text{¼-BPS}} + \ldots$$

where a smaller representation means fewer non-vanishing Fourier coefficients.

Recently, Miller-Sahi showed, using a similar framework, that all Fourier coefficients of automorphic forms in the minimal representation (½ BPS) are completely determined by maximally degenerate Whittaker vectors supported only on one instanton charge.

We complement and extend these results by showing, for $G = SL(3)$ and $SL(4)$, explicitly how the Fourier coefficients can be expressed for
½ BPS: as maximally degenerate Whittaker vectors
¼ BPS: as degenerate Whittaker vectors supported on at most two commuting charges.

Specifically, a Fourier coefficient on a maximal parabolic subgroup, parametrized by few instanton charges, is equal to a single G-translated maximally degenerate Whittaker vector.

Work in progress
[Olof Ahlén, Philipp Fleig, HG, Axel Kleinschmidt & Daniel Persson]

Outlook

We expect these results to hold for larger groups as well and plan to apply these methods to computing instanton effects for $D = 5, 4$ and 3 corresponding to the groups $G = E_6, E_7$ and E_8 respectively.

We also hope to extend this mathematical framework to also include the Kac-Moody groups E_9, E_{10} and E_{11} which are the symmetry groups for $D < 3$.

[Julia, Nicolai, Damour-Hennaux-Nicolai, West]

Although these groups do not allow for the usual definition of a small automorphic representation their initially infinite number of non-vanishing contributions collapses to only a few for certain families of automorphic forms just as for the larger dimensional cases.

[Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

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