Stanford University

String theory and automorphic representations

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Further corrected slides will be updated at: hgustafsson.se
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Preamble

Based on the book:
[FGKP] "Eisenstein series and automorphic representations –
With applications in string theory" (CUP, 2018)
Philipp Fleig, HG, Axel Kleinschmidt, Daniel Persson
Early version on arXiv:1511.04265. References to sections are for new version: TOC

PDF of slides also designed as a guide to the literature – references are clickable.

Some books about string and field theory in general:

- Quantum Fields and Strings : A Course for Mathematicians [Deligne–Kazhdan–Etingof–Morgan–Freed–Morrison–Witten]
- Superstring theory [Green–Schwarz-Witten Vol 1, Vol 2]
- String theory [Polchinski Vol 1, Vol 2]
- Basic Concepts of String Theory [Blumenhagen–Lüst–Theisen]
- Lectures on String Theory [Tong]
- Conformal Field Theory [Di Francesco–Mathieu–Sénéchal]

Outline – Part I

- Overview of different ways automorphic forms and modular forms appear in string theory.
- Focus on: low-energy expansion of 4 graviton scattering amplitudes.
- Why are the coefficients in this expansion automorphic?
- Automorphic representations from supersymmetry (first look)
- Extracting physics from Fourier coefficients in SL_2 example
- Automorphic forms and Fourier coefficients in the adelic framework

Outline – Part II

- Automorphic representations and the global wave-front set
- Different parabolic subgroups and their interpretations in physics
- BPS-orbits and character variety orbits
- Computing Fourier coefficients
  - Langlands' constant term formula
  - Casselman–Shalika formula
  - The subgroup reduction formula
  - Some new results
- Kac–Moody groups (in preparation for the last two talks on Friday)

PART I

Automorphic and modular forms in string theory

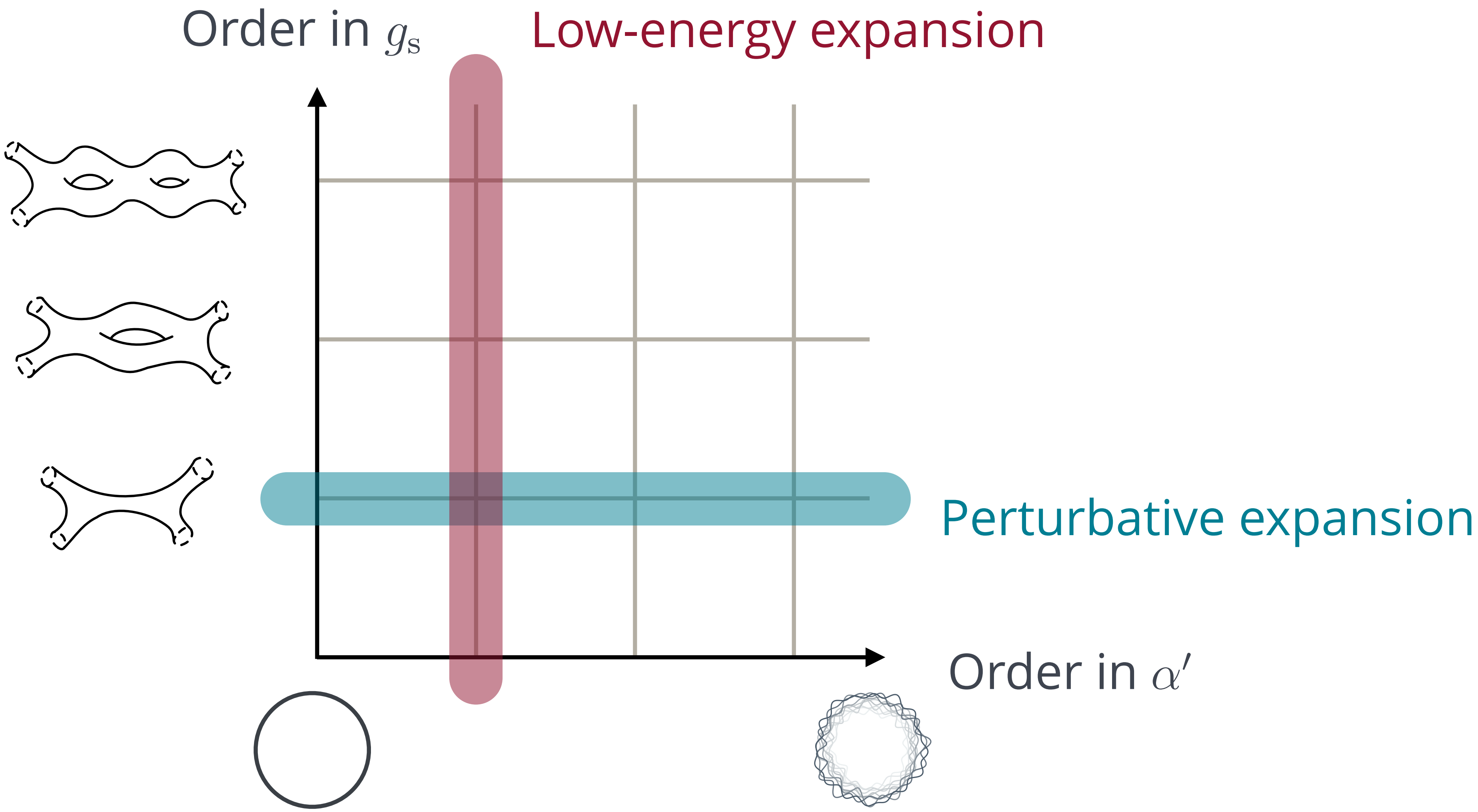
1. Generating functions for quantum state counting

   #(states with charge $\gamma) = d(\gamma^2/2)$ where $\frac{1}{\Delta} = \sum_{n=-1}^{\infty} d(n)q^n$

2. Perturbative expansion (in orders of the string coupling constant g_{s})
3. Low-energy expansion (in orders of the string area scale α')

Automorphic and modular forms in string theory

1. Generating functions for quantum state counting
2. Perturbative expansion (in orders of the string coupling constant g_s)
3. Low-energy expansion (in orders of the string area scale α')



Compactifications

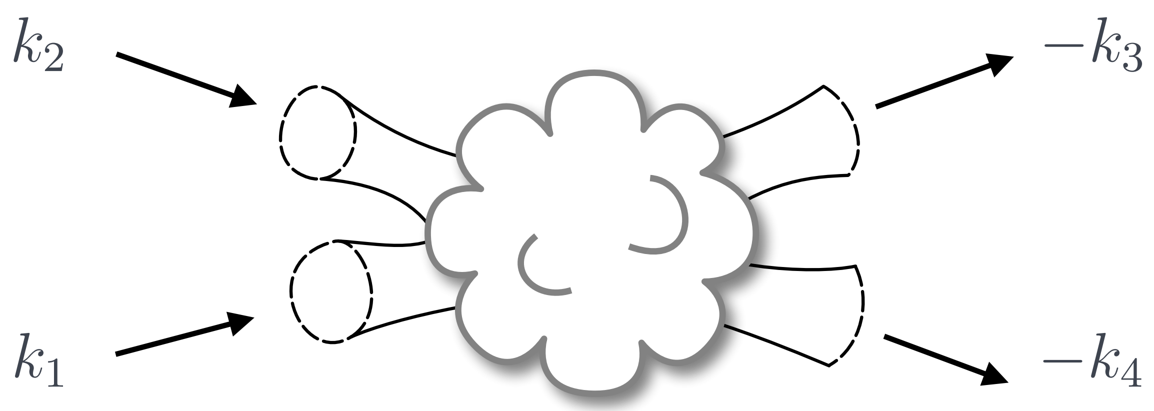
Scattering amplitude of four gravitons (two in, two out) in type IIB string theory on a manifold $X \times \mathbb{R}^{10-d}$

Compact d-dimensional

$X \times \mathbb{R}^{10-d}$	Conserved supercharges	Conserved supersymmetry spinors $\mathcal{N}$
$\mathbb{R}^{10}$	32	
$T^d \times \mathbb{R}^{10-d}$	32	
$T^6 \times \mathbb{R}^4$	32	$\mathcal{N} = 8$
$K3 \times T^2 \times \mathbb{R}^4$	16	$\mathcal{N} = 4$
$CY3 \times \mathbb{R}^4$	8	$\mathcal{N} = 2$

See details in [FGKP §15.4]

Scattering data



- Momenta

  Mandelstam variables: $s = -\frac{\alpha'}{4}(k_1 + k_2)^2, \quad t = -\frac{\alpha'}{4}(k_1 + k_3)^2, \quad u = -\frac{\alpha'}{4}(k_1 + k_4)^2$

- Polarizations

  $\epsilon_1, \ldots, \epsilon_4$

- Scalar parameters

  String moduli: string coupling g_s, parameters for X such as radii

Toroidal compactifications

- Scalar parameters

Toroidal compactifications $X = T^d$: moduli space is $G(\mathbb{R})/K(\mathbb{R})$

$T^d \times \mathbb{R}^D$	Split real form	Maximal compact subgroup
$D = 10 - d$	$G(\mathbb{R})$	$K(\mathbb{R})$
10	$SL_2(\mathbb{R})$	$SO_2(\mathbb{R})$
9	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$SO_2(\mathbb{R})$
8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$
7	$SL_5(\mathbb{R})$	$SO_5(\mathbb{R})$
6	$Spin_{5,5}(\mathbb{R})$	$\big(Spin_5(\mathbb{R}) \times Spin_5(\mathbb{R})\big)/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp_8(\mathbb{R})/\mathbb{Z}_2$
4	$E_7(\mathbb{R})$	$SU_8(\mathbb{R})/\mathbb{Z}_2$
3	$E_8(\mathbb{R})$	$Spin_{16}(\mathbb{R})/\mathbb{Z}_2$

[Cremmer–Julia 78]

Toroidal compactifications

Toroidal compactifications $X = T^d$: moduli space is $G(\mathbb{R})/K(\mathbb{R})$

$D = 10 - d$	$G(\mathbb{R})$	$K(\mathbb{R})$
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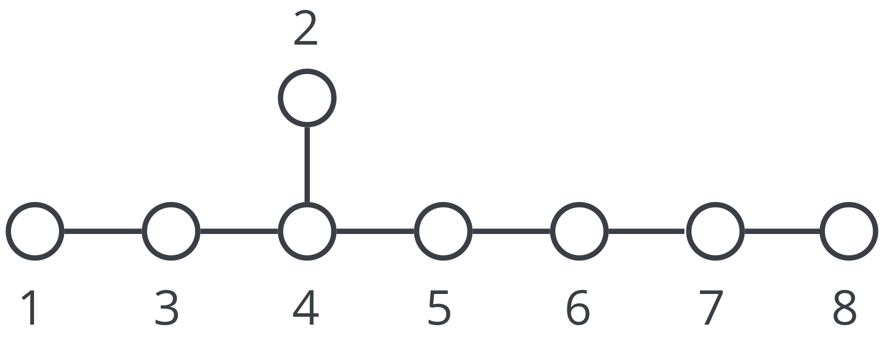

$G = E_{d+1}$

Low-energy expansion

4-graviton scattering amplitude on $T^d \times \mathbb{R}^D$ $(D = 10 - d)$

$$\mathcal{A}^{(D)}(s,t,u,\epsilon_i;g) = \left(\frac{3}{\sigma_3} + \sum_{p\geq 0}\sum_{q\geq 0} \mathcal{E}^{(D)}_{(p,q)}(g)\sigma_2^p\sigma_3^q\right)\mathcal{R}^4$$

Spherical functions on $G(\mathbb{R}) = E_{d+1}(\mathbb{R})$.
To be determined.

$\sigma_k = s^k + t^k + u^k$

Linearized Riemann curvature tensor
for each graviton $\propto k_{\mu_1}\epsilon_{\mu_2\nu_1}k_{\nu_2} +$ permutations

$$\mathcal{R}^4 = \mathcal{R}_{\mu_1\mu_2\nu_1\nu_2}\cdots\mathcal{R}_{\mu_7\mu_8\nu_7\nu_8}t^{\mu_1\ldots\mu_8}t^{\nu_1\ldots\nu_8}$$

Standard rank 8 tensor such that, for antisymmetric matrix M:
$t^{\mu_1\ldots\mu_8}M_{\mu_1\mu_2}\cdots M_{\mu_7\mu_8} = 4\mathrm{Tr}(M^4) - \left(\mathrm{Tr}(M^2)\right)^2$

Low-energy expansion

A convenient way to describe scattering amplitudes is by an *effective action*: a *field theory* whose *classical interactions* give rise to the same quantum corrected amplitudes obtained from string theory.

The classical solutions are given by the *stationary points* of the *action functional*.

momentum $\to \partial$ $\qquad$ $\sigma_2^p \sigma_3^q \mathcal{R}^4 \to \nabla^{4p+6q} R^4$

Same contraction but with full space-time Riemann curvature tensor

Space-time metric

$$S = S_{\rm EH} + \int d^D x \sqrt{-G} \Big((\alpha')^3 \underline{\mathcal{E}^{(D)}_{(0,0)}(g)} R^4 + (\alpha')^5 \underline{\mathcal{E}^{(D)}_{(1,0)}(g)} \nabla^4 R^4 + (\alpha')^6 \underline{\mathcal{E}^{(D)}_{(0,1)}(g)} \nabla^6 R^4 + \dots \Big)$$

Einstein-Hilbert action in general relativity

$$S_{\rm EH} = \int d^D x \sqrt{-G}\, R$$

$$R^4, \;\; D^4 R^4, \;\; D^6 R^4$$

Automorphic forms

In the next few slides we will list the conditions on the coefficient functions imposed by string theory and compare them to the definition of an automorphic form.

Automorphic invariance	U-duality ($G(\mathbb{Z})$-invariance)
K-finiteness	Spherical ($K(\mathbb{R})$-invariance)
$\mathcal{Z}$-finiteness	Supersymmetry
Growth condition	String theory limits e.g. perturbation theory

U-duality

Two physical theories are *dual* if they give the same physical observables, such as scattering amplitudes.

In string theory, *U-duality* implies that the coefficients $\mathcal{E}_{(p,q)}(g)$ in the 4-graviton scattering amplitudes are *invariant* under right-translations of $G(\mathbb{Z}) = E_{d+1}(\mathbb{Z})$

$$\mathcal{E}_{(p,q)}(\gamma g k) = \mathcal{E}_{(p,q)}(g) \qquad \gamma \in G(\mathbb{Z}),\, k \in K(\mathbb{R})$$

Why discrete? Should *preserve the lattice* of quantized charges.

[Hull–Townsend 95, Obers–Pioline 99]

U-duality

$$\mathcal{E}_{(p,q)}(\gamma g k) = \mathcal{E}_{(p,q)}(g) \qquad \gamma \in G(\mathbb{Z}),\ k \in K(\mathbb{R})$$

$D = 10 - d$	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$
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8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
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[Hull–Townsend 95, Obers–Pioline 99]

See also [Becker–Becker–Schwarz 06, Polchinski 07, Blumenhagen–Lüst–Theisen 13]

Supersymmetry

There are similar expansions based on scattering amplitudes of *other particles* which can be added to the effective action.
The different coefficients are related by *supersymmetry*.

Einstein gravity $\longrightarrow$ Supergravity

$$S_{\text{EH}} \longrightarrow S_{\text{SUGRA}}$$

Requiring that the effective action is supersymmetric leads to *differential equations* for the coefficients $\mathcal{E}_{(p,q)}(g)$

We will see later how these differential equations are connected to *automorphic representations*.

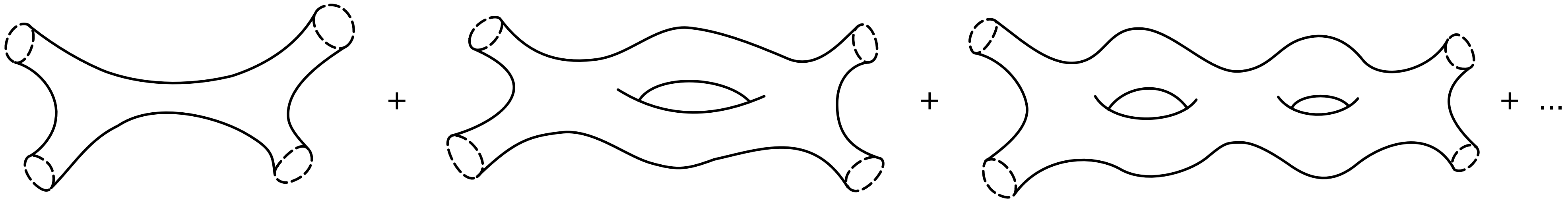
See the different results in later slides for references

Limits

String theory also tells us how $\mathcal{E}_{(p,q)}(g)$ behave in different limits.

Recall that the parameters for $G(\mathbb{R})/K(\mathbb{R})$ include:

- String coupling g_{s}.
  $g_{\mathrm{s}} \to 0$ determined by string perturbation theory.



- Radius r of compactified dimension.
  $r \to \infty$ recovers scattering amplitude for dimension $D + 1$.

Example for D=10, G=SL$_2$

Let us make the conditions more explicit in an example.

$G(\mathbb{R})/K(\mathbb{R}) = SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong$ upper half-plane $\mathbb{H}$ $\ni z = x + iy$

Relevant part of effective action: $\left(\mathcal{E}^{(10)}_{(p,q)} = \mathcal{E}_{(p,q)}\right)$

$$\mathcal{E}_{(0,0)}(z)R^4 + (\alpha')^2\mathcal{E}_{(1,0)}(z)\nabla^4R^4 + (\alpha')^3\mathcal{E}_{(1,0)}(z)\nabla^6R^4$$

- U-duality gives automorphic invariance:

$$\mathcal{E}_{(p,q)}(\gamma(z)) = \mathcal{E}_{(p,q)}(z) \qquad \gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

- Supersymmetry gives differential equations:

$$(\Delta_{\mathbb{H}} - \tfrac{3}{4})\mathcal{E}_{(0,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - \tfrac{15}{4})\mathcal{E}_{(1,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - 12)\mathcal{E}_{(0,1)}(z) = -\big(\mathcal{E}_{(0,0)}(z)\big)^2$$

$$\Delta_{\mathbb{H}} = 4y^2\partial_z\partial_{\bar{z}} = y^2(\partial_x^2 + \partial_y^2)$$

[Green–Sethi 99, Sinha 02, Green–Vanhove 06]

Example for D=10, G=SL$_2$

$$\mathcal{E}_{(0,0)}(z)R^4 + (\alpha')^2\mathcal{E}_{(1,0)}(z)\nabla^4R^4 + (\alpha')^3\mathcal{E}_{(1,0)}(z)\nabla^6R^4$$

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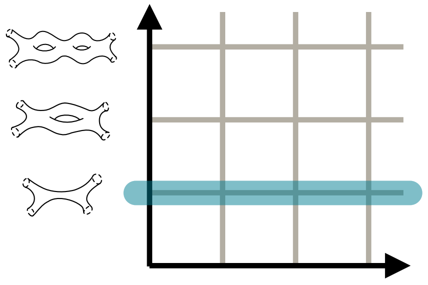
$$\Delta_{\mathbb{H}} = 4y^2\partial_z\partial_{\bar{z}} = y^2(\partial_x^2 + \partial_y^2)$$

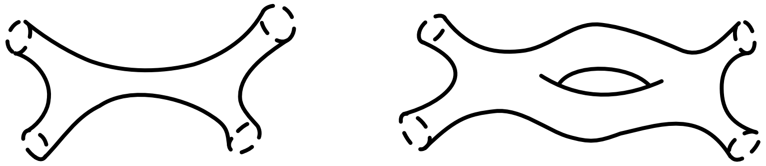
- Perturbative expansion gives limit $y^{-1} = g_{\mathrm{s}} \to 0$:



Example for D=10, G=SL₂

- Perturbative expansion gives limit $y^{-1} = g_{\text{s}} \to 0$:

$$\mathcal{A} \propto (\mathcal{A}_{\text{tree-level}} + g_{\text{s}}^2 \mathcal{A}_{\text{one-loop}} + \ldots)\mathcal{R}^4$$



Compare with z in the $SL_2(\mathbb{Z})$-invariant string moduli space $G(\mathbb{R})/K(\mathbb{R}) \cong \mathbb{H}$

$$\mathcal{A}_{\text{tree-level}} = \frac{1}{stu}\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

Modular invariant function in torus modulus τ.

$$\mathcal{A}_{\text{one-loop}} = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \mathcal{B}_1(s,t,u|\tau)$$

Fundamental domain of $SL_2(\mathbb{Z})$ acting on $\mathbb{H}$

$$\mathcal{B}_1(s,t,u|\tau) = \frac{1}{\tau_2^4} \prod_{i=1}^{4} \int_{\Sigma_1(\tau)} d^2 z_i \exp\Big(\sum_{1\le i\le j\le 4} s_{ij} G(z_i - z_j|\tau)\Big)$$

$s_{12} = s_{34} = s$

$s_{13} = s_{24} = t$

$s_{14} = s_{23} = u$

Torus with modulus τ

[Green–Schwarz–Witten 87, Green–Schwarz–Brink 82, Gross–Witten 86]

Example for D=10, G=SL$_2$

- Perturbative expansion gives limit $y^{-1} = g_{\mathrm{s}} \to 0$:

$$\mathcal{A}_{\text{tree-level}} = \frac{1}{stu}\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} = \frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\sigma_2 + \frac{2}{3}\zeta(3)^2\sigma_3 + \mathcal{O}(\alpha'^{\,4})$$

$$\mathcal{A}_{\text{one-loop}} = 2\pi \int\limits_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\mathcal{B}_1(s,t,u|\tau) = 4\zeta(2) + \frac{4}{3}\zeta(2)\zeta(3)\sigma_3 + \mathcal{O}(\alpha'^{\,4})$$

$$\sigma_k = s^k + t^k + u^k = \mathcal{O}(\alpha'^{\,k})$$

Collecting powers of α' and switching from string frame to Einstein frame giving extra powers of g_{s}:

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + \ldots$$
$$\mathcal{E}_{(1,0)}(z) = \zeta(5)y^{5/2} + \ldots$$
$$\mathcal{E}_{(0,1)}(z) = \frac{2}{3}\zeta(3)^2y^3 + \frac{4}{3}\zeta(2)\zeta(3)y + \ldots$$

[Green–Russo–Vanhove 08, D'Hoker–Green–Vanhove 15]

Example for D=10, G=SL₂

$$\mathcal{E}_{(0,0)}(z)R^4 + (\alpha')^2\mathcal{E}_{(1,0)}(z)\nabla^4R^4 + (\alpha')^3\mathcal{E}_{(1,0)}(z)\nabla^6R^4$$

- U-duality gives automorphic invariance:

$$\mathcal{E}_{(p,q)}(\gamma(z)) = \mathcal{E}_{(p,q)}(z) \qquad \gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

- Supersymmetry gives differential equations:

$$(\Delta_{\mathbb{H}} - \tfrac{3}{4})\mathcal{E}_{(0,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - \tfrac{15}{4})\mathcal{E}_{(1,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - 12)\mathcal{E}_{(0,1)}(z) = -\big(\mathcal{E}_{(0,0)}(z)\big)^2$$

$$\Delta_{\mathbb{H}} = 4y^2\partial_z\partial_{\bar{z}} = y^2(\partial_x^2 + \partial_y^2)$$

- Perturbative expansion gives limit $y^{-1} = g_{\mathrm{s}} \to 0$:

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + \dots$$
$$\mathcal{E}_{(1,0)}(z) = \zeta(5)y^{5/2} + \dots$$
$$\mathcal{E}_{(0,1)}(z) = \frac{2}{3}\zeta(3)^2y^3 + \frac{4}{3}\zeta(2)\zeta(3)y + \dots$$

Example for D=10, G=SL$_2$

$$\mathcal{E}_{(0,0)}(z)R^4+(\alpha')^2\mathcal{E}_{(1,0)}(z)\nabla^4R^4+(\alpha')^3\mathcal{E}_{(1,0)}(z)\nabla^6R^4$$

- U-duality gives automorphic invariance:
- Supersymmetry gives differential equations:

$$(\Delta_{\mathbb{H}}-\tfrac{3}{4})\mathcal{E}_{(0,0)}(z)=0 \qquad (\Delta_{\mathbb{H}}-\tfrac{15}{4})\mathcal{E}_{(1,0)}(z)=0 \qquad (\Delta_{\mathbb{H}}-12)\mathcal{E}_{(0,1)}(z)=-\big(\mathcal{E}_{(0,0)}(z)\big)^2$$

- Perturbative expansion gives limit $y^{-1}=g_{\mathrm{s}}\to 0$:

$\mathcal{E}_{(0,0)}(z)$ and $\mathcal{E}_{(1,0)}(z)$ are automorphic forms

$\mathcal{E}_{(0,1)}$ is automorphic invariant, but not $\mathcal{Z}$-finite. Thus, strictly not an automorphic form.

Example for D=10, G=SL₂

$$\mathcal{E}_{(0,0)}(z)R^4 + (\alpha')^2\mathcal{E}_{(1,0)}(z)\nabla^4R^4 + (\alpha')^3\mathcal{E}_{(1,0)}(z)\nabla^6R^4$$

- U-duality gives automorphic invariance:
- Supersymmetry gives differential equations:

$$(\Delta_{\mathbb{H}} - \tfrac{3}{4})\mathcal{E}_{(0,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - \tfrac{15}{4})\mathcal{E}_{(1,0)}(z) = 0 \qquad (\Delta_{\mathbb{H}} - 12)\mathcal{E}_{(0,1)}(z) = -\big(\mathcal{E}_{(0,0)}(z)\big)^2$$

- Perturbative expansion gives limit $y^{-1} = g_\mathrm{s} \to 0$:

Unique solutions:

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)E(\tfrac{3}{2};z)$$

$$\mathcal{E}_{(1,0)}(z) = \zeta(5)E(\tfrac{5}{2};z)$$

$$E(s;z) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma(z))^s$$

$$B(\mathbb{Z}) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \cap SL_2(\mathbb{Z})$$

$$\Delta_{\mathbb{H}}E(s;z) = s(s-1)E(s;z)$$

[Green–Gutperle 97, Green–Vanhove 97, Green–Gutperle–Vanhove 97, Pioline 98]

$\nabla^6 R^4$-term in D=10

$$\mathcal{E}_{(0,0)}(z) R^4 + (\alpha')^2 \mathcal{E}_{(1,0)}(z) \nabla^4 R^4 + (\alpha')^3 \mathcal{E}_{(1,0)}(z) \nabla^6 R^4$$

$(\Delta_{\mathbb{H}} - 12)\mathcal{E}_{(0,1)}(z) = -\big(\mathcal{E}_{(0,0)}(z)\big)^2$ $\quad$ $\mathcal{E}_{(0,1)}(z) = \frac{2}{3}\zeta(3)^2 y^3 + \frac{4}{3}\zeta(2)\zeta(3) y + \ldots$ as $y \to \infty$

Solution:

Not a character as for Eisenstein series

$$\mathcal{E}_{(0,1)}(z) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \Phi(\gamma(z))$$

Complicated expression involving Bessel functions and rational functions

$$\Phi(z) = \frac{2}{3}\zeta(3)^2 y^3 + \frac{1}{9}\pi^2 \zeta(3) y + \sum_{n \neq 0} c_n(y) e^{2\pi i n x}$$

[Green–Miller–Vanhove 15, Green–Vanhove 06, Bossard–Verschinin 15, D’Hoker–Green–Pioline–Russo 15, Bossard–Kleinschmidt 16]

Physical interpretation of Fourier expansions

Rewrite SL_2-Eisenstein series in terms of lattice:

$$E(s;z) = \frac{1}{2\zeta(2s)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}}$$

$$E(s;z) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma(z))^s$$

Poisson resummation gives:

Modified Bessel function of the second kind

$$E(s;z) = \underbrace{y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}}_{\text{Constant term}} + \underbrace{\frac{2y^{\frac{1}{2}}}{\xi(2s)} \sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{s-\frac{1}{2}} \sigma_{1-2s}(m) K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi i m x}}_{\text{Non-zero modes}}$$

$$z = x + iy \qquad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \qquad \sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s}$$

Physical interpretation of Fourier expansions

Poisson resummation gives:

$$E(s;z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{\frac{1}{2}}}{\xi(2s)} \sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{s-\frac{1}{2}} \sigma_{1-2s}(m) K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi i m x}$$

$$z = x + iy \qquad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \qquad \sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s}$$

In perturbartive expansion limit $y^{-1} \to 0$: ($g_s \to 0$)

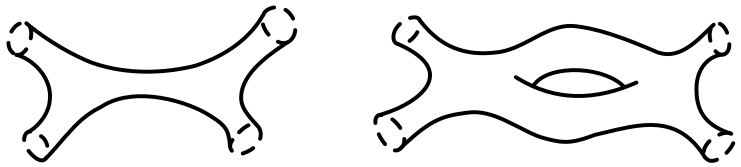
$$y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi\,|m|\,y) = e^{-2\pi|m|y}\left(\frac{1}{2\sqrt{|m|}} + \mathcal{O}(y^{-1})\right)$$

Physical interpretation of Fourier expansions

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)E(\tfrac{3}{2};z)$$

Exponentially suppressed in perturbative limit $y \to \infty$ ($g_s \to 0$)

$$\mathcal{E}_{(0,0)}(z) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}+\underbrace{4\zeta(2)y^{-1/2}}}^{\text{perturbative}}+\overbrace{2\pi\sum_{m\in\mathbb{Z}\setminus\{0\}}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\Big(1+\mathcal{O}(y^{-1})\Big)}^{\text{non-perturbative}}$$



amplitudes in presence of instantons

$$S_{\text{inst}}(z) = 2\pi\,|m|\,y - 2\pi i m x$$

The Fourier expansion of $\mathcal{E}_{(1,0)}$ is very similar, but the one for $\mathcal{E}_{(0,1)}$ is much more complicated involving Kloosterman sums and nested integrals over Bessel functions.

Instantons

Supergravity (classical field theory)

Non-trivial, “localized” solutions to the equations of motion which give *finite* values for the supergravity action: $S_{\mathrm{inst}}(z) = 2\pi\,|m|\,y - 2\pi i m x$

Instanton charge

String theory

D-instantons, which are D-branes localized to a *single point in space-time*.
Need to include *open* world-sheets with boundaries attaching to these.

Path integral now includes summing over the *number* of open world-sheets and their *topologies* weighted by the *Euler characteristic* as ${g_{\mathrm{s}}}^{-\chi}$ together with factors to *compensate for the interchange* of identical world-sheets and boundaries.

$$\sum_{d_1=0}^{\infty}\frac{1}{d_1!}\left(\frac{1}{g_{\mathrm{s}}}\langle\bigcirc\rangle\right)^{d_1}\sum_{d_2=0}^{\infty}\frac{1}{d_2!}\left(\frac{1}{2!}\langle\circledcirc\rangle\right)^{d_2}\cdots=\exp\left(\frac{1}{g_{\mathrm{s}}}\langle\bigcirc\rangle+\frac{1}{2!}\langle\circledcirc\rangle+\ldots\right)$$

More information in [Green 95, Green–Gutperle 97, Polchinski 94]

Instantons

Arguments from previous slide motivates

Instanton measure

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2}+4\zeta(2)y^{-1/2}+2\pi\sum_{m\in\mathbb{Z}\setminus\{0\}}\overbrace{\sqrt{|m|}\sigma_{-2}(m)}^{?}\overbrace{e^{-S_{\text{inst}}(z)}\Big(1+\mathcal{O}(y^{-1})\Big)}$$

The divisor sum $\sigma_{-2}(m) = \sum_{d|m} d^{-2}$ suggests that we are summing over different states with the same value for the action; a *degeneracy*.

Can be explained using a *dual theory* via *T-duality*.

[Green–Gutperle 97]

Instantons

$S_{\text{inst}}(z) = 2\pi\,|m|\,y - 2\pi i m x$

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2}+4\zeta(2)z^{-1/2}+2\pi\sum_{m\in\mathbb{Z}\setminus\{0\}}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}}\Big(1+\mathcal{O}(y^{-1})\Big)$$

Can be explained using a *dual theory* via *T-duality*.

Type IIA string theory on $\mathbb{R}^9 \times S^1_{(r)}$ $\xleftrightarrow{\text{T-duality}}$ Type IIB string theory on $\mathbb{R}^9 \times S^1_{(1/r)}$

D-particle with integer Ramond-Ramond charge n and worldline wrapping the circle d times.

$$\begin{tikzcd} S_{\text{particle}} = \dots \arrow[d, "r \to 0"] \\ S_{\text{instanton}} = 2\pi\,|nd|\,y - 2\pi i(nd)x \end{tikzcd}$$

D-instanton with charge $m = nd$.

Degeneracy $=$ number of divisors of m

[Green–Gutperle 97]

Instantons

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2} + 4\zeta(2)z^{-1/2} + 2\pi \sum_{m\in\mathbb{Z}\setminus\{0\}} \sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}}\left(1+\mathcal{O}(y^{-1})\right)$$

It is possible to derive the *appearance of the entire divisor sum* $\sigma_{-2}(m)$ in the instanton measure directly from physics.

The arguments are based on relating the D-particles of the previous slide to *Kaluza–Klein modes of M-theory compactified on a circle*. These can then be counted by the partition function of an $SU(m)$ *super Yang–Mills matrix model* and the *Witten index* of m D-particles.

For more information see [Kostov–Vanhove 98, Moore–Nekrasov–Shatashvili 00]

Lower dimensions – larger groups

Space-time: $T^d \times \mathbb{R}^D$

$$\mathcal{E}_{(p,q)}(\gamma g k) = \mathcal{E}_{(p,q)}(g) \qquad \gamma \in G(\mathbb{Z}),\ k \in K(\mathbb{R})$$

$D = 10 - d$	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$
✓ 10	$SL_2(\mathbb{R})$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z})$
9	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
7	$SL_5(\mathbb{R})$	$SO_5(\mathbb{R})$	$SL_5(\mathbb{Z})$
6	$Spin_{5,5}(\mathbb{R})$	$\big(Spin_5(\mathbb{R}) \times Spin_5(\mathbb{R})\big)/\mathbb{Z}_2$	$Spin_{5,5}(\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp_8(\mathbb{R})/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU_8(\mathbb{R})/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin_{16}(\mathbb{R})/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

Lower dimensions – larger groups

Space-time: $T^d \times \mathbb{R}^D$

Supersymmetry:

$$R^4: \left(\Delta_{G/K} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)}(g) = 6\pi\delta_{D,8}$$

$$\nabla^4 R^4: \left(\Delta_{G/K} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)}(g) = 40\zeta(2)\delta_{D,7} + 7\mathcal{E}^{(6)}_{(0,0)}\delta_{D,6}$$

$$\nabla^6 R^4: \left(\Delta_{G/K} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)}(g) = -\boxed{\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2} + 40\zeta(3)\delta_{D,6} + \frac{55}{3}\mathcal{E}^{(5)}_{(0,0)}\delta_{D,5} + \frac{85}{2\pi}\mathcal{E}^{(4)}_{(1,0)}\delta_{D,4}$$

[Green–Russo–Vanhove 10, Pioline 15]

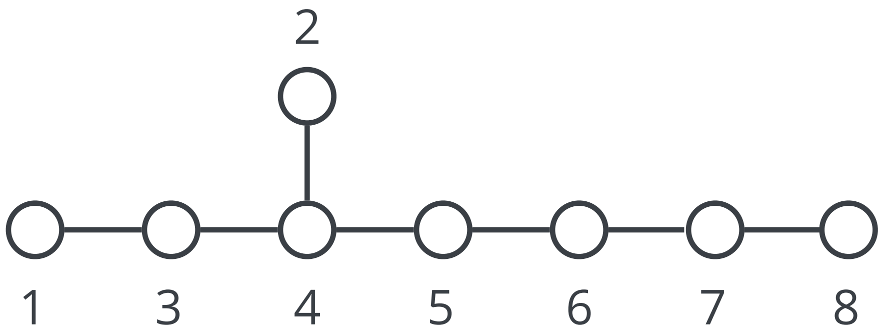
For discussions about the Kronecker deltas see [Pioline 15, Bossard–Kleinschmidt 15]

For equations with other G-invariant differential operators:
[Bossard–Verschinin 14, 15a ,15b]

Lower dimensions – larger groups

Space-time: $T^d \times \mathbb{R}^D$

$$\text{Solutions for } G = \begin{cases} E_6 & (D=5) \\ E_7 & (D=4) \\ E_8 & (D=3) \end{cases}$$



$$R^4: \qquad \mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3) E(\lambda_{s=3/2}, g), \qquad \lambda_s = 2s\Lambda_1 - \rho$$

$$\nabla^4 R^4: \qquad \mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5) E(\lambda_{s=5/2}, g).$$

Weyl vector

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} |\gamma g|^{\lambda+\rho}$$

Character on B: $|nak|^{\lambda'} = |a|^{\lambda'} \in \mathbb{C}$

$$|a_1 a_2|^{\lambda'} = |a_1|^{\lambda'} |a_2|^{\lambda'} \qquad \left|e^{tH_\alpha}\right|^{\lambda'} = |t|^{\lambda'(H_\alpha)}$$

Borel subgroup $\quad G = BK = NAK \quad$ N unipotent radical

For details and other dimensions see: [Green–Russo–Vanhove 10a, 10b, Pioline 10, Green–Miller–Russo–Vanhove 10, Green–Miller–Vanhove 15].

For D6R4 see [FGKP §15.1] and references therein.

Usually written with parabolic Eisenstein series. See Proposition 5.30 in [FGKP]

Remarks

New non-perturbative terms

Numerous consistency checks (also of new predictions).
See list of references at end of §2.4.3 [HG, Thesis]

Two different viewpoints:

The Fourier coefficients of automorphic forms give us perturbative and non-perturbative scattering amplitude corrections which are difficult to compute directly from string theory.

Theoretical physicist have devised a complex machinery (involving path integrals over geometries, supersymmetry and vertex operators) that generate Fourier coefficients of automorphic forms.

Small representations

$$R^4: \qquad \mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3)E(\lambda_{s=3/2}, g),$$ minimal representation

$$\nabla^4 R^4: \qquad \mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5)E(\lambda_{s=5/2}, g).$$ next-to-minimal representation

[Green–Miller–Russo–Vanhove 10, Pioline 10, Green–Miller–Vanhove 15]

The size of the representation is determined by: (real groups)

Irreducible representation ⟷ Annihilator ideal of universal env. alg. $\mathcal{U}(\mathfrak{g})$ ⟷ Closure of nilpotent orbit in $\mathfrak{g}$

Differential operators

min and ntm representations ← Partial ordering

[Joseph 85, Green–Miller–Vanhove 15]

Nilpotent orbits

For a nilpotent element $X \in \mathfrak{g}$, let

$$\mathcal{O}_X = \{gXg^{-1} : g \in G(\mathbb{C})\}$$

For classical groups, nilpotent orbits are labeled by certain integer partitions.

(with some caveats for D_n and very even partitions)

Partial ordering:

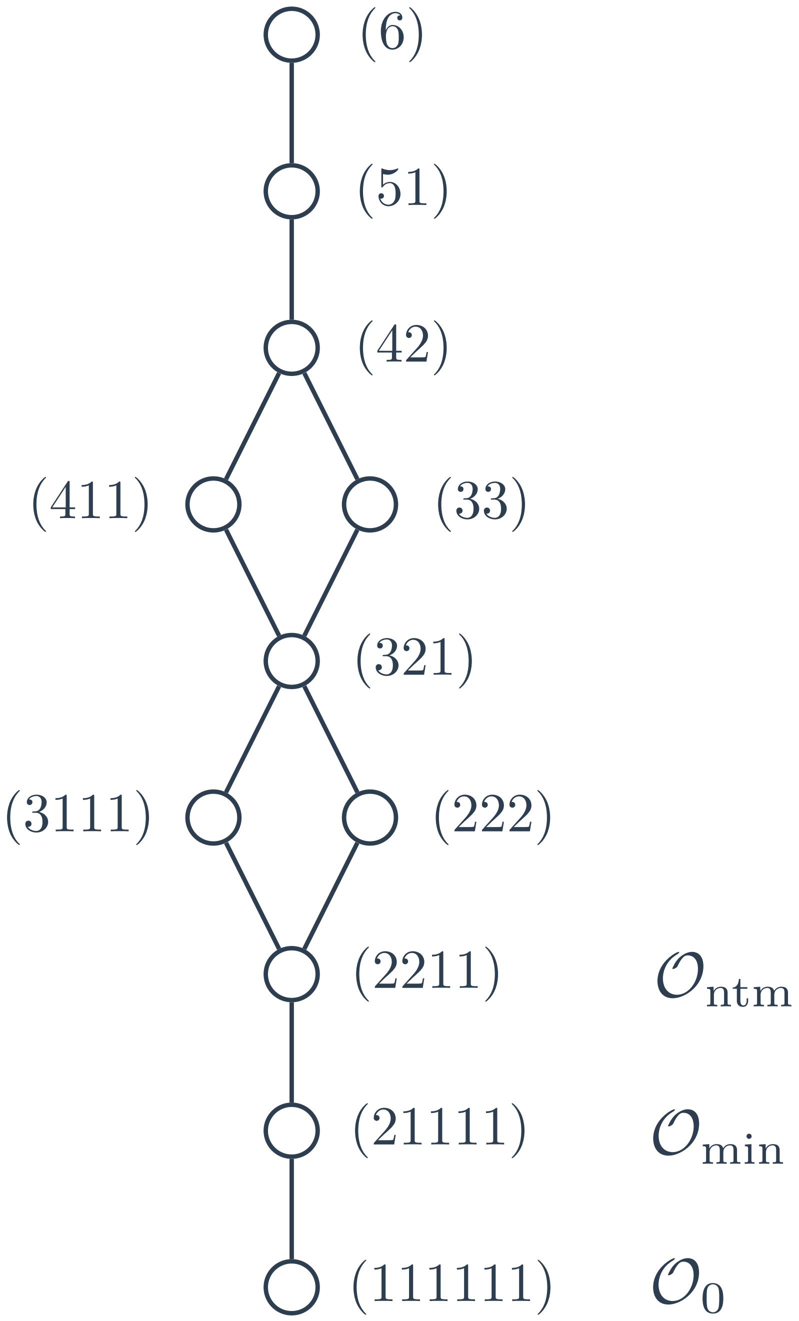
$$(\lambda_1, \ldots, \lambda_N) \leq (\mu_1, \ldots, \mu_N)$$

$$\iff$$

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \text{ for } 1 \leq k \leq N$$

Closure: $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$

SL_6



[Collingwood–McGovern 93]

Small representations

$$\begin{tikzcd} \text{Irreducible representation} \arrow[r, leftrightarrow] & \text{Annihilator ideal of universal env. alg. } U(\mathfrak{g}) \arrow[r, leftrightarrow] & \text{Closure of nilpotent orbit in } \mathfrak{g} \arrow[d, phantom, "\|"] \\ & & \overline{\mathrm{WF}}(\pi) \end{tikzcd}$$

Local wave-front set of representation $\pi = \pi_v$:

$$\text{Character } \chi_\pi \xrightarrow[\text{Infinitesimal around identity}]{} \sum_{\text{orbits } \mathcal{O}} c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}}$$

Fourier transforms of G-measures of orbits

$$\overline{\mathrm{WF}}(\pi) = \bigcup_{\substack{\text{orbit } \mathcal{O} \\ c_{\mathcal{O}}(\pi)\neq 0}} \overline{\mathcal{O}}$$

[Howe 78, Harish-Chandra 77, Barbasch–Vogan 80, Gourevitch–Sahi 18, §4]

Small representations

Irreducible representation ⟷ Annihilator ideal of universal env. alg. $U(\mathfrak{g})$ ⟷ Closure of nilpotent orbit in $\mathfrak{g}$

$$\parallel$$

$$\overline{\mathrm{WF}}(\pi)$$

Representation has implications for the vanishing properties of Fourier coefficients .

Archimedean (real): [Matumoto 87]
Global (adelic): [Ginzburg–Rallis–Soudry 03, Gomez–Gourevitch–Sahi 17, Jiang–Liu–Savin 16]

(Non-archimedean counterpart: [Mœglin–Waldspurger 87])

Automorphic forms in small representations have few non-vanishing Fourier coefficients

Fourier coefficients

To compute Fourier coefficients when $G = SL_2$ we had a simplification of the moduli space $G(\mathbb{R})/K(\mathbb{R}) \cong \mathbb{H}$ and could rewrite the Eisenstein series using a *lattice sums* (without extra constraints).

This will not always be possible for larger groups. To compute Fourier coefficients here we turn to the *adelic framework*.

The ring of adeles

$$\mathbb{Q} \xrightarrow[\text{Standard norm } |\cdot|]{\text{Completion of Cauchy sequences}} \mathbb{R} = \mathbb{Q}_\infty \qquad \mathbb{Q} \xrightarrow[p\text{-adic norm } |\cdot|_p]{} \mathbb{Q}_p$$

Archimedean

Non-archimedean

For each prime p there is another norm. With $q \in \mathbb{Q}$ prime factorized as $q = p_1^{k_1} \cdots p_n^{k_n}$ we define

$$|q|_p = \begin{cases} p_i^{-k_i} & \text{if } p = p_i \text{ for any } i \\ 1 & \text{otherwise} \end{cases}$$

The p-adic completion of $\mathbb{Z}$ is: $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

Ring of adeles:

$$\mathbb{A} = \mathbb{R} \times \prod_{\text{prime } p}{}' \mathbb{Q}_p$$

where the prime denotes a restriction to elements $a = (a_\infty; a_2, a_3, a_5, a_7, \ldots)$ such that, for all but a finite number of primes p, $a_p \in \mathbb{Z}_p$.

This ensures that the global norm $|a| = \prod_{p \leq \infty} |a_p|_p$ converges.

The ring of adeles

$$\mathbb{Q} \xrightarrow[\text{Standard norm } |\cdot|]{} \mathbb{R} = \mathbb{Q}_\infty \qquad \mathbb{Q} \xrightarrow[p\text{-adic norm } |\cdot|_p]{} \mathbb{Q}_p$$

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This ensures that the global norm $|a| = \prod_{p \leq \infty} |a_p|_p$ converges.

$\mathbb{Q}$ is embedded diagonally in $\mathbb{A}$, that is $q \in \mathbb{Q} \mapsto (q; q, q, \ldots) \in \mathbb{A}$.

$\mathbb{Q}$ is discrete in $\mathbb{A}$.

We also have that Q\A is compact

Adelization

$$\begin{tikzcd}
\text{Automorphic forms on } G(\mathbb{A}) \arrow[r, "\text{Compute}"] & \mathbb{Q}\backslash\mathbb{A}\text{-Fourier coefficients} \arrow[d, "\text{Restriction}"', "g = (g_{\mathbb{R}}; \mathbb{1}, \mathbb{1}, \ldots)"] \\
\text{Automorphic forms on } G(\mathbb{R}) \arrow[u, "\text{Adelization}", "\begin{array}{c} G(\mathbb{Z})\backslash G(\mathbb{R}) \\ \cong \\ G(\mathbb{Q})\backslash G(\mathbb{A}) / \prod_{\text{prime } p} G(\mathbb{Z}_p) \end{array}"'] & Z\backslash R\text{-Fourier coefficients}
\end{tikzcd}$$

For details see [FGKP §2, §6]

Adelization

Automorphic forms on $G(\mathbb{R})$ $\xrightarrow[\text{Adelization}]{}$ Automorphic forms on $G(\mathbb{A})$

$\mathbb{Z}$ is discrete in $\mathbb{R}$

$\mathbb{Q}$ is discrete in $\mathbb{A}$

In particular, for Eisenstein series:

$$E_{\mathbb{R}}(\lambda, g_{\mathbb{R}}) = \sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} |\gamma g_{\mathbb{R}}|^{\lambda+\rho} \longrightarrow E(\lambda, g) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} |\gamma g|^{\lambda+\rho}$$

$$E_{\mathbb{R}}(\lambda, g_{\mathbb{R}}) = E(\lambda, (g_{\mathbb{R}}; \mathbb{1}, \mathbb{1}, \ldots))$$

Fourier coefficients

Let $\mathbf{e}$ be an *additive character* on $\mathbb{A}$ trivial on $\mathbb{Q}$, let U be a *unipotent subgroup* of G and φ an automorphic form on $G(\mathbb{A})$.

Let ψ be a *unitary character* on $U(\mathbb{A})$, that is, a group homomorphism $U(\mathbb{A}) \to U(1)$ trivial on $U(\mathbb{Q})$.

$\{z \in \mathbb{C} : |z| = 1\}$

Since $\psi(u_1 u_2) = \psi(u_1)\psi(u_2)$ we only need to specify ψ on the abelianization $U(\mathbb{A})/[U,U](\mathbb{A})$ supported on roots $\Delta^{(1)}(\mathfrak{u}) = \Delta(\mathfrak{u}) \setminus \Delta([\mathfrak{u},\mathfrak{u}])$.

$$\psi : \prod_{\alpha \in \Delta^{(1)}(\mathfrak{u})} \exp(u_\alpha E_\alpha) \mapsto \mathbf{e}\Big(\sum_{\alpha \in \Delta^{(1)}(\mathfrak{u})} m_\alpha u_\alpha \Big)$$

$m_\alpha \in \mathbb{Q}$, charge or mode number

Fourier coefficients

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The associated Fourier coefficient:

$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\, du$$

Fourier coefficients

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$$\mathcal{F}_{U,\psi}[\varphi](g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\,du$$

If U is abelian, then

$$\varphi(g) = \mathcal{F}_{U,1}[\varphi](g) + \sum_{\psi\neq 1} \mathcal{F}_{U,\psi}[\varphi](g)$$

Otherwise, let $U^{(i+1)} = [U^{(i)}, U^{(i)}]$ with $U^{(0)} = U$ and $\psi^{(i)}$ unitary character on $U^{(i)}(\mathbb{A})$. Then,

$$\varphi(g) = \mathcal{F}_{U^{(0)},1}[\varphi](g) + \sum_{\psi^{(0)}\neq 1} \mathcal{F}_{U^{(0)},\psi^{(0)}}[\varphi](g) + \sum_{\psi^{(1)}\neq 1} \mathcal{F}_{U^{(1)},\psi^{(1)}}[\varphi](g) + \cdots$$

Fourier coefficients

If U is abelian, then

$$\varphi(g) = \mathcal{F}_{U,1}[\varphi](g) + \sum_{\psi \neq 1} \mathcal{F}_{U,\psi}[\varphi](g)$$

Otherwise, let $U^{(i+1)} = [U^{(i)}, U^{(i)}]$ with $U^{(0)} = U$ and $\psi^{(i)}$ unitary character on $U^{(i)}(\mathbb{A})$. Then,

$$\varphi(g) = \underbrace{\mathcal{F}_{U^{(0)},1}[\varphi](g)}_{\text{Constant term}} + \underbrace{\sum_{\psi^{(0)} \neq 1} \mathcal{F}_{U^{(0)},\psi^{(0)}}[\varphi](g)}_{\text{"Abelian coefficients"}} + \underset{\text{"Non-abelian coefficients"}}{\underset{\longmapsto}{\sum_{\psi^{(1)} \neq 1} \mathcal{F}_{U^{(1)},\psi^{(1)}}[\varphi](g) + \cdots}}$$

For details see [FGKP §6]

Whittaker pairs

When manipulating Fourier coefficients it is convenient to describe U and ψ with a pair $(S, f) \in \mathfrak{g}(\mathbb{Q}) \times \mathfrak{g}(\mathbb{Q})$ as follows.

Let $S \in \mathfrak{g}(\mathbb{Q})$ be *semisimple* and such that $\mathrm{ad}(S)$ has rational eigenvalues. For $r \in \mathbb{Q}$ define $\mathfrak{g}^S_r = \{x \in \mathfrak{g} : \mathrm{ad}(S)x = rx\}$ and let $f \in \mathfrak{g}^S_{-2}(\mathbb{Q})$ which is a *nilpotent* element.

Then, $U_{S,f}$ is defined by the Lie algebra

$$\mathfrak{u}_{S,f} = \mathfrak{g}^S_{>1} \oplus \mathfrak{g}^S_1 \cap \mathfrak{g}_f$$

where $\mathfrak{g}_f$ is the centralizer of f in $\mathfrak{g}$ under the adjoint action.

Additionally, with $\langle\ ,\ \rangle$ denoting the Killing form, ψ_f is defined by

$$\psi_f(u) = \mathbf{e}(\langle f, \log(u)\rangle), \qquad u \in U_{S,f}(\mathbb{A}),\ \mathbf{e} \text{ character on } \mathbb{Q}\backslash\mathbb{A}$$

[Gourevitch–Gomez–Sahi 17]

Whittaker pairs

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$$\psi_f(u) = \mathbf{e}(\langle f, \log(u)\rangle), \qquad u \in U_{S,f}(\mathbb{A}), \ \mathbf{e} \text{ character on } \mathbb{Q}\backslash\mathbb{A}$$

The associated Fourier coefficient is then

$$\mathcal{F}_{S,f}[\varphi](g) = \int\limits_{U_{S,f}(\mathbb{Q})\backslash U_{S,f}(\mathbb{A})} \varphi(ug)\psi_f(u)^{-1} du$$

[Gourevitch–Gomez–Sahi 17]

PART II

Outline – Part II

- Automorphic representations and the global wave-front set
- Different parabolic subgroups and their interpretations in physics
- BPS-orbits and character variety orbits
- Computing Fourier coefficients
  - Langlands' constant term formula
  - Casselman–Shalika formula
  - The subgroup reduction formula
  - Orbit methods
- Kac–Moody groups (in preparation for the last two talks on Friday)

Automorphic representations

Let $\mathcal{A}$ denote the space of automorphic forms on $G(\mathbb{A})$ and $G_f = \prod'_{p<\infty} G(\mathbb{Q}_p)$.

$g_f \in G_f$ acts on $\varphi \in \mathcal{A}$ by the *right-regular action*: $[\pi(g_f)\varphi](h) = \varphi(hg_f)$ for $h \in G(\mathbb{A})$.

But the same right-regular action for $g \in G(\mathbb{R})$ does not preserve the *K-finiteness* condition, and thus takes us outside the space of automorphic forms. We do however have a *right-regular action for $k \in K(\mathbb{R})$*.

Besides the right-regular actions by G_f and $K(\mathbb{R})$ we also have an action by the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g}_{\mathbb{R}})$ as *differential operators*.

The actions by $K(\mathbb{R})$ and $\mathcal{U}(\mathfrak{g}_{\mathbb{R}})$ both commute with the action by G_f but not with each other. Instead they give $\mathcal{A}$ the structure of a so called *$(\mathfrak{g}_{\mathbb{R}}, K(\mathbb{R}))$-module*.

An *automorphic representation* is an irreducible component of $\mathcal{A}$ under the simultaneous action by $(g_{\mathbb{R}}, K(\mathbb{R})) \times G_f$.

[Bump 09, FGKP §5]

Global wave-front set

In the adelic picture, the size of a representation is defined by the global wave-front set.

Jacobson–Morozov and Kostant:

$$\text{Nilpotent orbits} \quad \xleftrightarrow{1:1} \quad \text{Conjugacy classes of } SL_2\text{-triples } (f,h,e) \text{ in } \mathfrak{g}$$

For an (irreducible) automorphic representation π

$$\mathrm{WF}(\pi) = \{\mathcal{O}_e : \mathcal{F}_{h,f}[\varphi] \neq 0 \text{ for some } \varphi \in \pi \text{ and triple } (f,h,e)\}$$

Lemma 3.3.1 in [Gourevitch–HG–Kleinschmidt–Persson–Sahi]:

$$\mathcal{F}_{S,f}[\varphi](g) = \mathcal{F}_{\mathrm{Ad}(\gamma)S,\mathrm{Ad}(\gamma)f}[\varphi](\gamma g), \qquad \gamma \in G(\mathbb{Q})$$

Lemma 2.2.4 in [Gourevitch–HG–Kleinschmidt–Persson–Sahi]:
"The global notion supersedes the local notion. The former is not as restrictive."

[Collingwood–McGovern 93]
[Ginzburg–Rallis–Soudry 03, Gomez–Gourevitch–Sahi 17, Jiang–Liu– Savin 16]

Global wave-front set

This has consequences for all Fourier coefficients:

Theorem C, [Gomez–Gourevitch–Sahi 17]:

Let $\varphi \in \pi$ and $f \in \mathcal{O}$ with $\mathcal{O} \notin \overline{\mathrm{WF}}(\pi)$. Then, $\mathcal{F}_{S,f}[\varphi] = 0$ for any Whittaker pair (S, f).

That is, we only need to consider Fourier coefficients with f in the *global wave-front set* which is small for *small automorphic representations*.

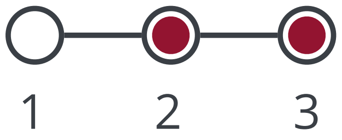
Parabolic Fourier coefficients

We will often consider the case where U is the unipotent radical of a *parabolic subgroup* P.

Levi decomposition: $P = LU$ where L is **reductive**.

In particular: *maximal* parabolic subgroup P_α for a simple root α defined by $L \approx GL_1 \times M$ where M is the semisimple subgroup obtained by removing α.

Example: $G = SL_4$, $\alpha = \alpha_1$



$$L = \left\{ \begin{pmatrix} * & & & \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{pmatrix} \right\} \cap SL_4 \qquad U = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\} \qquad P = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{pmatrix} \right\} \cap SL_4$$

Parabolic Fourier coefficients

In term of Whittaker pairs we can describe a Fourier coefficient on U for a *maximal parabolic* $P_\alpha = LU$ by S_α defined by $\alpha(S_\alpha) = 2$ and $\beta(S_\alpha) = 0$ for all other simple roots.

Then $L = \exp(\mathfrak{g}_0^{S_\alpha})$ and $U = \exp(\mathfrak{g}_{\geq 2}^{S_\alpha})$. We see that *$L$ normalizes U* (under conjugation), and, for $f \in \mathfrak{g}_{-2}^{S_\alpha}(\mathbb{Q})$ and $\gamma \in L(\mathbb{Q})$,

$$\mathcal{F}_{U,\psi_f}[\varphi](g) = \mathcal{F}_{S_\alpha,f}[\varphi](g) = \mathcal{F}_{\mathrm{Ad}(\gamma)S_\alpha,\mathrm{Ad}(\gamma)f}[\varphi](\gamma g) = \mathcal{F}_{U,\psi_{\mathrm{Ad}(\gamma)f}}[\varphi](\gamma g)$$

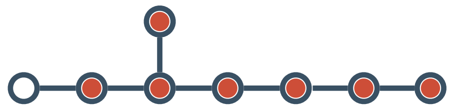
These L-orbits are called *character variety orbits* and we will soon see how they are related to *BPS-orbits*.

Different cusps and their physical interpretations

$L = GL_1 \times M$ GL_1 M (visualized for E_8)

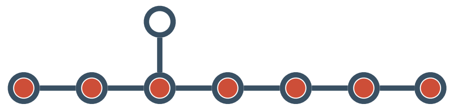
- String perturbation limit $g_s \to 0$

  D-instantons, NS5-instantons



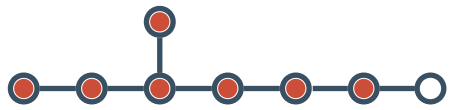
- M-theory limit volume of M-theory torus $\to \infty$

  M2, M5-instantons



- Decompactification limit radius of S^1 in $X \to \infty$

  Higher-dimensional BPS states, black holes



$E_{d+1} \to E_d$

[Pioline 10, Green–Miller–Vanhove 15]

Asymptotics of maximal parabolic Fourier coefficients

$$G = PK \text{ (not unique)}, \quad P = UL, \quad L = GL_1 \times M$$

Let φ be a *spherical automorphic form on $G(\mathbb{R})$* that is an *eigenfunction* to the Laplace–Beltrami operator $\Delta_{G/K}$ on the symmetric space $G(\mathbb{R})/K(\mathbb{R})$ with a real eigenvalue.

Then, for a suitable coordinate t for the GL_1 *factor* in L, we have the following *asymptotic behavior* of a Fourier coefficient

$$\mathcal{F}_U[\psi, \varphi](ua(t)m) \overset{t \to 0}{\sim} c_\psi(m) e^{-b_\psi/t} \psi(u), \quad u \in U, a(t) \in GL_1, m \in M$$

↑ Non-perturbative in t

[FGKP §14.2.4]

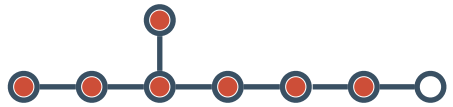
Different cusps and their physical interpretations

We will now focus on

$t = 1/r$

- Decompactification limit radius of S^1 in $X \to \infty$

  Higher-dimensional BPS states, black holes



$E_{d+1} \to E_d$

in the case where space-time is $T^6 \times \mathbb{R}^4$, that is, $G = E_7$. This will give us information about *BPS-states* in 5 *dimensions*.

BPS-states

Dimension: 5 Preserved supercharges: 32 $\mathcal{N} = 8$

We have point particles whose *electric charges* q^i furnish the *fundamental representation* $\mathbf{27}$ of E_6.

This is also a representation for the maximal compact subgroup USp_8 which has a unique *cubic invariant* I_3 that is also E_6 invariant.

BPS-states are classified by $(GL_1 \times E_6)$*-orbits* of charges which are determined by I_3:

Type	Conditions	Dimension
$\frac{1}{8}$-BPS	$I_3 \neq 0$	27
$\frac{1}{4}$-BPS	0= I_3 [redacted] $\frac{\partial I_3}{\partial q^i} \neq 0$	26
$\frac{1}{2}$-BPS	$\begin{cases} I_3 = \frac{\partial I_3}{\partial q^i} = 0 \\ \frac{\partial^2 I_3}{\partial q^i \partial q^j} \neq 0 \end{cases}$	17

[Ferrara–Maldacena 98, Becker–Becker–Schwarz 06, Green–Miller–Vanhove 15]

BPS-orbits and character orbits

These BPS-states in $D = 5$, when wrapped around a circle of radius r, give rise to instanton contributions to $\mathcal{E}^{(D\,=\,4)}_{(p,q)}$ in $D = 4$.

The contributions are *non-perturbative* in the limit $r \to \infty$ and appear in the *Fourier coefficients* of $\mathcal{E}^{(D=4)}_{(p,q)}$ corresponding to the *decompactification limit* where $L = GL_1 \times E_6$.

Fourier coefficients in L-orbits
(character variety orbits) ⟷ $(GL_1 \times E_6)$-BPS orbits

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Fourier coefficients in L-orbits
(character variety orbits) ⟷ $(GL_1 \times E_6)$-BPS orbits

27 modes or charges m_α

27 electric charges

Characters ψ_f with $f \in \mathfrak{g}^{S_\alpha}_{-2}(\mathbb{Q})$

$\dim(\mathfrak{g}^{S_\alpha}_{-2}) = 27$

↳ Becomes $\mathbb{Z}$

BPS-orbits and character orbits

The E_6 in L acts on the charges m_α by the *fundamental representation* **27** and the L-orbits of the (non-trivial) charges are of dimensions 17, 26 *and* 27.

[Miller–Sahi 12]

Compare with:

Type	Conditions	Dimension
$\frac{1}{8}$-BPS	$I_3 \neq 0$	27
$\frac{1}{4}$-BPS	0= I_3 [redacted] $\frac{\partial I_3}{\partial q^i} \neq 0$	26
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[Ferrara–Maldacena 98, Becker–Becker–Schwarz 06, Green–Miller–Vanhove 15]

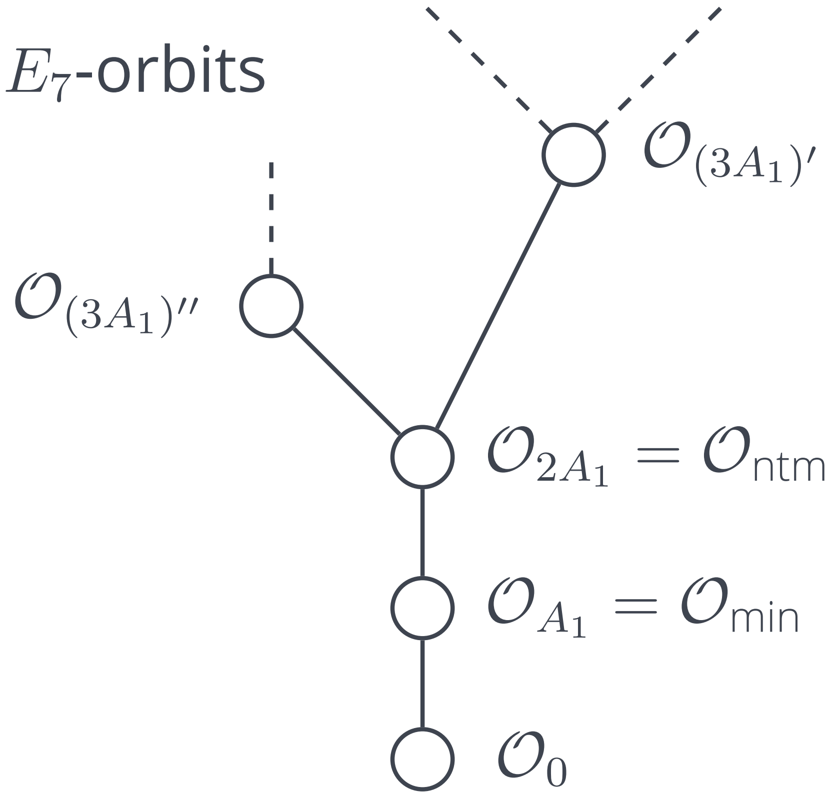
BPS-orbits and character orbits

The E_6 in L acts on the charges m_α by the *fundamental representation* $\mathbf{27}$ and the L-orbits of the (non-trivial) charges are of dimensions 17, 26 *and* 27.

These L-orbits are part of the following (coarser) G-orbits: $\mathcal{O}_{A_1}$, $\mathcal{O}_{2A_1}$, $\mathcal{O}_{(3A_1)''}$ respectively expressed with *Bala–Carter labels*.

BPS/L-orbit	$\dim(L\text{-orbit})$	Intersecting G-orbit
$\frac{1}{2}$-BPS	17	$\mathcal{O}_{A_1} = \mathcal{O}_{\text{min}}$
$\frac{1}{4}$-BPS	26	$\mathcal{O}_{2A_1} = \mathcal{O}_{\text{ntm}}$
$\frac{1}{8}$-BPS	27	$\mathcal{O}_{(3A_1)''}$

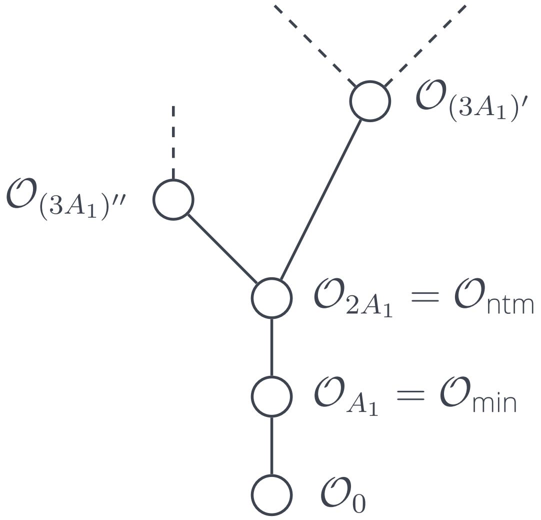
[Miller–Sahi 12]



BPS-orbits and character orbits

$$\mathcal{E}^{(D=4)}_{(0,0)}(g)R^4 + (\alpha')^2\mathcal{E}^{(D=4)}_{(1,0)}(g)\nabla^4R^4 + (\alpha')^3\mathcal{E}^{(D=4)}_{(1,0)}(g)\nabla^6R^4$$

BPS/L-orbit	$\dim(L\text{-orbit})$	Intersecting G-orbit
$\frac{1}{2}$-BPS	17	$\mathcal{O}_{A_1} = \mathcal{O}_{\text{min}}$
$\frac{1}{4}$-BPS	26	$\mathcal{O}_{2A_1} = \mathcal{O}_{\text{ntm}}$
$\frac{1}{8}$-BPS	27	$\mathcal{O}_{(3A_1)''}$



$\overline{\text{WF}}(\mathcal{E}^{(D=4)}_{(0,0)}) = \overline{\mathcal{O}_{\text{min}}}$ $\frac{1}{2}$-BPS contributions R^4 min rep

$\overline{\text{WF}}(\mathcal{E}^{(D=4)}_{(1,0)}) = \overline{\mathcal{O}_{\text{ntm}}}$ $\frac{1}{4}$-BPS contributions ∇^4R^4 ntm rep

$\overline{\text{WF}}(\mathcal{E}^{(D=4)}_{(0,1)}) \supseteq \overline{\mathcal{O}_{(3A_1)''}}$ $\frac{1}{8}$-BPS contributions ∇^6R^4

BPS degeneracies

Why do want to count all states that have a certain charge γ?
The *degeneracy* $d(\gamma)$ is related to the *entropy*:

$$S = k_B \log d(\gamma) = S_{\text{Bekenstein–Hawking}} + \text{quantum corrections}$$

Boltzmann's constant

$$S_{\text{Bekenstein–Hawking}} = \frac{\text{Area}}{4}$$

Relates microscopic quantum description to macroscopic thermodynamic quantity

As discussed in yesterday's overview talk, for BPS black holes in string theory we should *count BPS states*. We want to compare calculations done with a *D=5* BPS index with degeneracies from *D=4* Fourier coefficients.

For details see [FGKP §15.4]

BPS degeneracies

We will now consider the Fourier coefficients of the E_7 automorphic form $\mathcal{E}^{(D=4)}_{(0,0)}(g) = 2\zeta(3)E(\lambda_{s=3/2}, g)$ with respect to the decompactification limit $P_{\alpha_7} = LU$.

Will present results first and describe methods later.

Constant term:

Radius of compactified circle.
Parameter for GL_1 in L.

$$\int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{E}^{(D=4)}_{(0,0)}(ug)du = r^3\mathcal{E}^{(D=5)}_{(0,0)}(g) + 4\pi\xi(4)r^6$$

Constant term contains amplitude from higher dimension.

[Bossard–Pioline 17, Bossard–Kleinschmidt 16, FGKP §14.2.4]

BPS degeneracies

Constant term:

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{E}^{(D=4)}_{(0,0)}(ug)du = r^3\mathcal{E}^{(D=5)}_{(0,0)}(g) + 4\pi\xi(4)r^6$$

Since $\mathcal{E}^{(D=4)}_{(0,0)}$ is in a *minimal representation*, the only other non-trivial, non-vanishing Fourier modes are those attached to the minimal orbit $\mathcal{O}_{min}$ corresponding to $\frac{1}{2}$*-BPS* contributions.

Recall: Fourier coefficients in the *same BPS/L-orbit* are related by translation of the argument so we only need to compute *one case*. We pick ψ_f with $f = kE_{-\alpha_7}, k \in \mathbb{Z}$.

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{E}^{(D=4)}_{(0,0)}(u'g)\psi_f(u')^{-1}du' = \frac{8\pi r^{9/2}\left|k\right|^{-3/2}}{\left\|m^{-1}(f)\right\|^{3/2}}\sigma_3(k)K_{3/2}(2\pi\left|k\right|r\left\|m^{-1}(f)\right\|)\psi_f(u)$$

$g = ua(r)mk \in \big(U(GL_1 \times M)K\big)(\mathbb{R})$

Norm invariant under $K(\mathbb{R}) = USp_8(\mathbb{R})$

[Bossard–Pioline 17, Bossard–Kleinschmidt 16, FGKP §14.2.4]

BPS degeneracies

$$\int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{E}^{(D=4)}_{(0,0)}(u'g)\psi_f(u')^{-1}du' = \frac{8\pi r^{9/2}\left|k\right|^{-3/2}}{\left\|m^{-1}(f)\right\|^{3/2}}\sigma_3(k)K_{3/2}(2\pi\left|k\right|r\left\|m^{-1}(f)\right\|)\psi_f(u)$$

The (abelian) Fourier coefficients for $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ corresponding to the *decompactification limit* have been computed for $3 \leq D \leq 10$ in [Bossard–Pioline 17].

In all cases, the degeneracies given by the divisor sums match the corresponding *BPS-index* computed by a *helicity supertrace*.

However, method for computing Fourier coefficients was difficult to generalize to other maximal parabolic subgroups.

[Bossard–Pioline 17]

BPS degeneracies

Similar story for half-maximal supersymmetry.

Space-time: $(K3 \times T^2) \times \mathbb{R}^4$. Supersymmetry: $\mathcal{N} = 4$

The reciprocals of the *discriminant* η^{24} and the unique weight 10 Siegel modular cusp form Φ_{10}, *the Igusa cusp form*, are generating functions for $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS indices (degeneracies) respectively.

The same BPS indices are found in the Fourier coefficients for the *decompactification limit* of certain *automorphic forms* on $SO_{8,24}(\mathbb{R})/(SO_8(\mathbb{R}) \times SO_{24}(\mathbb{R}))$ that appear in the $D = 3$ effective action.

[Bossard–Cosnier-Horeau–Pioline 17]

Computing Fourier coefficients

Computing Fourier coefficients of automorphic forms is, in general, very difficult. Because of their importance in string theory (as well as in the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$) we focus on Eisenstein series.

Previously we have had simplifying circumstances such as:

- Symmetric space with additional structure (e.g. $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$)
- Lattice sum representation which allows for Poisson resummation
- Input from “external” sources (such as string theory)

We will now take a more general approach using the adelic framework discussed in previous slides.

Classification of Fourier coefficients

We have considered Fourier coefficients with respect to the unipotent radical U of different *parabolic subgroups* P.

The *minimal parabolic subgroup* is the *Borel subgroup* B (for a fixed choice of simple roots) whose unipotent radical N is a *maximal unipotent subgroup*.

A Fourier coefficient with respect to N and character ψ will be denoted by $\mathcal{W}_\psi$ and is called a *Whittaker coefficients* because of its transformation property:

$$\mathcal{W}_\psi[\varphi](ng) = \psi(n)\mathcal{W}_\psi[\varphi](g), \quad n \in N(\mathbb{A})$$

Classification of Fourier coefficients

A Fourier coefficient with respect to N and character ψ will be denoted by $\mathcal{W}_\psi$ and is called a *Whittaker coefficients* because of its transformation property:

$$\mathcal{W}_\psi[\varphi](ng) = \psi(n)\mathcal{W}_\psi[\varphi](g), \quad n \in N(\mathbb{A})$$

If φ is spherical, $\mathcal{W}_\psi[\varphi](g)$ is then determined by $\mathcal{W}_\psi[\varphi](a)$ where $g = nak$.

Short-hand notation: $\mathcal{W}_\psi[\lambda] = \mathcal{W}_\psi[E(\lambda, \cdot)]$

These are easier to compute than Fourier coefficients of smaller unipotent subgroups

Classification of Fourier coefficients

Character ψ on N is determined by $m_\alpha \in \mathbb{Q}$ for simple roots $\alpha \in \Pi$:

$$\prod_{\alpha\in\Pi} \exp(n_\alpha E_\alpha) \mapsto \mathbf{e}\Big(\sum_{\alpha\in\Pi} m_\alpha n_\alpha\Big)$$

(Assume $\ker(\mathbf{e}) = \mathbb{Q}$)

Condition	Name
All $m_\alpha = 0$	Constant term
All $m_\alpha = 1$	Unramified
All $m_\alpha \neq 0$	Generic
Otherwise	Degenerate

Langlands' constant term formula

Constant term ($\psi = 1$):

$g = nak$

Weyl group W

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, n'g)\, dn' = \sum_{w\in W} |a|^{w\lambda+\rho} M(w,\lambda)$$

$$M(w,\lambda) = \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}$$

[Langlands 76]

Similar formula for the constant term for a *maximal parabolic subgroup* P_α also exists due to [Mœglin–Waldspurger 95] with an additional factor on the right-hand side: an Eisenstein series on M in $L = GL_1 \times M$.

See also [FGKP §8.9]

Generic Whittaker coefficients

Using the Bruhat decomposition

$$G(\mathbb{Q}) = \bigcup_{w \in W} B(\mathbb{Q}) w B(\mathbb{Q})$$

one can show that, for generic Whittaker coefficients,

$$\mathcal{W}_\psi[\lambda](g) := \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)\psi(n)^{-1} dn = \prod_{p \leq \infty} \mathcal{W}_{p,\psi}[\lambda](g)$$

Eulerian

↑ Primes p where $\mathbb{Q}_\infty = \mathbb{R}$

$$\mathcal{W}_{p,\psi}[\lambda](g) = \int_{N(\mathbb{Q}_p)} |w_{\text{long}} n a|_p^{\lambda+\rho} \psi_p(n)^{-1} dn$$

$\mathcal{W}_{\infty,\psi}$ needs to be computed by hand, but $\mathcal{W}_{p,\psi}$ for $p < \infty$ can be computed using the *Casselman–Shalika formula*.

Casselman–Shalika formula

$$\mathcal{W}_{p,\psi}[\lambda](a) = \int\limits_{N(\mathbb{Q}_p)} |w_{\text{long}} n a|_p^{\lambda+\rho} \, \psi_p(n)^{-1} dn = \frac{1}{\zeta(\lambda)} \sum_{w \in W} \epsilon(w\lambda) \, |a|_p^{w\lambda+\rho}$$

$$\zeta(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{-(\langle\lambda|\alpha\rangle+1)}} \qquad \epsilon(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}}$$

[Casselman–Shalika 80]

Presented here for unramified character. See [FGKP §9] for generic case.

Reduction formula

Degenerate Whittaker coefficients *do not factorize* in general and the Casselman–Shalika formula cannot be used.

However, the coefficient can be reduced to a *generic* Whittaker coefficient on a *subgroup* G' with Weyl group W'. Using carefully chosen representatives $w_c w'_{\text{long}}$ of W/W':

$$\mathcal{W}_\psi[\lambda](a) = \sum_{w_c w'_{\text{long}} \in W/W'} |a|^{(w_c w'_{\text{long}})^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) \mathcal{W}'_{\psi^a}[\lambda'](\mathbb{1})$$

$\uparrow$

Generic Whittaker coefficient on G'

$$M(w, \lambda) = \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}$$

$$\psi^a(n) = \psi(ana^{-1})$$

$$\mathcal{W}_{\psi^a}[\lambda](\mathbb{1}) = |a|^{-(w_{\text{long}}\lambda+\rho)} \mathcal{W}_\psi(\lambda, a)$$

[Hashizume 82, Fleig–Kleinschmidt–Persson 14]

Fourier coefficients in terms of Whittaker coefficients

For G simply-laced we have the following results from
[Gourevitch–HG–Kleinschmidt–Persson–Sahi] which supersedes
[Ahlén–HG–Kleinschmidt–Liu–Persson 18] $(G = SL_n,\ n \geq 5)$
[HG–Kleinschmidt–Persson 16] $(G = SL_3$ or $SL_4)$

Let $P_{\alpha_m} = LU$ be a maximal parabolic subgroup of G. Let φ_{min} and φ_{ntm} be automorphic forms in a minimal and next-to-minimal representation of $G(\mathbb{A})$ respectively. Then, for

- $f \in \mathcal{O}_{\text{min}} : \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{min}}](g) = \mathcal{W}_{\psi_{f'}}[\varphi_{\text{min}}](\gamma_0 g)$ $\qquad \gamma_0 \in L(\mathbb{Q})$ such that $f' := \mathrm{Ad}(\gamma_0) f \in \mathfrak{g}_{-\alpha_m}$

- $f \notin \overline{\mathcal{O}_{\text{min}}} : \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{min}}](g) = 0$

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- $f \notin \overline{\mathcal{O}_{\text{min}}} : \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{min}}](g) = 0$
- $f \in \mathcal{O}_{\text{min}} : \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{ntm}}](g) = \mathcal{W}_{\psi_{f'}}[\varphi_{\text{min}}](\gamma_0 g) + \sum_{i \in I^{\perp m}} \sum_{\gamma \in \Gamma_i} \sum_{f'' \in \mathfrak{g}^{\times}_{-\alpha_i}} \mathcal{W}_{\psi_{f'+f''}}[\varphi_{\text{ntm}}](\gamma\gamma_0 g)$

Certain subsets of $L(\mathbb{Q})$ ↑

$\gamma_0 \in L(\mathbb{Q})$ such that $f' := \mathrm{Ad}(\gamma_0) f \in \mathfrak{g}_{-\alpha_m}$ $\qquad I^{\perp m} = \{i : \alpha_i \perp \alpha_m\}$

[Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Fourier coefficients in terms of Whittaker coefficients

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- $f \in \mathcal{O}_{\text{min}}: \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{ntm}}](g) = \mathcal{W}_{\psi_{f'}}[\varphi_{\text{ntm}}](\gamma_0 g) + \sum_{i\in I^{\perp m}} \sum_{\gamma\in\Gamma_i} \sum_{f''\in\mathfrak{g}^{\times}_{-\alpha_i}} \mathcal{W}_{\psi_{f'+f''}}[\varphi_{\text{ntm}}](\gamma\gamma_0 g)$
- $f \in \mathcal{O}_{\text{ntm}}: \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{ntm}}](g) = \int\limits_{V(\mathbb{A})} \mathcal{W}_{\psi_{\tilde{f}}}[\varphi_{\text{ntm}}](v\tilde{\gamma}_0 g)dv$
- $f \notin \overline{\mathcal{O}_{\text{ntm}}}: \quad \mathcal{F}_{U,\psi_f}[\varphi_{\text{ntm}}](g) = 0$

$\gamma_0 \in L(\mathbb{Q})$ such that $f' := \text{Ad}(\gamma_0)f \in \mathfrak{g}_{-\alpha_m}$ $\qquad I^{\perp m} = \{i : \alpha_i \perp \alpha_m\}$

$\tilde{\gamma}_0 \in L(\mathbb{Q})$ such that $\tilde{f} := \text{Ad}(\gamma_0)f \in \oplus_{i=1}^{r}\mathfrak{g}_{-\alpha_i}$

[Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Fourier coefficients in terms of Whittaker coefficients

The same paper also gives the *full expansion* of such automorphic forms in terms of Whittaker coefficients. Generalizes Piatetski-Shapiro and Shalika formula for cusp forms to small automorphic representations.

For *non-simply-laced groups*, or *larger automorphic representations*, Whittaker coefficients are not always enough.
We give a description of the set of Fourier coefficients that would replace these in theorems similar to above.

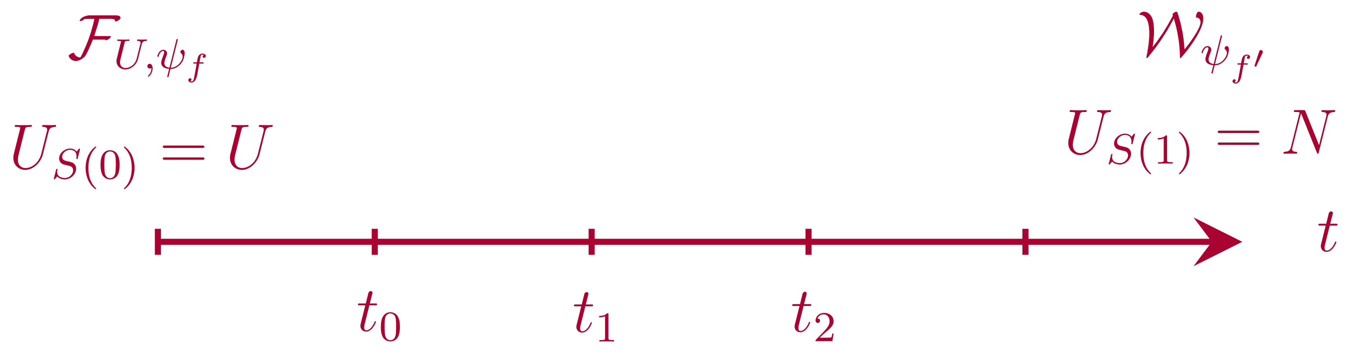
Finally, we present some *sufficient conditions* for a Fourier coefficient to be *Eulerian*.

[Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Fourier coefficients in terms of Whittaker coefficients

Earlier papers for SL_n relied on *matrix manipulations*. In this paper for reductive groups we used *Whittaker pairs* (S, f) which give a convenient way to describe both the unipotent subgroup and the character.

By *deformations* $S(t) = S + tZ$ of these pairs we could relate Fourier coefficients to the left and right of *critical points* t_i where the unipotent subgroup $U_{S(t)}$ jumps.



$\mathcal{F}_{U,\psi_f}$

$U_{S(0)} = U$

$\mathcal{W}_{\psi_{f'}}$

$U_{S(1)} = N$

t

t_0 t_1 t_2

[Gourevitch–HG–Kleinschmidt–Persson–Sahi]

Kac–Moody groups

$D = 10 - d$	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$
10	$SL_2(\mathbb{R})$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z})$
9	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
7	$SL_5(\mathbb{R})$	$SO_5(\mathbb{R})$	$SL_5(\mathbb{Z})$
6	$Spin_{5,5}(\mathbb{R})$	$\big(Spin_5(\mathbb{R}) \times Spin_5(\mathbb{R})\big)/\mathbb{Z}_2$	$Spin_{5,5}(\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp_8(\mathbb{R})/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU_8(\mathbb{R})/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin_{16}(\mathbb{R})/\mathbb{Z}_2$	$E_8(\mathbb{Z})$
	$E_9(\mathbb{R})$		
	$E_{10}(\mathbb{R})$		
	$E_{11}(\mathbb{R})$		

Kac–Moody groups

Eisenstein series can formally be defined in the same way for Kac–Moody groups

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} |\gamma g|^{\lambda+\rho}$$

Convergence is established for the affine case [Garland 06] and for rank 2 hyperbolic [Carbone–Lee–Liu 17], but open question for general case.

However, B and B^- are not conjugate (no longest Weyl word w_{long}) and there is no known relationship between the corresponding Eisenstein series in the general case.

Kac–Moody groups

Langlands' constant term formula generalized for affine case in [Garland 01]

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, n'g)\, dn' = \sum_{w\in W} |a|^{w\lambda+\rho} M(w,\lambda) \qquad M(w,\lambda) = \prod_{\substack{\alpha>0\\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}$$

↑ Infinite order

Infinitely many roots

String theory predicts a finite number of terms for the zero-mode, that is, a finite number of perturbative contributions.

This puzzle was resolved in [Fleig–Kleinschmidt 12]:

For $\lambda = \lambda_s = 2s\Lambda_1\rho$ with $s = 3/2$ and $s = 5/2$ (corresponding to the R^4 and $\nabla^4 R^4$ coefficients) the sum collapses to a finite number of terms due to $M(w,\lambda)$ eventually vanishing.

Kac–Moody groups

Because of the lack of longest Weyl word, generic Whittaker coefficients vanish.

Recall the following rewriting for a generic Whittaker coefficient in the finite case:

$$\mathcal{W}_\psi[\lambda](g) := \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)\psi(n)^{-1}dn = \prod_{p\leq\infty} \mathcal{W}_{p,\psi}[\lambda](g) \qquad \mathcal{W}_{p,\psi}[\lambda](g) = \int\limits_{N(\mathbb{Q}_p)} |w_{\text{long}}na|_p^{\lambda+\rho}\, \psi_p(n)^{-1}dn$$

In the Kac–Moody case, one *defines* a corresponding coefficient by replacing the local factors with

$$\mathcal{W}_{p,\psi}[\lambda](g) = \int\limits_{N^-(\mathbb{Q}_p)} |n'a|_p^{\lambda+\rho}\, \psi_p(n')^{-1}dn'$$

which allows for a generalization of the *Casselman–Shalika formula* in the affine case [Patnaik 17]

Kac–Moody groups

In the same way as for the finite case, the computation of degenerate Whittaker coefficient reduces to that of *generic coefficients on a subgroup*.

In [Fleig–Kleinschmidt–Persson 14] it was shown that the only non-vanishing (non-zero mode) Whittaker coefficients for $E(\lambda_s)$ on E_9, E_{10} and E_{11} with $s = 3/2$ are those that are *maximally degenerate*, that is, are generic on an SL_2 *subgroup*.

For the finite-dimensional case, this behavior is typical for an automorphic form in a *minimal representation*, but such a representation has not been defined for Kac–Moody groups. There is, for example, no notion of a minimal nilpotent orbit.

The story is similar for $s = 5/2$ with only Whittaker coefficients that are generic on at most an $SL_2 \times SL_2$ *subgroup* are non-vanishing, which is typical for an automorphic form in a *next-to-minimal representation*.

Conjecture: These Eisenstein series generate a generalization of a minimal and next-to-minimal representation respectively.

[Fleig–Kleinschmidt–Persson 14]

Thank you!

Slides available at:

hgustafsson.se